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Volumetric bounds for intersections of congruent balls in Euclidean spaces

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Abstract. We investigate the intersections of balls of radius r, called r-ball bodies, in Euclidean d-space. An r-lense (resp., r-spindle) is the intersection of two balls of radius r (resp., balls of radius r containing a given pair of points). We prove that among r-ball bodies of a given volume, the r-lense (resp., r-spindle) has the smallest inradius (resp., largest circumradius). In general, we upper (resp., lower) bound the intrinsic volumes of r-ball bodies of a given inradius (resp., circumradius). This complements and extends some earlier results on volumetric estimates for r-ball bodies.

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1. Introduction

Let $\mathbb{E}^d$ denote the d-dimensional Euclidean vector space, with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Its unit sphere centered at the origin $\mathbf{o}$ is $\mathbb{S}^{d-1} := \{\mathbf{x} \in \mathbb{E}^d \mid \|\mathbf{x}\| = 1\}$. The closed Euclidean ball of radius r centered at $\mathbf{p} \in \mathbb{E}^d$ is denoted by $\mathbf{B}^d[\mathbf{p}, r] := \{\mathbf{q} \in \mathbb{E}^d \mid |\mathbf{p} - \mathbf{q}| \leq r\}$. The Lebesgue measure on $\mathbb{E}^d$ is denoted by $V_d(\cdot)$ and the spherical Lebesgue measure on $\mathbb{S}^{d-1}$ by $SV_{d-1}(\cdot)$. If $A \subset \mathbb{E}^d$ is a compact convex set, and $0 \leq k < d$, then we denote the *kth intrinsic volume* of A by $V_k(A)$, which can be defined via the Steiner formula:

$$V_d(A + \epsilon\mathbf{B}^d[\mathbf{o}, 1]) = \sum_{i=1}^{d} \omega_{d-i} V_i(A) \epsilon^{d-i}. \tag{1}$$

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Here $V_d(A)$ (resp.,$V_d(A+\epsilon\mathbf{B}^d[\mathbf{o},1])$) is called the *volume* of A (resp., $A+\epsilon\mathbf{B}^d[\mathbf{o},1])$), $2V_{d-1}(A)$ is the *surface area* of A, $\frac{2\omega_{d-1}}{d\omega_d}V_1(A)$ is equal to the *mean width* of A, and $V_0(A)=1$, where ω_d stands for the volume of a d-dimensional unit ball, i.e., $\omega_d:=\frac{\pi^{\frac{d}{2}}}{\Gamma(1+\frac{d}{2})}$.

Definition 1. For a set $\emptyset\neq X\subseteq\mathbb{E}^d$, and $r>0$ let the r-ball body X^r generated by X be defined by $X^r:=\bigcap_{\mathbf{x}\in X}\mathbf{B}^d[\mathbf{x},r]$. If $X\subset\mathbb{E}^d$ is a finite set, then we call X^r the r-ball polyhedron generated by X in $\mathbb{E}^d$.

We note that r-ball bodies and r-ball polyhedra have been intensively studied (under various names) from the point of view of convex and discrete geometry in a number of publications (see the recent papers [2,14,16,17,19], and the references mentioned there). In particular, the following *Blaschke–Santaló-type inequalities* have been proved by Paouris and Pivovarov (Theorem 3.1 in [20]) as well as the author (Theorem 1 in [7]) for r-ball bodies in $\mathbb{E}^d$. Let $\mathbf{A}\subset\mathbb{E}^d$, $d>1$ be a compact set of volume $V_d(\mathbf{A})>0$ and $r>0$. If $\mathbf{B}\subset\mathbb{E}^d$ is a ball with $V_d(\mathbf{A})=V_d(\mathbf{B})$, then

$$V_k(\mathbf{A}^r)\leq V_k(\mathbf{B}^r) \tag{2}$$

holds for all $1\leq k\leq d$. In order to state an extension of (2) to non-Euclidean spaces we recall the following. Let $\mathbb{M}^d$, $d>1$ denote the d-dimensional Euclidean, hyperbolic, or spherical space, i.e., one of the simply connected complete Riemannian manifolds of constant sectional curvature. Since simply connected complete space forms, the sectional curvature of which have the same sign are similar, we may assume without loss of generality that the sectional curvature κ of $\mathbb{M}^d$ is $0,-1$, or 1. Let $\mathbf{R}_+$ denote the set of positive real numbers for $\kappa\leq 0$ and the half-open interval $(0,\frac{\pi}{2}]$ for $\kappa=1$. Let $\mathrm{dist}_{\mathbb{M}^d}(\mathbf{x},\mathbf{y})$ stand for the *geodesic distance* between the points $\mathbf{x}\in\mathbb{M}^d$ and $\mathbf{y}\in\mathbb{M}^d$. Furthermore, let $\mathbf{B}_{\mathbb{M}^d}[\mathbf{x},r]$ denote the closed d-dimensional ball with center $\mathbf{x}\in\mathbb{M}^d$ and radius $r\in\mathbf{R}_+$ in $\mathbb{M}^d$, i.e., let $\mathbf{B}_{\mathbb{M}^d}[\mathbf{x},r]:=\{\mathbf{y}\in\mathbb{M}^d\ |\mathrm{dist}_{\mathbb{M}^d}(\mathbf{x},\mathbf{y})\leq r\}$. Finally, for a set $X\subseteq\mathbb{M}^d$, $d>1$ and $r\in\mathbf{R}_+$ let the *r-ball body* X^r generated by X be defined by $X^r:=\bigcap_{\mathbf{x}\in X}\mathbf{B}_{\mathbb{M}^d}[\mathbf{x},r]$. The following extension of (2) to $\mathbb{M}^d$ has been proved by the author in [6]. Let $\mathbf{A}\subseteq\mathbb{M}^d$, $d>1$ be a compact set of volume $V_{\mathbb{M}^d}(\mathbf{A})>0$ and $r\in\mathbf{R}_+$. If $\mathbf{B}\subseteq\mathbb{M}^d$ is a ball with $V_{\mathbb{M}^d}(\mathbf{A})=V_{\mathbb{M}^d}(\mathbf{B})$, then

$$V_{\mathbb{M}^d}(\mathbf{A}^r)\leq V_{\mathbb{M}^d}(\mathbf{B}^r). \tag{3}$$

We note that some years before the publication of [6], Gao et al. [12] proved a special case of (3) namely, when $\mathbb{M}^d=\mathbb{S}^d$ and $r=\frac{\pi}{2}$. On the other hand, (2) and (3) were used in [6] and [7] to prove the Kneser–Poulsen conjecture for uniform contractions of sufficiently many congruent balls in $\mathbb{M}^d$ (see also Theorem 1.4 and its proof in [5]). Next, we discuss the following related result of Schneider and the author [3], which is again on upper bounding the volume of r-ball bodies for $r = \frac{\pi}{2}$ in $\mathbb{S}^d$. In order to state it, recall that a *spherically convex body* $\mathbf{K}$ is a closed, spherically convex subset of $\mathbb{S}^d$ with interior points and lying in some closed hemisphere, that is, $\mathbf{K}$ is the intersection of $\mathbb{S}^d$ with a $(d+1)$-dimensional closed convex cone of $\mathbb{E}^{d+1}$ different from $\mathbb{E}^{d+1}$. The *inradius* $r_{in}(\mathbf{K})$ of $\mathbf{K}$ is the angular radius of the largest spherical ball contained in $\mathbf{K}$. Also, recall that a *lune* in $\mathbb{S}^d$ is the d-dimensional intersection of $\mathbb{S}^d$ with two closed halfspaces of $\mathbb{E}^{d+1}$ with the origin $\mathbf{o}$ in their boundaries. Evidently, the inradius of a lune is half the interior angle between the two defining hyperplanes. Now, the main result of [3] on *volume maximizing lunes* can be stated as follows. For a more direct proof by Akopyan and Karasev see Section 6 in [1] (as well as Section 8.4 in [4]). If $\mathbf{K}$ is a spherically convex body in $\mathbb{S}^d, d \geq 2$, then

$$\mathrm{Svol}_d(\mathbf{K}) \leq \frac{(d+1)\omega_{d+1}}{\pi} r_{in}(\mathbf{K}). \tag{4}$$

Equality holds if and only if $\mathbf{K}$ is a lune. For the sake of completeness we note that (4) is used in [3] to derive the following spherical version of a Tarski-type theorem of Kadets [15]. If the spherically convex bodies $\mathbf{K}_1, \ldots, \mathbf{K}_n$ cover the spherical ball $\mathbf{B}$ of radius $r_{in}(\mathbf{B}) \geq \frac{\pi}{2}$ in $\mathbb{S}^d, d \geq 2$, then $\sum_{i=1}^{n} r_{in}(\mathbf{K}_i) \geq r_{in}(\mathbf{B})$.

The main goal of this note is to extend (4) to Euclidean spaces as follows. Let $\mathbf{K} \subset \mathbb{E}^d$ be a convex body, i.e., let $\mathbf{K}$ be a compact convex set with nonempty interior in $\mathbb{E}^d$. Then its *inradius* $r_{in}(\mathbf{K})$ (resp., *circumradius* $r_{cr}(\mathbf{K})$) is the radius of the largest (resp., smallest) ball contained in (resp., containing) $\mathbf{K}$. Furthermore, if $\mathbf{K}$ is an intersection of two balls of radius r, then we call it an r-*lense* of $\mathbb{E}^d$. In particular, we are going to use the notation $L_{r,\rho,d}$ for an r-lense whose inradius is ρ in $\mathbb{E}^d$, where $r \geq \rho > 0$.

Theorem 1. *Let $r > r_0 > 0$, $N > 1, d > 1$, and let $P := \{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N\} \subset \mathbb{E}^d$ with $r_{cr}(P) = r_0$. Then*

$$V_d(P^r) \leq V_d(L_{r,r-r_0,d}). \tag{5}$$

Remark 2. We note that $r_{in}(P^r) = r - r_0$ in Theorem 1. Thus, it follows that Theorem 1 is equivalent to the statement that among r-ball polyhedra (resp., r-ball bodies) of a given volume in $\mathbb{E}^d$, the r-lense has the smallest inradius.

One can derive from Theorem 1 the following weaker version of Kadets's theorem. (It is worth emphasizing that our proof of Corollary 3 is volumetric while the proof of Kadets's theorem published in [15] is not.)

Corollary 3. *Let $\mathbf{B}$ be a ball of radius $r > 0$ in $\mathbb{E}^d, d > 1$ and let $\mathbf{C}_i$ be an r_i-ball body with $r_i \leq r$ for $1 \leq i \leq n$ in $\mathbb{E}^d$ such that $\mathbf{B} \subseteq \bigcup_{i=1}^{n} \mathbf{C}_i$. Then*

$$r \leq \sum_{i=1}^{n} r_{in}(\mathbf{C}_i \cap \mathbf{B}). \tag{6}$$

Theorem 4. *Let* $r > r_0 > 0$, $N > 1, d > k > 0$, *and let* $P := \{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N\} \subset \mathbb{E}^d$ *with* $r_{cr}(P) = r_0$. *Then*

$$V_k(P^r) \le V_k\left(L_{r, r-\sqrt{\frac{d+1}{2d}}r_0, d}\right). \tag{7}$$

In connection with Theorems 1 and 4 it is natural to raise

Conjecture 5. *Let* $r > r_0 > 0$, $N > 1, d > k > 0$, *and let* $P := \{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N\} \subset \mathbb{E}^d$ *with* $r_{cr}(P) = r_0$. *Then*

$$V_k(P^r) \le V_k\left(L_{r, r-r_0, d}\right). \tag{8}$$

Remark 6. Recall that according to [9] (see also [11]) the r-lense has maximal perimeter among r-ball bodies of equal area in $\mathbb{E}^2$. This statement and Theorem 1 imply Conjecture 5 for $d = 2$ and $k = 1$. Hence, if $r > r_0 > 0$, $N > 1$, $d = 2$, $k = 1$, and $P := \{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N\} \subset \mathbb{E}^2$ with $r_{cr}(P) = r_0$, then

$$V_1\left(P^r\right) \le V_1\left(L_{r, r-r_0, 2}\right). \tag{9}$$

Definition 2. Let $\emptyset \ne K \subset \mathbb{E}^d$, $d > 1$ and $r > 0$. Then the r-ball convex hull $\mathrm{conv}_r K$ of K is defined by $\mathrm{conv}_r K := \bigcap\{\mathbf{B}^d[\mathbf{x}, r] \mid K \subseteq \mathbf{B}^d[\mathbf{x}, r]\}$. Moreover, let the r-ball convex hull of $\mathbb{E}^d$ be $\mathbb{E}^d$. Furthermore, we say that $K \subseteq \mathbb{E}^d$ is r-ball convex if $K = \mathrm{conv}_r K$.

We note that clearly, $\mathrm{conv}_r K = \emptyset$ if and only if $K^r = \emptyset$. Moreover, $\emptyset \ne K \subset \mathbb{E}^d$ is r-ball convex if and only if K is an r-ball body. If $K := \{\mathbf{x}, \mathbf{y}\} \subset \mathbb{E}^d$ with $0 < \|\mathbf{x} - \mathbf{y}\| \le 2r$, then $\mathrm{conv}_r K$ is called an r-*spindle* with $r_{cr} = \frac{1}{2}\|\mathbf{x} - \mathbf{y}\|$. In particular, we are going to use the notation $S_{r,\lambda,d}$ for an r-spindle whose circumradius is λ in $\mathbb{E}^d$, where $r \ge \lambda > 0$.

Theorem 7. *Let* $r > r_0 > 0$, $N > 1, d > 1$, *and let* $P := \{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N\} \subset \mathbb{E}^d$ *with* $r_{cr}(P) = r_0$. *Then*

$$V_d(S_{r,r_0,d}) \le V_d\left(\mathrm{conv}_r P\right). \tag{10}$$

Remark 8. Clearly, Theorem 7 is equivalent to the statement that among r-ball bodies of a given volume in $\mathbb{E}^d$, the r-spindle has the largest circumradius.

Corollary 9. *Let* $r > r_0 > 0$, $N > 1, d > k > 0$, *and let* $P := \{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N\} \subset \mathbb{E}^d$ *with* $r_{cr}(P) = r_0$. *Then*

$$\frac{\binom{d}{k}\omega_d^{1-\frac{k}{d}}}{\omega_{d-k}}\left(V_d(S_{r,r_0,d})\right)^{\frac{k}{d}} \le V_k\left(\mathrm{conv}_r P\right). \tag{11}$$

Moreover, if $r > r_0 > 0$, $N > 1$, $d = 2$, $k = 1$, *and* $P := \{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N\} \subset \mathbb{E}^2$ *with* $r_{cr}(P) = r_0$, *then*

$$V_1\left(S_{r,r_0,2}\right) \le V_1\left(\mathrm{conv}_r P\right). \tag{12}$$

We conclude this section by raising

Conjecture 10. *Let* $r > r_0 > 0$, $N > 1, d > k > 0$, *and let* $P := \{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N\} \subset \mathbb{E}^d$ *with* $r_{cr}(P) = r_0$. *Then*

$$V_k(S_{r,r_0,d}) \leq V_k\left(\mathrm{conv}_r P\right). \tag{13}$$

Remark 11. Conjecture 10 for $k = 1$ states that among r-ball bodies of a given circumradius the r-spindle possesses the smallest mean width. If true, then this result could be regarded as an extension of the corresponding inequality of Linhart (see inequality (1) in [18]) from convexity to r-convexity.

In the rest of the paper we prove the theorems stated.

2. Proof of Theorem 1

Lemma 12. *Let* $r > r_0 > 0$, $N > 1, d > 1$, *and let* $P := \{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N\} \subset \mathbf{B}^d[\mathbf{o}, r_0]$ *with* $r_{cr}(P) = r_0$. *Then*

$$P^r \subset \mathbf{B}^d\left[\mathbf{o}, \sqrt{r^2 - r_0^2}\right] \tag{14}$$

and so, $r_{cr}(P^r) \leq r_{cr}(L_{r,r-r_0,d}) = \sqrt{r^2 - r_0^2}$.

Proof. First, recall that Lemma 5 of [6] and (ii) of Corollary 3.4 of [2] imply

$$P^r = \left(\mathrm{conv}_r P\right)^r \text{ and } \left(P^r\right)^r = \mathrm{conv}_r P \tag{15}$$

from which it follows in a straightforward way that

$$r_{in}^{\mathbf{o}}\left(\mathrm{conv}_r P\right) + r_{cr}^{\mathbf{o}}(P^r) \leq r, \tag{16}$$

where $r_{in}^{\mathbf{o}}\left(\mathrm{conv}_r P\right) := \max\{\rho \mid B^d[\mathbf{o}, \rho] \subset \mathrm{conv}_r P\}$ and $r_{cr}^{\mathbf{o}}\left(P^r\right) := \min\{\lambda \mid P^r \subset B^d[\mathbf{o}, \lambda]\}$. Thus, (16) implies that in order to prove (14) it is sufficient to show

$$r_{in}\left(S_{r,r_0,d}\right) = r - \sqrt{r^2 - r_0^2} \leq r_{in}^{\mathbf{o}}\left(\mathrm{conv}_r P\right), \tag{17}$$

where $S_{r,r_0,d}$ is an r-spindle with circumradius r_0. Next, without loss of generality, we may assume that the circumscribed ball of $S_{r,r_0,d}$ is $\mathbf{B}^d[\mathbf{o}, r_0]$ and the inscribed ball of $S_{r,r_0,d}$ is $\mathbf{B}^d\left[\mathbf{o}, r - \sqrt{r^2 - r_0^2}\right]$. As $\mathbf{B}^d[\mathbf{o}, r_0]$ is the smallest ball containing the *convex hull* $\mathrm{conv}P$ of P (resp., $\mathrm{conv}_r P$), there must exist a simplex Δ of dimension l ($1 \leq l \leq d$) spanned by $l+1$ points of P lying on $r_0\mathbb{S}^{d-1} = \mathrm{bd}(\mathbf{B}^d[\mathbf{o}, r_0])$ such that $\mathbf{o} \in \mathrm{relint}(\Delta)$, where $\mathrm{bd}(\cdot)$ (resp., $\mathrm{relint}(\cdot)$) refers to the *boundary* (resp., *relative interior*) of the corresponding set in $\mathbb{E}^d$. (Clearly, the circumscribed ball of Δ (resp., $\mathrm{conv}_r\Delta$) is $\mathbf{B}^d[\mathbf{o}, r_0]$.) Without loss of generality, we may assume that $\Delta = \mathrm{conv}\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{l+1}\}$ with $\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{l+1}\} \subset r_0\mathbb{S}^{d-1}$. As $\mathrm{conv}_r\Delta \subseteq \mathrm{conv}_r P$, if

$$\mathbf{B}^d\left[\mathbf{o}, r - \sqrt{r^2 - r_0^2}\right] \subseteq \mathrm{conv}_r\Delta \tag{18}$$

holds, then (17) follows. So, we are left to show that indeed, (18) holds. In order to see this, recall Lemma 3.1 and Corollary 3.4 of [2] according to which for each boundary point $\mathbf{p}$ of $\mathrm{conv}_r\Delta$ there exists a $(d-1)$-dimensional sphere S of radius r (called *supporting r-sphere of* $\mathrm{conv}_r\Delta$) that bounds a ball $\mathbf{B}$ (called *supporting r-ball of* $\mathrm{conv}_r\Delta$) in $\mathbb{E}^d$ such that $\mathrm{conv}_r\Delta \subseteq \mathbf{B}$ and $\mathbf{p} \in S \cap \mathrm{conv}_r\Delta$. Moreover, $\mathrm{conv}_r\Delta$ is the intersection of its supporting r-balls. Thus, (18) follows if one can prove that

$$\mathbf{B}^d\left[\mathbf{o}, r - \sqrt{r^2 - r_0^2}\right] \subseteq \mathbf{B} \tag{19}$$

holds for any supporting r-ball $\mathbf{B}$ of $\mathrm{conv}_r\Delta$. Finally, we prove (19) as follows. First, we note that clearly, $\mathbf{p}_i \in S = \mathrm{bd}(\mathbf{B})$ for some $1 \leq i \leq l+1$. Moreover, $\{\mathbf{p}_i, -\mathbf{p}_i\} \subset r_0\mathbb{S}^{d-1}$ and $\mathrm{conv}_r\{\mathbf{p}_i, -\mathbf{p}_i\}$ is an r-spindle of inradius $r - \sqrt{r^2 - r_0^2}$. Hence, if $-\mathbf{p}_i \in \mathbf{B}$, then one obtains (19) in a straightforward way. So, the case left is when $-\mathbf{p}_i \notin \mathbf{B}$. But then, $\mathbf{B} \cap r_0\mathbb{S}^{d-1}$ is a spherical cap of angular radius $< \frac{\pi}{2}$ on $r_0\mathbb{S}^{d-1}$ containing $\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{l+1}\}$ and clearly implying that $\mathbf{o} \notin \mathrm{relint}(\Delta)$, a contradiction. This completes the proof of Lemma 12. □

For the purpose of the next statement recall that $\mathbf{B}_{\mathbb{S}^d}[\mathbf{x}, \epsilon]$ denotes the closed ball of angular radius $\epsilon \leq \frac{\pi}{2}$ centered at the point $\mathbf{x}$ in $\mathbb{S}^d$. Furthermore, for any subset X of $\mathbb{S}^d$ let $X_\epsilon := \cup_{\mathbf{x}\in X}\mathbf{B}_{\mathbb{S}^d}[\mathbf{x}, \epsilon]$ be called the *ϵ-neighbourhood* of X in $\mathbb{S}^d$. The following statement, which we need for the proof of Theorem 1, has been proved by Akopyan and Karasev (see Lemma 7 in [1] as well as Lemma 8.4.3 in [4]). In what follows, we reprove it in a similar but simpler way.

Lemma 13. *Let $\mathbf{X}$ be a closed subset of $\mathbb{S}^d$ not lying on an open hemisphere of $\mathbb{S}^d$. Then for any $\epsilon \leq \frac{\pi}{2}$ the inequality*

$$SV_d(\mathbf{X}_\epsilon) \geq SV_d(\hat{\mathbf{X}}_\epsilon) \tag{20}$$

holds, where $\hat{\mathbf{X}}$ is a pair of antipodal points of $\mathbb{S}^d$.

Proof. It is sufficient to prove Lemma 13 for finite $\mathbf{X}$ say, for $\mathbf{X} := \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\} \subset \mathbb{S}^d$. Take the (nearest point) Voronoi tiling of $\mathbb{S}^d$ generated by $\mathbf{X}$ with $\mathbf{V}_i$ standing for the Voronoi cell assigned to the point $\mathbf{x}_i$, $1 \leq i \leq m$. Let $\mathbf{H}_i$ be the closed hemisphere of $\mathbb{S}^d$ centered at $\mathbf{x}_i$, $1 \leq i \leq m$. As, by assumption, $\mathbf{X}$ does not lie on an open hemisphere of $\mathbb{S}^d$, $\mathbf{V}_i \subseteq \mathbf{H}_i$ holds for all $1 \leq i \leq m$. The following lower bound for the *density* $\frac{SV_d(\mathbf{B}_{\mathbb{S}^d}[\mathbf{x}_i,\epsilon]\cap\mathbf{V}_i)}{SV_d(\mathbf{V}_i)}$ of $\mathbf{B}_{\mathbb{S}^d}[\mathbf{x}_i, \epsilon] \cap \mathbf{V}_i$ within $\mathbf{V}_i$, is the core part of our proof of (20).

Proposition 14.

$$\frac{SV_d(\mathbf{B}_{\mathbb{S}^d}[\mathbf{x}_i, \epsilon] \cap \mathbf{V}_i)}{SV_d(\mathbf{V}_i)} \geq \frac{SV_d(\mathbf{B}_{\mathbb{S}^d}[\mathbf{x}_i, \epsilon])}{SV_d(\mathbf{H}_i)} \tag{21}$$

holds for all $1 \leq i \leq m$.

Proof. For any $\mathbf{x}, \mathbf{y} \in \mathbb{S}^d$ with $\mathbf{x} \neq -\mathbf{y}$ let $[\mathbf{x}, \mathbf{y}]_{\mathbb{S}^d}$ denote the *geodesic segment* connecting $\mathbf{x}$ and $\mathbf{y}$, i.e., let $[\mathbf{x}, \mathbf{y}]_{\mathbb{S}^d}$ stand for the shorter closed unit circle arc connecting $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{S}^d$.

Definition 3. For $\mathbf{a} \in \mathrm{bd}(\mathbf{H}_i)$, $1 \leq i \leq m$ let $\mathbf{b} := [\mathbf{a}, \mathbf{x}_i]_{\mathbb{S}^d} \cap \mathrm{bd}(\mathbf{B}_{\mathbb{S}^d}[\mathbf{x}_i, \epsilon])$ and $\mathbf{c} := [\mathbf{a}, \mathbf{x}_i]_{\mathbb{S}^d} \cap \mathrm{bd}(\mathbf{V}_i)$, where $\mathrm{bd}(\cdot)$ refers to the *boundary* of the corresponding set in $\mathbb{S}^d$. Then let $\mathbf{A}_i := \bigcup\{[\mathbf{a}, \mathbf{x}_i]_{\mathbb{S}^d} | \mathbf{a} \in \mathrm{bd}(\mathbf{H}_i) \text{ with } \mathbf{c} \in [\mathbf{b}, \mathbf{x}_i]\}$. Moreover, let $\mathbf{A}'_i := \mathrm{bd}(\mathbf{H}_i) \setminus \mathbf{A}_i$.

Clearly, for any $\mathbf{a} \in \mathbf{A}'_i$, $1 \leq i \leq m$ we have $\mathbf{b} \in \mathrm{relint}([\mathbf{c}, \mathbf{x}_i])$, where $\mathrm{relint}(\cdot)$ denotes the *relative interior* of the corresponding set in $\mathbb{S}^d$. Moreover, we note that $\mathbf{A}_i$ as well as $\mathbf{A}'_i$ are starshaped sets with respect to $\mathbf{x}_i$ in $\mathbb{S}^d$, where $1 \leq i \leq m$. Thus, it follows in a rather straightforward way that

$$\begin{aligned}
\frac{SV_d(\mathbf{B}_{\mathbb{S}^d}[\mathbf{x}_i, \epsilon] \cap \mathbf{V}_i)}{SV_d(\mathbf{V}_i)} &= \frac{SV_d(\mathbf{A}_i \cap \mathbf{V}_i) + SV_d(\mathbf{A}'_i \cap \mathbf{B}_{\mathbb{S}^d}[\mathbf{x}_i, \epsilon])}{SV_d(\mathbf{V}_i)} \\
&= \frac{SV_d(\mathbf{A}_i \cap \mathbf{V}_i) + SV_d(\mathbf{A}'_i \cap \mathbf{V}_i)\frac{SV_d(\mathbf{A}'_i \cap \mathbf{B}_{\mathbb{S}^d}[\mathbf{x}_i, \epsilon])}{SV_d(\mathbf{A}'_i \cap \mathbf{V}_i)}}{SV_d(\mathbf{V}_i)} \\
&\geq \left(\frac{SV_d(\mathbf{A}_i \cap \mathbf{V}_i) + SV_d(\mathbf{A}'_i \cap \mathbf{V}_i)}{SV_d(\mathbf{V}_i)}\right) \frac{SV_d(\mathbf{B}_{\mathbb{S}^d}[\mathbf{x}_i, \epsilon])}{SV_d(\mathbf{H}_i)} \\
&= \frac{SV_d(\mathbf{B}_{\mathbb{S}^d}[\mathbf{x}_i, \epsilon])}{SV_d(\mathbf{H}_i)},
\end{aligned}$$

finishing the proof of Proposition 14. □

Thus, Proposition 14 yields that

$$SV_d(\mathbf{B}_{\mathbb{S}^d}[\mathbf{x}_i, \epsilon] \cap \mathbf{V}_i) SV_d(\mathbf{H}_i) \geq SV_d(\mathbf{B}_{\mathbb{S}^d}[\mathbf{x}_i, \epsilon]) SV_d(\mathbf{V}_i),$$

or equivalently

$$SV_d(\mathbf{X}_\epsilon \cap \mathbf{V}_i) \frac{SV_d(\mathbb{S}^d)}{2} \geq SV_d(\mathbf{B}_{\mathbb{S}^d}[\mathbf{x}_i, \epsilon]) SV_d(\mathbf{V}_i) \tag{22}$$

holds for all $1 \leq i \leq m$. As $\sum_{i=1}^m SV_d(\mathbf{X}_\epsilon \cap \mathbf{V}_i) = SV_d(\mathbf{X}_\epsilon)$ and $\sum_{i=1}^m SV_d(\mathbf{V}_i) = SV_d(\mathbb{S}^d)$, (22) implies in a straightforward way that

$$SV_d(\mathbf{X}_\epsilon) \geq 2SV_d(\mathbf{B}_{\mathbb{S}^d}[\mathbf{x}, \epsilon]) = SV_d(\hat{\mathbf{X}}_\epsilon)$$

holds for $\hat{\mathbf{X}} = \{\mathbf{x}, -\mathbf{x}\}$ with $\mathbf{x} \in \mathbb{S}^d$. This completes the proof of Lemma 13. □

Now, we turn to the proof of Theorem 1. Without loss of generality we may assume that $P = \{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N\} \subset \mathbf{B}^d[\mathbf{o}, r_0]$ with $r_{cr}(P) = r_0$ implying that there exists a simplex of dimension l ($1 \leq l \leq d$) spanned by some points of P say, by $Q := \{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{l+1}\}$ lying on $r_0\mathbb{S}^{d-1} = \mathrm{bd}(\mathbf{B}^d[\mathbf{o}, r_0])$ such that $\mathbf{o} \in \mathrm{relint}(\mathrm{conv}(Q))$. As $P^r \subseteq Q^r$ and $r_{cr}(P) = r_{cr}(Q) = r_0$, Theorem 1 follows from the inequality

$$V_d(Q^r) \leq V_d(L_{r, r-r_0, d}), \tag{23}$$

where the inscribed ball of Q^r as well as $L_{r,r-r_0,d}$ is $\mathbf{B}^d[\mathbf{o}, r-r_0]$. Clearly,

$$r_{cr}(L_{r,r-r_0,d}) = \sqrt{r^2 - r_0^2} \text{ and } L_{r,r-r_0,d} \subset \mathbf{B}^d\left[\mathbf{o}, \sqrt{r^2 - r_0^2}\right]. \tag{24}$$

Moreover, Lemma 12 implies that

$$Q^r \subset \mathbf{B}^d\left[\mathbf{o}, \sqrt{r^2 - r_0^2}\right] \left(\text{and therefore } r_{cr}(Q^r) \leq \sqrt{r^2 - r_0^2}\right). \tag{25}$$

Thus, (24) and (25) yield that

$$\begin{aligned} V_d(Q^r) &= \int_0^{\sqrt{r^2-r_0^2}} \sigma(x\mathbb{S}^{d-1} \cap Q^r)dx \text{ and } V_d(L_{r,r-r_0,d}) \\ &= \int_0^{\sqrt{r^2-r_0^2}} \sigma(x\mathbb{S}^{d-1} \cap L_{r,r-r_0,d})dx, \end{aligned} \tag{26}$$

where σ denotes the proper spherical Lebesgue measure on $x\mathbb{S}^{d-1}$. Hence, using (26) we get that in order to prove (23) it is sufficient to show that

$$\sigma(x\mathbb{S}^{d-1} \cap Q^r) \leq \sigma(x\mathbb{S}^{d-1} \cap L_{r,r-r_0,d}) \tag{27}$$

holds for all x with $0 \leq x \leq \sqrt{r^2 - r_0^2}$. Now, (27) holds trivially for all $0 \leq x \leq r - r_0 = r_{in}(Q^r) = r_{in}(L_{r,r-r_0,d})$. So, we are left with the case when $r - r_0 < x \leq \sqrt{r^2 - r_0^2}$. Next, notice that according to (24) the subset $x\mathbb{S}^{d-1} \cap L_{r,r-r_0,d}$ of $x\mathbb{S}^{d-1}$ is the complement of the union of a pair of antipodal (open) spherical caps of angular radius $0 < \epsilon \leq \frac{\pi}{2}$. On the other hand, the subset $x\mathbb{S}^{d-1} \cap Q^r$ of $x\mathbb{S}^{d-1}$ is the complement of the union of $l+1$ (open) spherical caps of angular radius ϵ centered at the points $-\frac{x}{r_0}\mathbf{p}_i$, $1 \leq i \leq l+1$, which do not lie on an open hemisphere of $x\mathbb{S}^{d-1}$ because $\mathbf{o} \in \operatorname{relint}(\operatorname{conv}(Q))$. Thus, Lemma 13 implies (27) in a straightforward way. This completes the proof of Theorem 1.

3. Proof of Corollary 3

Clearly, $\mathbf{C}_i \cap \mathbf{B}$ is an r-ball body in $\mathbb{E}^d$ for all $1 \leq i \leq n$. Thus, Theorem 1 and $\mathbf{B} \subseteq \bigcup_{i=1}^n \mathbf{C}_i$ imply that

$$V_d(\mathbf{B}) \leq \sum_{i=1}^n V_d(\mathbf{C}_i \cap \mathbf{B}) \leq \sum_{i=1}^n V_d\left(L_{r, r_{in}(\mathbf{C}_i \cap \mathbf{B}), d}\right). \tag{28}$$

Finally, we note that in order to have (28) one must have $r \leq \sum_{i=1}^n r_{in}(\mathbf{C}_i \cap \mathbf{B})$, finishing the proof of Corollary 3.

4. Proof of Theorem 4

As in the proof of Theorem 1, we may assume without loss of generality that $P = \{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N\} \subset \mathbf{B}^d[\mathbf{o}, r_0]$ with $r_{cr}(P) = r_0$. It follows that there exists a simplex Δ of dimension l ($1 \leq l \leq d$) spanned by $l+1$ points of P say, by $Q := \{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{l+1}\}$ lying on $r_0\mathbb{S}^{d-1} = \mathrm{bd}(\mathbf{B}^d[\mathbf{o}, r_0])$ such that $\mathbf{o} \in \mathrm{relint}(\Delta)$. Clearly, the circumscribed ball of $\Delta = \mathrm{conv}Q$ is $\mathbf{B}^d[\mathbf{o}, r_0]$ and

$$r_{cr}(P) = r_{cr}(Q) = r_0,\ r_{in}(P^r) = r_{in}(Q^r) = r - r_0,\ \text{and}\ P^r \subseteq Q^r. \tag{29}$$

Definition 4. Let $\emptyset \neq X \subseteq \mathbb{E}^d$. Then the central symmetral (called also Minkowski symmetral) $M_{\mathbf{o}}(X)$ of X is defined by $M_{\mathbf{o}}(X) := \frac{1}{2}\left(X + (-X)\right)$.

For properties of central symmetrization we refer the interested reader to [8]. In particular, recall that the Brunn–Minkowski inequality for intrinsic volumes [13] and (29) yield

$$V_k(P^r) \leq V_k\left(M_{\mathbf{o}}(P^r)\right) \leq V_k\left(M_{\mathbf{o}}(Q^r)\right), \tag{30}$$

where $0 < k \leq d$.

Lemma 15.

$$M_{\mathbf{o}}(Q^r) = \left(M_{\mathbf{o}}(Q)\right)^r. \tag{31}$$

Proof. Clearly, (31) is equivalent to

$$M_{\mathbf{o}}(Q^r) = \bigcap\left\{\mathbf{B}^d\left[\frac{1}{2}(\mathbf{p}_i - \mathbf{p}_j), r\right] \middle| 1 \leq i, j \leq l+1\right\}, \tag{32}$$

which we prove as follows. Let $\mathbf{z} \in M_{\mathbf{o}}(Q^r) = \frac{1}{2}\left(Q^r + (-Q^r)\right)$. Then there exist $\mathbf{x}, \mathbf{y} \in Q^r$ such that $\mathbf{z} = \frac{1}{2}(\mathbf{x} - \mathbf{y})$. It follows that $\mathbf{x} \in \mathbf{B}^d[\mathbf{p}_i, r]$ and $\mathbf{y} \in \mathbf{B}^d[\mathbf{p}_j, r]$ for all $1 \leq i, j \leq l+1$ and therefore

$$\mathbf{z} = \frac{1}{2}(\mathbf{x} - \mathbf{y}) \in \frac{1}{2}\mathbf{B}^d[\mathbf{p}_i, r] + \frac{1}{2}\mathbf{B}^d[-\mathbf{p}_j, r] = \mathbf{B}^d\left[\frac{1}{2}(\mathbf{p}_i - \mathbf{p}_j), r\right] \tag{33}$$

holds for all $1 \leq i, j \leq l+1$. Clearly, (33) yields $M_{\mathbf{o}}(Q^r) \subseteq \bigcap\left\{\mathbf{B}^d\left[\frac{1}{2}(\mathbf{p}_i - \mathbf{p}_j), r\right] \middle| 1 \leq i, j \leq l+1\right\}$. On the other hand, let $\mathbf{z}' \in \bigcap\left\{\mathbf{B}^d\left[\frac{1}{2}(\mathbf{p}_i - \mathbf{p}_j), r\right] \middle| 1 \leq i, j \leq l+1\right\}$. Then $\mathbf{z}' \in \mathbf{B}^d\left[\frac{1}{2}(\mathbf{p}_i - \mathbf{p}_j), r\right] = \frac{1}{2}\left(\mathbf{B}^d[\mathbf{p}_i, r] + (-\mathbf{B}^d[\mathbf{p}_j, r])\right)$ holds for all $1 \leq i, j \leq l+1$ and therefore $\mathbf{z}' \in \frac{1}{2}\left(Q^r + (-Q^r)\right)$ implying that $\bigcap\left\{\mathbf{B}^d\left[\frac{1}{2}(\mathbf{p}_i - \mathbf{p}_j), r\right] \middle| 1 \leq i, j \leq l+1\right\} \subseteq M_{\mathbf{o}}(Q^r)$. This completes the proof of Lemma 15. □

Corollary 16. *Lemma 15 implies that $M_{\mathbf{o}}(Q^r)$ is an $\mathbf{o}$-symmetric r-ball polyhedron and therefore it is contained in an r-lense of inradius equal to*

$$r_{in}\left[M_{\mathbf{o}}(Q^r)\right] = r_{in}\left[(M_{\mathbf{o}}(Q))^r\right] = r_{in}^{\mathbf{o}}\left[(M_{\mathbf{o}}(Q))^r\right] = r - r_{cr}^{\mathbf{o}}\left[M_{\mathbf{o}}(Q)\right]. \tag{34}$$

Hence,

$$V_k\left(M_{\mathbf{o}}(Q^r)\right) \leq V_k\left(L_{r, r-r_{cr}^{\mathbf{o}}[M_{\mathbf{o}}(Q)], d}\right) \tag{35}$$

holds for all $0 < k \leq d$.

Lemma 17. *Let $Q = \{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_{l+1}\} \subset r_0\mathbb{S}^{d-1} = \mathrm{bd}(\mathbf{B}^d[\mathbf{o}, r_0])$ be given so that $\mathrm{conv}Q$ is an l-dimensional simplex with $\mathbf{o} \in \mathrm{relint}(\mathrm{conv}Q)$ in $\mathbb{E}^d$, where $1 \leq l \leq d$. Then*

$$\sqrt{\frac{d+1}{2d}}r_0 \leq \sqrt{\frac{l+1}{2l}}r_0 \leq r_{cr}^{\mathbf{o}}\left[M_{\mathbf{o}}(Q)\right]. \tag{36}$$

Proof. In fact, one may assume that $Q = \{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_{l+1}\} \subset r_0\mathbb{S}^{l-1} \subset \mathbb{E}^l$ and $\mathrm{conv}Q$ is an l-dimensional simplex with the origin lying in its interior in $\mathbb{E}^l$ (i.e., $\mathbf{o} \in \mathrm{int}(\mathrm{conv}Q)$). Clearly, $r_{cr}(Q) = r_{cr}^{\mathbf{o}}(Q) = r_0$ and

$$r_{cr}^{\mathbf{o}}\left[M_{\mathbf{o}}(Q)\right] = \max\left\{\frac{1}{2}\|\mathbf{p}_i - \mathbf{p}_j\| \middle| 1 \leq i, j \leq l+1\right\} =: \frac{1}{2}\mathrm{diam}(Q), \tag{37}$$

where $M_{\mathbf{o}}(Q)$ stands for the central symmetral of Q in $\mathbb{E}^l \subseteq \mathbb{E}^d$ and $\mathrm{diam}(Q)$ denotes the *diameter* of Q. Next, recall Jung's theorem stated as follows (see Theorem 1 in [10]): Let $C \subset \mathbb{E}^l$ be a compact set having unit circumradius. Then $2\sqrt{\frac{l+1}{2l}} \leq \mathrm{diam}(C)$. Finally, this theorem of Jung and (37) imply in a straightforward way that $\sqrt{\frac{l+1}{2l}}r_0 \leq \frac{1}{2}\mathrm{diam}(Q) = r_{cr}^{\mathbf{o}}\left[M_{\mathbf{o}}(Q)\right]$ and so, (36) follows. This completes the proof of Lemma 17. $\square$

Hence, (30), (35), and (36) yield (7), finishing the proof of Theorem 4.

5. Proof of Theorem 7

As in the proof of Theorem 1, we may assume without loss of generality that $P = \{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N\} \subset \mathbf{B}^d[\mathbf{o}, r_0]$ with $r_{cr}(P) = r_0$. It follows that there exists a simplex Δ of dimension l ($1 \leq l \leq d$) spanned by $l+1$ points of P say, by $Q := \{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_{l+1}\}$ lying on $r_0\mathbb{S}^{d-1} = \mathrm{bd}(\mathbf{B}^d[\mathbf{o}, r_0])$ such that $\mathbf{o} \in \mathrm{relint}(\Delta)$. Clearly, the circumscribed ball of $\mathrm{conv}_r\Delta = \mathrm{conv}_r Q$ is $\mathbf{B}^d[\mathbf{o}, r_0]$ and so,

$$r_{cr}^{\mathbf{o}}\left(\mathrm{conv}_r Q\right) = r_0 \text{ with } \mathrm{conv}_r Q \subseteq \mathrm{conv}_r P \text{ implying } V_d\left(\mathrm{conv}_r Q\right) \leq V_d\left(\mathrm{conv}_r P\right). \tag{38}$$

Furthermore, (17) and (18) imply

$$r_{in}^{\mathbf{o}}\left(S_{r,r_0,d}\right)=r-\sqrt{r^2-r_0^2}\leq r_{in}^{\mathbf{o}}\left(\mathrm{conv}_r Q\right), \tag{39}$$

where $S_{r,r_0,d}$ is an r-spindle having $r_{cr}^{\mathbf{o}}\left(S_{r,r_0,d}\right)=r_0$ in $\mathbb{E}^d$. Thus, it follows that

$$V_d(S_{r,r_0,d})=\int_0^{r_0}\sigma(x\mathbb{S}^{d-1}\cap S_{r,r_0,d})dx \text{ and } V_d(\mathrm{conv}_r Q)=\int_0^{r_0}\sigma(x\mathbb{S}^{d-1}\cap \mathrm{conv}_r Q)dx, \tag{40}$$

where σ denotes the proper spherical Lebesgue measure on $x\mathbb{S}^{d-1}$. Clearly, (39) yields that

$$\sigma(x\mathbb{S}^{d-1}\cap S_{r,r_0,d})=\sigma(x\mathbb{S}^{d-1}\cap \mathrm{conv}_r Q) \tag{41}$$

holds for all $0\leq x\leq r-\sqrt{r^2-r_0^2}$. Finally, let $r-\sqrt{r^2-r_0^2}<x\leq r_0$. On the one hand, notice that the subset $x\mathbb{S}^{d-1}\cap S_{r,r_0,d}$ of $x\mathbb{S}^{d-1}$ is the union of a pair of antipodal spherical caps of angular radius $0\leq\epsilon<\frac{\pi}{2}$. On the other hand, the subset $x\mathbb{S}^{d-1}\cap\mathrm{conv}_r Q$ of $x\mathbb{S}^{d-1}$ contains the union of $l+1$ spherical caps of angular radius ϵ centered at the points $\frac{x}{r_0}\mathbf{p}_i$, $1\leq i\leq l+1$, which do not lie on an open hemisphere of $x\mathbb{S}^{d-1}$ because $\mathbf{o}\in\mathrm{relint}(\mathrm{conv}(Q))$. Hence, Lemma 13 implies that

$$\sigma(x\mathbb{S}^{d-1}\cap S_{r,r_0,d})\leq\sigma(x\mathbb{S}^{d-1}\cap \mathrm{conv}_r Q) \tag{42}$$

holds for all $r-\sqrt{r^2-r_0^2}<x\leq r_0$. Thus, (38), (40), (41), and (42) yield (10) in a straightforward way. This completes the proof of Theorem 7.

6. Proof of Corollary 9

On the one hand, the extended isoperimetric inequality (see for example, (1.1) in [20]) yields

$$\left(\frac{V_d(\mathrm{conv}_r P)}{V_d(\mathbf{B}^d[\mathbf{o},1])}\right)^{\frac{1}{d}}\leq\left(\frac{V_k(\mathrm{conv}_r P)}{V_k(\mathbf{B}^d[\mathbf{o},1])}\right)^{\frac{1}{k}}, \tag{43}$$

where $1\leq k<d$. On the other hand, recall ([21]) that

$$V_k(\mathbf{B}^d[\mathbf{o},1])=\frac{\binom{d}{k}\omega_d}{\omega_{d-k}} \tag{44}$$

holds for all $1\leq k\leq d$. Hence, (43), (44), and Theorem 7 imply

$$V_k(\mathrm{conv}_r P)\geq V_k(\mathbf{B}^d[\mathbf{o},1])\frac{1}{\omega_d^{\frac{k}{d}}}\left[V_d(\mathrm{conv}_r P)\right]^{\frac{k}{d}}\geq\frac{\binom{d}{k}\omega_d^{1-\frac{k}{d}}}{\omega_{d-k}}\left(V_d(S_{r,r_0,d})\right)^{\frac{k}{d}} \tag{45}$$

for all $1\leq k<d$, finishing the proof of (11).

Now, we turn to the proof of (12). Proposition 2.5 of [5] and (15) imply

$$\mathbf{B}^2[\mathbf{o}, r] = \mathrm{conv}_r P - (\mathrm{conv}_r P)^r = \mathrm{conv}_r P - P^r, \tag{46}$$

from which one obtains

$$V_1(\mathbf{B}^2[\mathbf{o}, r]) = V_1(\mathrm{conv}_r P) + V_1(P^r). \tag{47}$$

Using Remark 6 and (47) we get that

$$V_1(\mathbf{B}^2[\mathbf{o}, r]) - V_1\left(L_{r,r-r_0,2}\right) \leq V_1(\mathbf{B}^2[\mathbf{o}, r]) - V_1(P^r) = V_1(\mathrm{conv}_r P). \tag{48}$$

Next, notice that (47) for $\mathrm{conv}_r P = S_{r,r_0,2}$ and $P^r = L_{r,r-r_0,2}$ yields

$$V_1\left(S_{r,r_0,2}\right) = V_1(\mathbf{B}^2[\mathbf{o}, r]) - V_1\left(L_{r,r-r_0,2}\right). \tag{49}$$

Finally, (48) and (49) imply (12) in a straightforward way. This completes the proof of Corollary 9.

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