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Illuminating spiky balls and cap bodies

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ABSTRACT

The convex hull of a ball with an exterior point is called a spike (or cap). A union of finitely many spikes of a ball is called a spiky ball. If a spiky ball is convex, then we call it a cap body. In this note we upper bound the illumination numbers of 2-illuminable spiky balls as well as centrally symmetric cap bodies. In particular, we prove the Illumination Conjecture for centrally symmetric cap bodies in sufficiently large dimensions. In fact, we do a bit more by showing that any d-dimensional centrally symmetric cap body can be illuminated by $< 2^d$ directions in Euclidean d-space for $d = 3, 4, 9$ and $d \geq 19$. Furthermore, we strengthen the latter result for 1-unconditionally symmetric cap bodies.

1. Introduction

Let $\mathbb{E}^d$ denote the d-dimensional Euclidean vector space, with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let $\mathbf{e}_1, \dots, \mathbf{e}_d$ be its standard basis. Its unit sphere centered at the origin $\mathbf{o}$ is $\mathbb{S}^{d-1} := \{\mathbf{x} \in \mathbb{E}^d \mid \|\mathbf{x}\| = 1\}$. A *greatcircle* of $\mathbb{S}^{d-1}$ is an intersection of $\mathbb{S}^{d-1}$ with a plane of $\mathbb{E}^d$ passing through $\mathbf{o}$. Two points are called *antipodes* if they can be obtained as an intersection of $\mathbb{S}^{d-1}$ with a line through $\mathbf{o}$ in $\mathbb{E}^d$. If $\mathbf{a}, \mathbf{b} \in \mathbb{S}^{d-1}$ are two points that are not antipodes, then we label the (uniquely determined) shortest geodesic arc of $\mathbb{S}^{d-1}$ connecting $\mathbf{a}$ and $\mathbf{b}$ by $\widehat{\mathbf{ab}}$. In other words, $\widehat{\mathbf{ab}}$ is the shorter circular arc with endpoints $\mathbf{a}$ and $\mathbf{b}$ of the greatcircle $\mathbf{ab}$ that passes through $\mathbf{a}$ and $\mathbf{b}$. The length of $\widehat{\mathbf{ab}}$ is called the *spherical* (or *angular*) *distance* between $\mathbf{a}$ and $\mathbf{b}$ and it is labeled by $l(\widehat{\mathbf{ab}})$, where $0 < l(\widehat{\mathbf{ab}}) < \pi$. The set $C_{\mathbb{S}^{d-1}}[\mathbf{x}, \alpha] := \{\mathbf{y} \in \mathbb{S}^{d-1} \mid l(\widehat{\mathbf{x},\mathbf{y}}) \leq \alpha\} = \{\mathbf{y} \in \mathbb{S}^{d-1} | \langle\mathbf{x},\mathbf{y}\rangle \geq \cos\alpha\}$ (resp., $C_{\mathbb{S}^{d-1}}(\mathbf{x}, \alpha) := \{\mathbf{y} \in \mathbb{S}^{d-1} \mid l(\widehat{\mathbf{x},\mathbf{y}}) < \alpha\} = \{\mathbf{y} \in \mathbb{S}^{d-1} | \langle\mathbf{x},\mathbf{y}\rangle > \cos\alpha\}$) is called the closed (resp., open) *spherical cap* of angular radius α centered at $\mathbf{x} \in \mathbb{S}^{d-1}$ for $0 < \alpha \leq \frac{\pi}{2}$. The closed Euclidean ball of radius r centered at $\mathbf{p} \in \mathbb{E}^d$ is denoted by $\mathbf{B}^d[\mathbf{p}, r] := \{\mathbf{q} \in \mathbb{E}^d \mid |\mathbf{p} - \mathbf{q}| \leq r\}$. A d-dimensional *convex body* $\mathbf{K}$ is a compact convex subset of $\mathbb{E}^d$ with non-empty interior. Then $\mathbf{K}$ is said to be $\mathbf{o}$-*symmetric* if $\mathbf{K} = -\mathbf{K}$ and $\mathbf{K}$ is called *centrally symmetric* if some translate of $\mathbf{K}$ is $\mathbf{o}$-symmetric. A light source at a point $\mathbf{p}$ outside a convex body $\mathbf{K} \subset \mathbb{E}^d$, *illuminates* a point $\mathbf{x}$ on the boundary of $\mathbf{K}$ if the halfline originating from $\mathbf{p}$ and passing through $\mathbf{x}$ intersects the interior of $\mathbf{K}$ at a point not lying between $\mathbf{p}$ and $\mathbf{x}$. The set of points $\{\mathbf{p}_i : i = 1, \dots, n\}$ in the exterior of $\mathbf{K}$ is said to *illuminate* $\mathbf{K}$ if every boundary point of $\mathbf{K}$ is illuminated by some $\mathbf{p}_i$. The *illumination number* $I(\mathbf{K})$ of $\mathbf{K}$ is the smallest n for which $\mathbf{K}$ can be illuminated by n point light sources. One can also consider illumination of $\mathbf{K} \subset \mathbb{E}^d$ by directions instead of by exterior points. We say that a point $\mathbf{x}$ on the boundary of $\mathbf{K}$ is

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illuminated in the direction $\mathbf{v} \in \mathbb{S}^{d-1}$ if the halfline originating from $\mathbf{x}$ and with direction vector $\mathbf{v}$ intersects the interior of $\mathbf{K}$. The former notion of illumination was introduced by Hadwiger [15], while the latter notion is due to Boltyanski [5]. It may not come as a surprise that the two concepts are equivalent in the sense that a convex body $\mathbf{K}$ can be illuminated by n point sources if and only if it can be illuminated by n directions. The following conjecture of Boltyanski [5] and Hadwiger [15] has become a central problem of convex and discrete geometry and inspired a significant body of research.

Conjecture 1 (***Illumination Conjecture***). *The illumination number $I(\mathbf{K})$ of any d-dimensional convex body $\mathbf{K}$, $d \geq 2$, is at most 2^d and $I(\mathbf{K}) = 2^d$ only if $\mathbf{K}$ is an affine d-cube.*

While Conjecture 1 has been proved in the plane ([5], [14], [15], and [19]), it is open for dimensions larger than 2. On the other hand, there are numerous partial results supporting Conjecture 1 in dimensions greater than 2. For details we refer the interested reader to the recent survey article [4] and the references mentioned there. Here we highlight only the following results. Let $\mathbf{K}$ be an arbitrary d-dimensional convex body with $d > 1$. Rogers [25] (see also [26]) has proved that $I(\mathbf{K}) \leq \binom{2d}{d} d(\ln d + \ln\ln d + 5) = O(4^d\sqrt{d}\ln d)$. Huang, Slomka, Tkocz, and Vritsiou [16] improved this bound of Rogers for sufficiently large values of d to $c_1 4^d e^{-c_2\sqrt{d}}$, where $c_1, c_2 > 0$ are universal constants. Lassak [18] improved the upper bound of Rogers for some small values of d to $(d+1)d^{d-1} - (d-1)(d-2)^{d-1}$. In fact, the best upper bounds for the illumination numbers of convex bodies in dimensions $3, 4, 5, 6$ are 14 ([23]), $96, 1091, 15373$ ([24]). The best upper bound for the illumination numbers of *centrally symmetric* convex bodies of $\mathbb{E}^d$, $d > 1$ is $2^d d(\ln d + \ln\ln d + 5)$ proved by Rogers ([25] and [26]). In connection with this upper bound we note that [28] proves Conjecture 1 for *unit balls of* 1*-symmetric norms* in $\mathbb{R}^d$ provided that d is sufficiently large. We also mention in passing that Conjecture 1 has been confirmed for certain classes of convex bodies such as *wide ball-bodies* including *convex bodies of constant width* ([1], [2], [3], [8], [27]), *convex bodies of Helly dimension* 2 ([7]), and *belt-bodies* including *zonoids* and *zonotopes* ([6]). The present article has been motivated by the investigations in [21] and it aims at proving Conjecture 1 for sufficiently high dimensional *centrally symmetric cap bodies* studied under the name centrally symmetric spiky balls in [21]. Actually, we do a bit more. The details are as follows.

Definition 1. Let $\mathbf{B}^d := \mathbf{B}^d[\mathbf{o}, 1]$ and let $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{E}^d \setminus \mathbf{B}^d$. Then

$$\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1, \ldots, \mathbf{x}_n] := \bigcup_{i=1}^{n} \mathrm{conv}(\mathbf{B}^d \cup \{\mathbf{x}_i\})$$

is called a spiky (unit) ball, where $\mathrm{conv}(\cdot)$ refers to the convex hull of the corresponding set. If $\mathbf{x}_i \notin \bigcup_{1\leq j\leq n, j\neq i} \mathrm{conv}(\mathbf{B}^d \cup \{\mathbf{x}_j\})$ holds for some $1 \leq i \leq n$, then $\mathbf{x}_i$ is called a vertex of $\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1, \ldots, \mathbf{x}_n]$. A point $\mathbf{x}$ on the boundary of the spiky ball $\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1, \ldots, \mathbf{x}_n]$ is illuminated in the direction $\mathbf{v} \in \mathbb{S}^{d-1}$ if the halfline originating from $\mathbf{x}$ and with direction vector $\mathbf{v}$ intersects the interior of $\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1, \ldots, \mathbf{x}_n]$ in points arbitrarily close to $\mathbf{x}$. Furthermore, the set of directions $\{\mathbf{v}_i : i = 1, \ldots, m\} \subset \mathbb{S}^{d-1}$ is said to illuminate $\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1, \ldots, \mathbf{x}_n]$ if every boundary point of $\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1, \ldots, \mathbf{x}_n]$ is illuminated by some $\mathbf{v}_i$. The illumination number $I(\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1, \ldots, \mathbf{x}_n])$ of $\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1, \ldots, \mathbf{x}_n]$ is the smallest m for which $\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1, \ldots, \mathbf{x}_n]$ can be illuminated by m directions. Moreover, we say that the spiky ball $\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1, \ldots, \mathbf{x}_n]$ with vertices $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is 2-illuminable if any two of its vertices can be simultaneously illuminated by a direction in $\mathbb{E}^d$. Finally, $\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1, \ldots, \mathbf{x}_n]$ is called a cap body if it is a convex body in $\mathbb{E}^d$. (See Fig. 1.)

We note that cap bodies were first studied by Minkowski [20]. On the other hand, the family of 2-illuminable spiky balls seems to be a new family of spiky balls that have not been investigated before.

Definition 2. If $0 < \alpha \leq \frac{\pi}{2}$, then let $N_{\mathbb{S}^{d-1}}(\alpha)$ denote the minimum number of closed spherical caps of angular radius α that can cover $\mathbb{S}^{d-1}$.

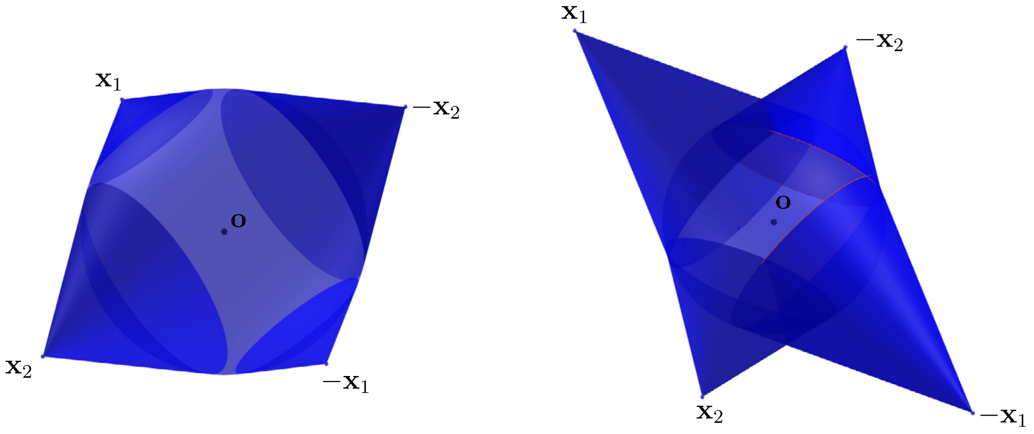
Our first result upper bounds the illumination numbers of 2-illuminable spiky balls. We note that spiky balls without being 2-illuminable can have arbitrarily large illumination numbers.

Theorem 2. *Suppose that $\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1, \ldots, \mathbf{x}_n]$ is a 2-illuminable spiky ball with vertices $\mathbf{x}_1, \ldots, \mathbf{x}_n$ in $\mathbb{E}^d$.*

(i) *If $d = 2$, then $I(\mathrm{Sp}_{\mathbf{B}^2}[\mathbf{x}_1, \ldots, \mathbf{x}_n]) = 3$.*
(ii) *If $d = 3$, then $I(\mathrm{Sp}_{\mathbf{B}^3}[\mathbf{x}_1, \ldots, \mathbf{x}_n]) \leq 5$.*
(iii) *If $d \geq 4$, then $I(\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1, \ldots, \mathbf{x}_n]) \leq 3 + N_{\mathbb{S}^{d-2}}(\frac{\pi}{6})$.*

Corollary 3. *Let $\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1, \ldots, \mathbf{x}_n]$ be a 2-illuminable spiky ball with vertices $\mathbf{x}_1, \ldots, \mathbf{x}_n$ in $\mathbb{E}^d$, $d \geq 4$. If $d = 4$, then $I(\mathrm{Sp}_{\mathbf{B}^4}[\mathbf{x}_1, \ldots, \mathbf{x}_n]) \leq 23$. If $d \geq 5$, then*

$$I(\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1, \ldots, \mathbf{x}_n]) \leq 3 + 2^{d-2}\sqrt{2\pi(d-1)}\left(\frac{1}{2} + \frac{3\ln\ln(d-2)}{\ln(d-2)} + \frac{3}{\ln(d-2)}\right)(d-2)\ln(d-2) < 2^{d+1}d^{\frac{3}{2}}\ln d.$$



(a) An **o**-symmetric convex spiky ball, i.e., a cap body. (b) An **o**-symmetric non-convex spiky ball.

Fig. 1. Centrally symmetric spiky balls.

Remark 4. Note that an arbitrary spiky ball $\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1,\dots,\mathbf{x}_n]$ is starshaped with respect to **o** and even if $\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1,\dots,\mathbf{x}_n]$ is 2-illuminable it is not necessarily a convex set. Still, one may wonder whether any d-dimensional 2-illuminable spiky ball can be illuminated by less than 2^d directions in $\mathbb{E}^d$, $d \geq 4$. In general, one can introduce the family of k-illuminable spiky balls for given $k \geq 2$ by a natural extension of Definition 1 and then ask whether Conjecture 1 holds for that family. The case $k = 2$ seems to be the most difficult one.

Remark 5. There exists a 2-illuminable spiky ball $\mathrm{Sp}_{\mathbf{B}^3}[\mathbf{x}_1,\dots,\mathbf{x}_{10}]$ in $\mathbb{E}^3$ with $I(\mathrm{Sp}_{\mathbf{B}^3}[\mathbf{x}_1,\dots,\mathbf{x}_{10}]) = 5$. Furthermore, there exists d_0 such that for any $d \geq d_0$ one possesses a 2-illuminable spiky ball $\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1,\dots,\mathbf{x}_n]$ in $\mathbb{E}^d$ with $I(\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1,\dots,\mathbf{x}_n]) > 1.0645^{d-1}$.

Before we state our main result let us recall the following very interesting theorem of Naszódi [21]: Let $1 < D < 1.116$. Then for any sufficiently large dimension d there exists a centrally symmetric cap body $\mathbf{K}$ such that $I(\mathbf{K}) \geq 0.05D^d$ and $\frac{1}{D}\mathbf{B}^d \subset \mathbf{K} \subset \mathbf{B}^d$. This raises the natural question whether Conjecture 1 holds for centrally symmetric cap bodies in sufficiently large dimensions. We give a positive answer this question as follows.

Theorem 6. *Let $\mathrm{Sp}_{\mathbf{B}^d}[\pm\mathbf{x}_1,\dots,\pm\mathbf{x}_n]$ be an **o**-symmetric cap body with vertices $\pm\mathbf{x}_1,\dots,\pm\mathbf{x}_n$ in $\mathbb{E}^d$, $d \geq 3$. Then*

$$I(\mathrm{Sp}_{\mathbf{B}^d}[\pm\mathbf{x}_1,\dots,\pm\mathbf{x}_n]) \leq 2 + N_{\mathbb{S}^{d-2}}\left(\frac{\pi}{4}\right).$$

Corollary 7. *Any 3-dimensional centrally symmetric cap body can be illuminated by 6 ($< 2^3$) directions in $\mathbb{E}^3$. (This is not a new result. It was proved via a dual method in [17].) On the other hand, any 4-dimensional centrally symmetric cap body can be illuminated by 12 ($< 2^4$) directions in $\mathbb{E}^4$. Moreover, if $\mathrm{Sp}_{\mathbf{B}^d}[\pm\mathbf{x}_1,\dots,\pm\mathbf{x}_n]$ is an **o**-symmetric cap body with vertices $\pm\mathbf{x}_1,\dots,\pm\mathbf{x}_n$ in $\mathbb{E}^d, d \geq 5$, then*

$$I(\mathrm{Sp}_{\mathbf{B}^d}[\pm\mathbf{x}_1,\dots,\pm\mathbf{x}_n]) \leq 2 + 2^{\frac{d-2}{2}}\sqrt{2\pi(d-1)}\left(\frac{1}{2} + \frac{3\ln\ln(d-2)}{\ln(d-2)} + \frac{3}{\ln(d-2)}\right)(d-2)\ln(d-2),$$

where $2 + 2^{\frac{d-2}{2}}\sqrt{2\pi(d-1)}\left(\frac{1}{2} + \frac{3\ln\ln(d-2)}{\ln(d-2)} + \frac{3}{\ln(d-2)}\right)(d-2)\ln(d-2) < 2^d$ holds for all $d \geq 19$.

Remark 8. Clearly, Corollary 7 proves the Illumination Conjecture for centrally symmetric cap bodies of dimension d for $d = 3, 4$ and $d \geq 19$. We note that based on Theorem 6, in order to prove the Illumination Conjecture for centrally symmetric cap bodies of dimension d for $5 \leq d \leq 18$, it is sufficient to show that $N_{\mathbb{S}^{d-2}}\left(\frac{\pi}{4}\right) \leq 2^d - 2$ holds for all $5 \leq d \leq 18$. It seems that the method of the recent paper [8] has the potential to achieve this goal. Indeed, this goal has already been achieved for $d = 9$ in [8] by showing that $N_{\mathbb{S}^7}\left(\frac{\pi}{4}\right) \leq 240 < 2^9 - 2 = 510$.

Definition 3. The cap body $\mathbf{K} \subset \mathbb{E}^d$ is called 1-unconditionally symmetric if it symmetric about each coordinate hyperplane of $\mathbb{E}^d$.

We close this section with a strengthening of Corollary 7 for 1-unconditionally symmetric cap bodies. Recall that according to [17] if $\mathbf{K}$ is a 1-unconditionally symmetric cap body in $\mathbb{E}^4$, then $I(\mathbf{K}) \leq 8$.

Theorem 9. *Let* $\mathbf{K}$ *be a* 1*-unconditionally symmetric cap body in* $\mathbb{E}^d$, $d \geq 5$*. Then* $I(\mathbf{K}) \leq 4d$*.*

While this proves Conjecture 1 for 1-unconditionally symmetric cap bodies in dimensions $d \geq 5$, the $4d$ estimate does not seem to be sharp, and, in fact, we propose

Conjecture 10. *Every* 1*-unconditionally symmetric cap body of* $\mathbb{E}^d$ *can be illuminated by* $2d$ *directions for all* $d \geq 5$*.*

In the rest of the paper we prove Theorems 2, 6, and 9, Corollaries 3 and 7, and Remark 5.

2. Proof of Theorem 2

We start with

Definition 4. If $\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1, \dots, \mathbf{x}_n]$ is a spiky ball with vertices $\mathbf{x}_1, \dots, \mathbf{x}_n$ in $\mathbb{E}^d$, then let $\mathbf{y}_i$ and $0 < \alpha_i < \frac{\pi}{2}$ be defined for $1 \leq i \leq n$ by $C_{\mathbb{S}^{d-1}}(\mathbf{y}_i, \alpha_i) = \mathrm{int}\left(\mathrm{conv}(\mathbf{B}^d \cup \{\mathbf{x}_i\})\right) \cap \mathbb{S}^{d-1}$, where $\mathrm{int}(\cdot)$ refers to the interior of the corresponding set in $\mathbb{E}^d$. We are going to refer to $C_{\mathbb{S}^{d-1}}(\mathbf{y}_i, \alpha_i)$ as the open spherical cap assigned to the vertex $\mathbf{x}_i$ of $\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1, \dots, \mathbf{x}_n]$.

It is easy to see that the direction $\mathbf{v} \in \mathbb{S}^{d-1}$ illuminates the vertex $\mathbf{x}_i$ of the spiky ball $\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1, \dots, \mathbf{x}_n]$ if and only if $\mathbf{v} \in C_{\mathbb{S}^{d-1}}(-\mathbf{y}_i, \frac{\pi}{2} - \alpha_i)$. Thus, by observing that the set $\{\mathbf{v}_k : 1 \leq k \leq m\} \subset \mathbb{S}^{d-1}$ of directions whose positive hull $\mathrm{pos}(\{\mathbf{v}_k : 1 \leq k \leq m\}) := \{\sum_{k=1}^{m} \lambda_k \mathbf{v}_k \mid \lambda_k > 0 \text{ for all } 1 \leq k \leq m\}$ is $\mathbb{E}^d$, illuminates the spiky ball $\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1, \dots, \mathbf{x}_n]$ if and only if it illuminates the vertices $\mathbf{x}_1, \dots, \mathbf{x}_n$ of $\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1, \dots, \mathbf{x}_n]$, the following statement is immediate.

Lemma 11. *Let* $\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1, \dots, \mathbf{x}_n]$ *be a spiky (unit) ball with vertices* $\mathbf{x}_1, \dots, \mathbf{x}_n$ *in* $\mathbb{E}^d$*. Then*

(a) $\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1, \dots, \mathbf{x}_n]$ *is* 2*-illuminable if and only if* $C_{\mathbb{S}^{d-1}}(-\mathbf{y}_i, \frac{\pi}{2} - \alpha_i) \cap C_{\mathbb{S}^{d-1}}(-\mathbf{y}_j, \frac{\pi}{2} - \alpha_j) \neq \emptyset$ *holds for all* $1 \leq i < j \leq n$ *moreover,*

(b) $\{\mathbf{v}_k : 1 \leq k \leq m\} \subset \mathbb{S}^{d-1}$ *with* $\mathrm{pos}(\{\mathbf{v}_k : 1 \leq k \leq m\}) = \mathbb{E}^d$ *illuminates* $\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1, \dots, \mathbf{x}_n]$ *if and only if* $C_{\mathbb{S}^{d-1}}(-\mathbf{y}_i, \frac{\pi}{2} - \alpha_i) \cap \{\mathbf{v}_k : 1 \leq k \leq m\} \neq \emptyset$ *holds for all* $1 \leq i \leq n$*.*

Now, we are set to prove Theorem 2.

Part (i): Let $\mathrm{Sp}_{\mathbf{B}^2}[\mathbf{x}_1, \dots, \mathbf{x}_n]$ be a 2-illuminable spiky (unit) disk with vertices $\mathbf{x}_1, \dots, \mathbf{x}_n$ in $\mathbb{E}^2$. Let $\mathcal{C} := \{C_{\mathbb{S}^1}(-\mathbf{y}_i, \frac{\pi}{2} - \alpha_i) \mid 1 \leq i \leq n\}$ be the family of open circular arcs (of length $< \pi$) assigned to the vertices of $\mathrm{Sp}_{\mathbf{B}^2}[\mathbf{x}_1, \dots, \mathbf{x}_n]$. Without loss of generality we may assume that $C_{\mathbb{S}^1}(-\mathbf{y}_1, \frac{\pi}{2} - \alpha_1)$ contains no other open circular arc of $\mathcal{C}$. As by Part (a) of Lemma 11 $C_{\mathbb{S}^1}(-\mathbf{y}_i, \frac{\pi}{2} - \alpha_i) \cap C_{\mathbb{S}^1}(-\mathbf{y}_j, \frac{\pi}{2} - \alpha_j) \neq \emptyset$ holds for all $1 \leq i < j \leq n$ therefore, there exist $\mathbf{v}_1, \mathbf{v}_2 \in C_{\mathbb{S}^1}(-\mathbf{y}_1, \frac{\pi}{2} - \alpha_1)$ with each of them lying sufficiently close to one of the two endpoints of $C_{\mathbb{S}^1}(-\mathbf{y}_1, \frac{\pi}{2} - \alpha_1)$ such that $C_{\mathbb{S}^1}(-\mathbf{y}_i, \frac{\pi}{2} - \alpha_i) \cap \{\mathbf{v}_1, \mathbf{v}_2\} \neq \emptyset$ holds for all $1 \leq i \leq n$. Clearly, $\mathbf{v}_1 \neq -\mathbf{v}_2$ and so, one can choose $\mathbf{v}_3 \in \mathbb{S}^1$ such that $\mathrm{pos}(\{\mathbf{v}_k : 1 \leq k \leq 3\}) = \mathbb{E}^2$. Hence, by Part (b) of Lemma 11 $\{\mathbf{v}_k : 1 \leq k \leq 3\} \subset \mathbb{S}^1$ illuminates $\mathrm{Sp}_{\mathbf{B}^2}[\mathbf{x}_1, \dots, \mathbf{x}_n]$, implying $I(\mathrm{Sp}_{\mathbf{B}^2}[\mathbf{x}_1, \dots, \mathbf{x}_n]) = 3$ in a straightforward way.

Part (ii): Let $\mathrm{Sp}_{\mathbf{B}^3}[\mathbf{x}_1, \dots, \mathbf{x}_n]$ be a 2-illuminable spiky (unit) ball with vertices $\mathbf{x}_1, \dots, \mathbf{x}_n$ in $\mathbb{E}^3$. Let $\mathcal{C} := \{C_{\mathbb{S}^2}(-\mathbf{y}_i, \frac{\pi}{2} - \alpha_i) \mid 1 \leq i \leq n\}$ be the family of open spherical caps assigned to the vertices of $\mathrm{Sp}_{\mathbf{B}^3}[\mathbf{x}_1, \dots, \mathbf{x}_n]$. By Part (a) of Lemma 11 any two members of $\mathcal{C}$ intersect. Next, recall the following theorem of Danzer [10]: If $\mathcal{F}$ is a family of finitely many closed spherical caps on $\mathbb{S}^2$ such that every two members of $\mathcal{F}$ intersect, then there exist 4 points on $\mathbb{S}^2$ such that each member of $\mathcal{F}$ contains at least one of them (i.e., 4 needles are always sufficient to pierce all members of $\mathcal{F}$). Now, applying Danzer's theorem to $\mathcal{C}$ (or rather to the corresponding family of closed spherical caps with each closed spherical cap being somewhat smaller and concentric to an open spherical cap of $\mathcal{C}$) one obtains the existence of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbb{S}^2$ with the property that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent and $C_{\mathbb{S}^2}(-\mathbf{y}_i, \frac{\pi}{2} - \alpha_i) \cap \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} \neq \emptyset$ holds for all $1 \leq i \leq n$. Finally, let us choose $\mathbf{v}_5 \in \mathbb{S}^2$ such that $\mathrm{pos}(\{\mathbf{v}_k : 1 \leq k \leq 5\}) = \mathbb{E}^3$. (See Fig. 2.) Thus, Part (b) of Lemma 11 implies in a straightforward way that $\{\mathbf{v}_k : 1 \leq k \leq 5\} \subset \mathbb{S}^2$ illuminates $\mathrm{Sp}_{\mathbf{B}^3}[\mathbf{x}_1, \dots, \mathbf{x}_n]$ and therefore $I(\mathrm{Sp}_{\mathbf{B}^3}[\mathbf{x}_1, \dots, \mathbf{x}_n]) \leq 5$.

Part (iii): Let $\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1, \dots, \mathbf{x}_n]$ be a 2-illuminable spiky (unit) ball with vertices $\mathbf{x}_1, \dots, \mathbf{x}_n$ in $\mathbb{E}^d$, $d \geq 4$. Let $\mathcal{C} := \{C_{\mathbb{S}^{d-1}}(-\mathbf{y}_i, \frac{\pi}{2} - \alpha_i) \mid 1 \leq i \leq n\}$ be the family of open spherical caps assigned to the vertices of $\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1, \dots, \mathbf{x}_n]$. By Part (a) of Lemma 11 any two members of $\mathcal{C}$ intersect. We need

Definition 5. Let $G(2, \mathbf{B}^d)$ denote the smallest positive integer k such that any finite family of pairwise intersecting d-dimensional closed balls in $\mathbb{E}^d$ is k-pierceable (i.e., the finite family of balls can be partitioned into k subfamilies each having a non-empty intersection).

Now, recall Danzer's estimate (see page 361 in [13]) according to which $G(2, \mathbf{B}^d) \leq 1 + N_{\mathbb{S}^{d-1}}(\frac{\pi}{6})$. Let $\mathbf{s} \in \mathbb{S}^{d-1}$ be a point which is not a boundary point of any member of $\mathcal{C}$. If $\mathcal{C}'$ (resp., $\mathcal{C}''$) consists of those members of $\mathcal{C}$ that contain $\mathbf{s}$ as an interior (resp., exterior) point, then clearly $\mathcal{C} = \mathcal{C}' \cup \mathcal{C}''$. Let H be the hyperplane tangent to $\mathbb{S}^{d-1}$ at $-\mathbf{s}$ in $\mathbb{E}^d$. If we take

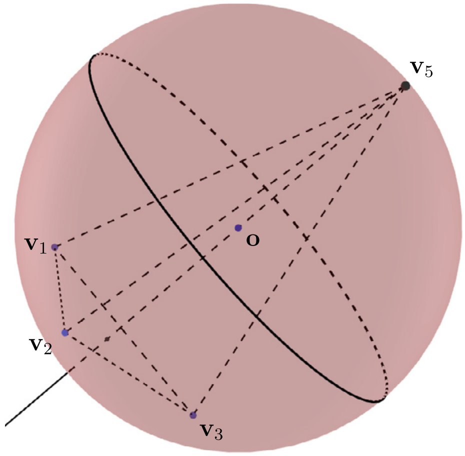
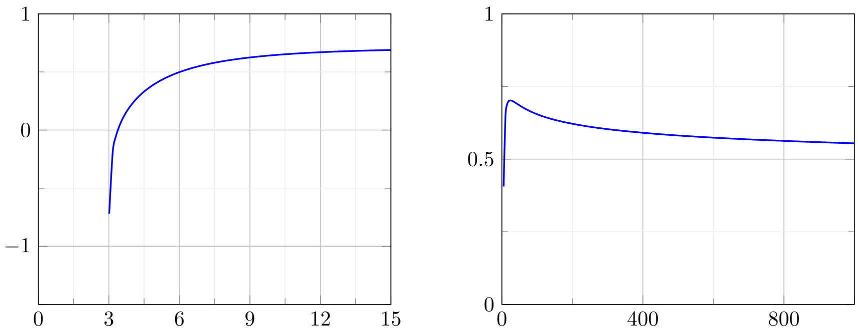


Fig. 2. Constructing $\mathbf{v}_5 \in \mathbb{S}^2$ in the proof of Part (ii) of Theorem 2.



(a) The graph of $f(x)$ for small x.

(b) The graph of $f(x)$ for large x.

Fig. 3. The graph of $f(x)$ showing that $f(x) < 1$ holds for all $x \geq 5$.

the stereographic projection with center $\mathbf{s}$ that maps $\mathbb{S}^{d-1} \setminus \mathbf{s}$ onto H, then applying Danzer's estimate to the images of $\mathcal{C}''$ in H we get that there are $1 + N_{\mathbb{S}^{d-2}}(\frac{\pi}{6})$ points of $\mathbb{S}^{d-1}$ piercing the members of $\mathcal{C}''$ in $\mathbb{S}^{d-1}$. Hence, $\mathcal{C}$ is pierceable by $2 + N_{\mathbb{S}^{d-2}}(\frac{\pi}{6})$ points (including $\mathbf{s}$) in $\mathbb{S}^{d-1}$. As members of $\mathcal{C}$ are open spherical caps of $\mathbb{S}^{d-1}$ therefore there are $3 + N_{\mathbb{S}^{d-2}}(\frac{\pi}{6})$ points in $\mathbb{S}^{d-1}$ whose positive hull is $\mathbb{E}^d$ such that they pierce the members of $\mathcal{C}$. Thus, by Part (b) of Lemma 11 we get that $I(\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1, \ldots, \mathbf{x}_n]) \leq 3 + N_{\mathbb{S}^{d-2}}(\frac{\pi}{6})$. This completes the proof of Theorem 2.

3. Proof of Corollary 3

First, we recall that according to [29] there exists a covering of $\mathbb{S}^2$ using 20 (closed) spherical caps of angular radius $\frac{\pi}{6}$. Thus, by Part (iii) of Theorem 2 if $\mathrm{Sp}_{\mathbf{B}^4}[\mathbf{x}_1, \ldots, \mathbf{x}_n]$ is a 2-illuminable spiky ball with vertices $\mathbf{x}_1, \ldots, \mathbf{x}_n$ in $\mathbb{E}^4$, then $I(\mathrm{Sp}_{\mathbf{B}^4}[\mathbf{x}_1, \ldots, \mathbf{x}_n]) \leq 23$.

Second, recall that Theorem 1 of [12] implies in a straightforward way that

$$3 + N_{\mathbb{S}^{d-2}}\left(\frac{\pi}{6}\right) \leq 3 + \frac{1}{\Omega_{d-2}(\frac{\pi}{6})}\left(\frac{1}{2} + \frac{3\ln\ln(d-2)}{\ln(d-2)} + \frac{3}{\ln(d-2)}\right)(d-2)\ln(d-2), \tag{1}$$

where $\Omega_{d-2}(\frac{\pi}{6})$ is the fraction of the surface of $\mathbb{S}^{d-2}$ covered by a closed spherical cap of angular radius $\frac{\pi}{6}$. Next, the estimate $\Omega_{d-2}(\frac{\pi}{6}) > \frac{1}{2^{d-2}\sqrt{2\pi(d-1)}}$ (see for example, Lemma 2.1 in [21]) combined with (1) yields that

$$3 + N_{\mathbb{S}^{d-2}}\left(\frac{\pi}{6}\right) \leq 3 + 2^{d-2}\sqrt{2\pi(d-1)}\left(\frac{1}{2} + \frac{3\ln\ln(d-2)}{\ln(d-2)} + \frac{3}{\ln(d-2)}\right)(d-2)\ln(d-2) < 2^{d+1}d^{\frac{3}{2}}\ln d \tag{2}$$

holds for all $d \geq 5$. Indeed, see Fig. 3 for the graph of the function

$$f(x) := \frac{3 + 2^{x-2}\sqrt{2\pi(x-1)}\left(\frac{1}{2} + \frac{3\ln\ln(x-2)}{\ln(x-2)} + \frac{3}{\ln(x-2)}\right)(x-2)\ln(x-2)}{2^{x+1}x^{\frac{3}{2}}\ln x}, \; x > 3$$

which clearly implies the last inequality of (2). For more details on this see the Appendix. Finally, if $\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1, \dots, \mathbf{x}_n]$ is a 2-illuminable spiky (unit) ball with vertices $\mathbf{x}_1, \dots, \mathbf{x}_n$ in $\mathbb{E}^d$, $d \ge 5$, then (2) combined with Part (iii) of Theorem 2 finishes the proof of Corollary 3.

4. Proof of Remark 5

Recall the following construction of Danzer [10]: there exist 10 closed circular disks in $\mathbb{E}^2$ such that any two of them intersect and it is impossible to pierce them by 3 needles. It follows in a straightforward way that there exists a family $\mathcal{C}$ of 10 open circular disks in $\mathbb{E}^2$ (each being somewhat larger and concentric to a closed circular disk of the previous family) such that any two of them intersect and it is impossible to pierce them by 3 needles. Now, Let H be the plane tangent to $\mathbb{S}^2$ at the point say, $-\mathbf{s}$ with $\mathcal{C}$ lying in H. If we take the stereographic projection with center $\mathbf{s}$ that maps H onto $\mathbb{S}^2 \setminus \mathbf{s}$ and label the image of the family $\mathcal{C}$ by $\mathcal{C}'$, then $\mathcal{C}'$ is a family of 10 open spherical caps in $\mathbb{S}^2$ such that any two of them intersect and it is impossible to pierce them by 3 needles. By choosing $\mathcal{C}$ within a small neighborhood $B_H(-\mathbf{s})$ of $-\mathbf{s}$ in H, we get that each member of $\mathcal{C}'$ is an open spherical cap of angular radius $< \frac{\pi}{2}$. Next, let us take the spiky unit ball $\mathrm{Sp}_{\mathbf{B}^3}[\mathbf{x}_1, \dots, \mathbf{x}_{10}]$ with $\{C_{\mathbb{S}^2}(-\mathbf{y}_i, \frac{\pi}{2} - \alpha_i) \mid 1 \le i \le 10\} = \mathcal{C}'$. Clearly, due to Part (a) of Lemma 11, $\mathrm{Sp}_{\mathbf{B}^3}[\mathbf{x}_1, \dots, \mathbf{x}_{10}]$ is 2-illuminable. Finally, if we choose $B_H(-\mathbf{s})$ sufficiently small, such that the spherical caps of $\mathcal{C}'$ all lie in a hemisphere, then Part (b) of Lemma 11 and Part (ii) of Theorem 2 yield that $I(\mathrm{Sp}_{\mathbf{B}^3}[\mathbf{x}_1, \dots, \mathbf{x}_{10}]) = 5$.

Next, we recall the following construction of Bourgain and Lindenstrauss [9]: there exists d^* such that for any $d \ge d^*$ one possesses a finite point set P of diameter 1 in $\mathbb{E}^d$ whose any covering by unit diameter closed balls requires at least 1.0645^d balls. Hence, if we take the unit diameter closed balls centered at the points of P in $\mathbb{E}^d$, then any two balls intersect and it is impossible to pierce them by fewer than $\lfloor 1.0645^d \rfloor$ needles. It follows in a straightforward way that for any $d \ge d^*$ there exists a family $\mathcal{C}_d$ of open balls centered at the points of P in $\mathbb{E}^d$ (each being somewhat larger and concentric to a unit diameter closed ball of the previous family) such that any two of them intersect and it is impossible to pierce them by fewer than $\lfloor 1.0645^d \rfloor$ needles. Now, Let H be the hyperplane tangent to $\mathbb{S}^{d-1}$ at the point say, $-\mathbf{s}$ with $\mathcal{C}_{d-1}$ lying in H. If we take the stereographic projection with center $\mathbf{s}$ that maps H onto $\mathbb{S}^{d-1} \setminus \mathbf{s}$ and label the image of the family $\mathcal{C}_{d-1}$ by $\mathcal{C}'_{d-1}$, then $\mathcal{C}'_{d-1}$ is a family of open spherical caps in $\mathbb{S}^{d-1}$ such that any two of them intersect and it is impossible to pierce them by fewer than $\lfloor 1.0645^{d-1} \rfloor$ needles. By choosing $\mathcal{C}_{d-1}$ within a small neighborhood $B_H(-\mathbf{s})$ of $-\mathbf{s}$ in H, we get that each member of $\mathcal{C}'_{d-1}$ is an open spherical cap of angular radius $< \frac{\pi}{2}$. Next, let us take the spiky unit ball $\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1, \dots, \mathbf{x}_n]$ with $\{C_{\mathbb{S}^{d-1}}(-\mathbf{y}_i, \frac{\pi}{2} - \alpha_i) \mid 1 \le i \le n\} = \mathcal{C}'_{d-1}$. Clearly, due to Part (a) of Lemma 11, $\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1, \dots, \mathbf{x}_n]$ is 2-illuminable. Finally, if we choose $B_H(-\mathbf{s})$ sufficiently small, such that the spherical caps of $\mathcal{C}'_{d-1}$ all lie in a hemisphere, then Part (b) of Lemma 11 yields that $I(\mathrm{Sp}_{\mathbf{B}^d}[\mathbf{x}_1, \dots, \mathbf{x}_n]) \ge 1 + \lfloor 1.0645^{d-1} \rfloor$, where $d \ge d^* + 1$.

5. Proof of Theorem 6

First, using Definition 4 we prove

Lemma 12. *Let $\mathrm{Sp}_{\mathbf{B}^d}[\pm\mathbf{x}_1, \dots, \pm\mathbf{x}_n]$ be an $\mathbf{o}$-symmetric cap body with vertices $\pm\mathbf{x}_1, \dots, \pm\mathbf{x}_n$ in $\mathbb{E}^d$, $d \ge 3$. Then*

(a) *$C_{\mathbb{S}^{d-1}}[\pm\mathbf{y}_i, \frac{\pi}{2} - \alpha_i] \cap C_{\mathbb{S}^{d-1}}[\pm\mathbf{y}_j, \frac{\pi}{2} - \alpha_j] \ne \emptyset$ holds for all $1 \le i < j \le n$ moreover,*
(b) *$\{\mathbf{v}_k : 1 \le k \le m\} \subset \mathbb{S}^{d-1}$ with $\mathrm{pos}(\{\mathbf{v}_k : 1 \le k \le m\}) = \mathbb{E}^d$ illuminates $\mathrm{Sp}_{\mathbf{B}^d}[\pm\mathbf{x}_1, \dots, \pm\mathbf{x}_n]$ if and only if $C_{\mathbb{S}^{d-1}}(\pm\mathbf{y}_i, \frac{\pi}{2} - \alpha_i) \cap \{\mathbf{v}_k : 1 \le k \le m\} \ne \emptyset$ holds for all $1 \le i \le n$.*

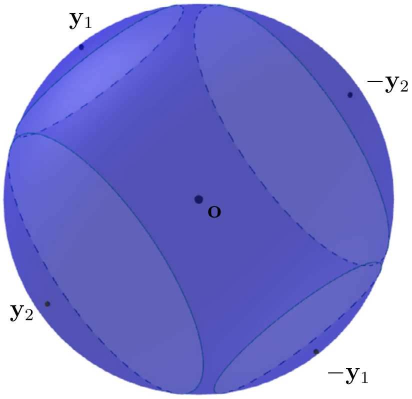
Proof. Due to convexity and symmetry of $\mathrm{Sp}_{\mathbf{B}^d}[\pm\mathbf{x}_1, \dots, \pm\mathbf{x}_n]$, the underlying spherical caps $C_{\mathbb{S}^{d-1}}[\pm\mathbf{y}_i, \alpha_i]$, $1 \le i \le n$ form a packing in $\mathbb{S}^{d-1}$ (see the Fig. 4 for the examples of the spiky ball cap configurations). Now, let $1 \le i < j \le n$. For Part (a) it is sufficient to show that

$$C_{\mathbb{S}^{d-1}}\left[-\mathbf{y}_i, \frac{\pi}{2} - \alpha_i\right] \cap C_{\mathbb{S}^{d-1}}\left[\mathbf{y}_j, \frac{\pi}{2} - \alpha_j\right] \ne \emptyset. \tag{3}$$

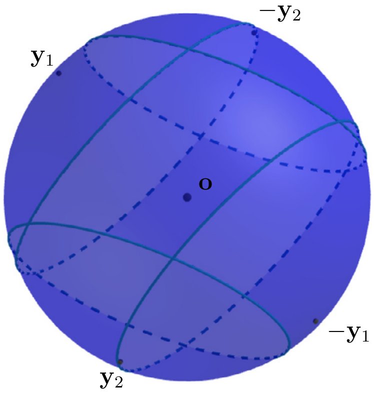
(Namely, the same argument and symmetry will imply that $C_{\mathbb{S}^{d-1}}[\pm\mathbf{y}_i, \frac{\pi}{2} - \alpha_i] \cap C_{\mathbb{S}^{d-1}}[\pm\mathbf{y}_j, \frac{\pi}{2} - \alpha_j] \ne \emptyset$.) Let H_{ij} be a hyperplane passing through $\mathbf{o}$ and separating $C_{\mathbb{S}^{d-1}}[\mathbf{y}_i, \alpha_i]$ and $C_{\mathbb{S}^{d-1}}[\mathbf{y}_j, \alpha_j]$. Furthermore, let $\mathbf{n}_{ij} \in \mathbb{S}^{d-1}$ (resp., $-\mathbf{n}_{ij} \in \mathbb{S}^{d-1}$) be on the same side of H_{ij} as $C_{\mathbb{S}^{d-1}}[\mathbf{y}_i, \alpha_i]$ (resp., $C_{\mathbb{S}^{d-1}}[\mathbf{y}_j, \alpha_j]$) such that $\langle \pm\mathbf{n}_{ij}, \mathbf{z}\rangle = 0$ for all $\mathbf{z} \in H_{ij}$. Clearly $-\mathbf{n}_{ij} \in C_{\mathbb{S}^{d-1}}\left[-\mathbf{y}_i, \frac{\pi}{2} - \alpha_i\right]$ moreover, $\mathbf{n}_{ij} \in C_{\mathbb{S}^{d-1}}\left[-\mathbf{y}_j, \frac{\pi}{2} - \alpha_j\right]$ implying $-\mathbf{n}_{ij} \in C_{\mathbb{S}^{d-1}}\left[\mathbf{y}_j, \frac{\pi}{2} - \alpha_j\right]$. Thus, (3) follows, finishing the proof of Part (a). Finally, Part (b) follows from Part (b) of Lemma 11 in a straightforward way. □

Second, based on Lemma 12, in order to prove Theorem 6 it is sufficient to show

Theorem 13. *Let $\{C_{\mathbb{S}^{d-1}}[\pm\mathbf{z}_i, \beta_i] \mid 1 \le i \le n\} \subset \mathbb{S}^{d-1}$ be an $\mathbf{o}$-symmetric family of $2n$ closed spherical caps with $d \ge 3$ and $0 < \beta_i < \frac{\pi}{2}$, $1 \le i \le n$ such that*



(a) Underlying packing of spherical caps of a cap body.



(b) Underlying arrangement of spherical caps of a non-convex spiky ball.

Fig. 4. Underlying caps corresponding to the spiky balls in Fig. 1.

$$C_{\mathbb{S}^{d-1}}[\pm\mathbf{z}_i,\beta_i]\cap C_{\mathbb{S}^{d-1}}[\pm\mathbf{z}_j,\beta_j]\neq\emptyset \tag{4}$$

holds for all $1\le i<j\le n$*. Then there exist* $\mathbf{u}_1,\dots,\mathbf{u}_N\in\mathbb{S}^{d-1}$ *with* $N=2+N_{\mathbb{S}^{d-2}}\left(\frac{\pi}{4}\right)$ *and* $\operatorname{pos}(\{\mathbf{u}_k:1\le k\le N\})=\mathbb{E}^d$ *such that* $C_{\mathbb{S}^{d-1}}(\pm\mathbf{z}_i,\beta_i)\cap\{\mathbf{u}_k:1\le k\le N\}\neq\emptyset$ *holds for all* $1\le i\le n$*.*

Proof. Without loss of generality we may assume that the points $\{\pm\mathbf{z}_i\mid 1\le i\le n\}$ are pairwise distinct and

$$0<\beta_1\le\beta_2\le\dots\le\beta_n<\frac{\pi}{2}. \tag{5}$$

Let H be the hyperplane of $\mathbb{E}^d$ with normal vectors $\pm\mathbf{z}_1$ passing through $\mathbf{o}$, and let $\mathbb{S}^{d-2}:=H\cap\mathbb{S}^{d-1}$.

Sublemma 1. $\mathbb{S}^{d-2}\cap C_{\mathbb{S}^{d-1}}[\pm\mathbf{z}_i,\beta_i]$ *is a* $(d-2)$*-dimensional closed spherical cap of angular radius at least* $\frac{\pi}{4}$ *for all* $2\le i\le n$*.*

Proof. Let H^+ be the closed halfspace of $\mathbb{E}^d$ bounded by H that contains $\mathbf{z}_1$. Let i be fixed with $2\le i\le n$. Without loss of generality we may assume that $\mathbf{z}_i\in H^+$ and our goal is to show that $\mathbb{S}^{d-2}\cap C_{\mathbb{S}^{d-1}}[\mathbf{z}_i,\beta_i]$ is a $(d-2)$-dimensional closed spherical cap of angular radius at least $\frac{\pi}{4}$. Let β be the smallest positive real such that

$$\beta_1\le\beta\le\beta_i \text{ and } C_{\mathbb{S}^{d-1}}[\mathbf{z}_i,\beta]\cap C_{\mathbb{S}^{d-1}}[-\mathbf{z}_1,\beta_1]\neq\emptyset \text{ (and therefore also } C_{\mathbb{S}^{d-1}}[\mathbf{z}_i,\beta]\cap C_{\mathbb{S}^{d-1}}[\mathbf{z}_1,\beta_1]\neq\emptyset). \tag{6}$$

Thus, either $C_{\mathbb{S}^{d-1}}[\mathbf{z}_i,\beta]$ is tangent to $C_{\mathbb{S}^{d-1}}[-\mathbf{z}_1,\beta_1]$ at some point of $\widehat{\mathbf{z}_i(-\mathbf{z}_1)}$ (Case 1) or $\beta_1=\beta$ (Case 2).

Case 1: Let $\mathbf{b}_i:=\widehat{\mathbf{z}_i(-\mathbf{z}_1)}\cap\mathbb{S}^{d-2}$ and $\mathbf{a}_i\in\operatorname{bd}\left(C_{\mathbb{S}^{d-1}}[\mathbf{z}_i,\beta]\right)\cap\mathbb{S}^{d-2}$, where $\operatorname{bd}(\cdot)$ denotes the boundary of the corresponding set in $\mathbb{S}^{d-1}$. If $\mathbf{z}_i\in H$, then $\mathbf{z}_i=\mathbf{b}_i$ and $\beta=l(\widehat{\mathbf{a}_i\mathbf{b}_i})$ and therefore, (6) yields $\frac{\pi}{4}=\frac{2\beta_1+2\beta}{4}\le\beta$, finishing the proof of Sublemma 1. So, we are left with the case when $\mathbf{a}_i,\mathbf{b}_i$, and $\mathbf{z}_i$ are pairwise distinct points on $\mathbb{S}^{d-1}$ and the spherical triangle with vertices $\mathbf{a}_i,\mathbf{b}_i$, and $\mathbf{z}_i$ has a right angle at $\mathbf{b}_i$. Clearly, $l(\widehat{\mathbf{a}_i\mathbf{z}_i})=\beta$ and $l(\widehat{\mathbf{b}_i\mathbf{z}_i})=\beta_1+\beta-\frac{\pi}{2}$. Let $\gamma:=l(\widehat{\mathbf{a}_i\mathbf{b}_i})$. According to Napier's trigonometric rule for the side lengths of a spherical right triangle we have $\cos\beta=\cos\left(\beta_1+\beta-\frac{\pi}{2}\right)\cos\gamma$. As $\frac{\pi}{2}<\beta_1+\beta<\pi$ and $\beta_1\le\beta<\frac{\pi}{2}$, it follows that

$$\cos\gamma=\frac{\cos\beta}{\sin(\beta_1+\beta)}\le\frac{\cos\beta}{\sin(2\beta)}=\frac{1}{2\sin\beta}<\frac{1}{2\sin\frac{\pi}{4}}=\frac{1}{\sqrt{2}}. \tag{7}$$

Thus, $\gamma>\frac{\pi}{4}$, implying that the angular radius of $\mathbb{S}^{d-2}\cap C_{\mathbb{S}^{d-1}}[\mathbf{z}_i,\beta_i]$ is $>\frac{\pi}{4}$. This completes the proof of Sublemma 1 in Case 1.

Case 2: Move $C_{\mathbb{S}^{d-1}}[\mathbf{z}_i,\beta]$ without changing its radius such that $\mathbf{z}_i$ moves along $\widehat{\mathbf{z}_i\mathbf{z}_1}$ and arrives at $\mathbf{z}_i^*\in\widehat{\mathbf{z}_i\mathbf{z}_1}$ with the property that $C_{\mathbb{S}^{d-1}}[\mathbf{z}_i^*,\beta]$ is tangent to $C_{\mathbb{S}^{d-1}}[-\mathbf{z}_1,\beta_1]$ at some point of $\widehat{\mathbf{z}_i^*(-\mathbf{z}_1)}$. Clearly,

$$\mathbb{S}^{d-2}\cap C_{\mathbb{S}^{d-1}}[\mathbf{z}_i^*,\beta]\subset\mathbb{S}^{d-2}\cap C_{\mathbb{S}^{d-1}}[\mathbf{z}_i,\beta].$$

Thus, the proof of Case 1 applied to $C_{\mathbb{S}^{d-1}}[\mathbf{z}_i^*,\beta]$ finishes the proof of Sublemma 1. □

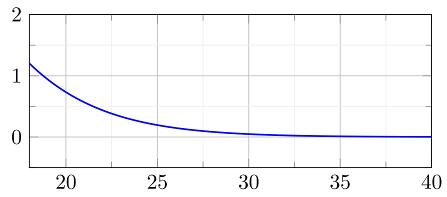


Fig. 5. The graph of $\frac{g(x)}{h(x)}$ showing that $\frac{g(x)}{h(x)} < 1$ holds for all $x \geq 19$.

Now, let $\mathbf{u}_1, \ldots, \mathbf{u}_{N_{\mathbb{S}^{d-2}}(\frac{\pi}{4})} \in \mathbb{S}^{d-2}$ such that the $(d-2)$-dimensional closed spherical caps $C_{\mathbb{S}^{d-2}}[\mathbf{u}_j, \frac{\pi}{4}]$, $1 \leq j \leq N_{\mathbb{S}^{d-2}}\left(\frac{\pi}{4}\right)$ cover $\mathbb{S}^{d-2}$. It follows via Sublemma 1 that $\{\mathbf{u}_1, \ldots, \mathbf{u}_{N_{\mathbb{S}^{d-2}}(\frac{\pi}{4})}\} \cap \left(\mathbb{S}^{d-2} \cap C_{\mathbb{S}^{d-1}}[\pm\mathbf{z}_i, \beta_i]\right) \neq \emptyset$ holds for all $2 \leq i \leq n$. If necessary one can reposition the points $\mathbf{u}_1, \ldots, \mathbf{u}_{N_{\mathbb{S}^{d-2}}(\frac{\pi}{4})}$ by a properly chosen isometry in $\mathbb{S}^{d-2}$ such that the stronger condition $\{\mathbf{u}_1, \ldots, \mathbf{u}_{N_{\mathbb{S}^{d-2}}(\frac{\pi}{4})}\} \cap \left(\mathbb{S}^{d-2} \cap C_{\mathbb{S}^{d-1}}(\pm\mathbf{z}_i, \beta_i)\right) \neq \emptyset$ holds as well for all $2 \leq i \leq n$. Finally, adding the points $\pm\mathbf{z}_1$ to $\{\mathbf{u}_1, \ldots, \mathbf{u}_{N_{\mathbb{S}^{d-2}}(\frac{\pi}{4})}\}$ completes the proof of Theorem 13. □

6. Proof of Corollary 7

Using $N_{\mathbb{S}^1}\left(\frac{\pi}{4}\right) = 4$ and Theorem 6 one obtains in a straightforward way that any 3-dimensional centrally symmetric cap body can be illuminated by 6 directions in $\mathbb{E}^3$. Next, recall that $9 \leq N_{\mathbb{S}^2}\left(\frac{\pi}{4}\right) \leq 10$ ([29]). This statement combined with Theorem 6 yields that any 4-dimensional centrally symmetric cap body can be illuminated by 12 directions in $\mathbb{E}^4$.

Remark 14. We note that the proof of Theorem 6 combined with the observation that $\mathbb{S}^1$ can be covered by 4 closed circular arcs of length $\frac{\pi}{2}$ forming an **o**-symmetric family implies that any 3-dimensional **o**-symmetric cap body can be illuminated by 6 **o**-symmetric directions in $\mathbb{E}^3$. However, a similar argument is not likely to work for the 4-dimensional setting because on the one hand, $9 \leq N_{\mathbb{S}^2}\left(\frac{\pi}{4}\right) \leq 10$ ([29]) on the other hand, it does not seem to be possible to cover $\mathbb{S}^2$ neither with 9 nor with 10 closed spherical caps of angular radius $\frac{\pi}{4}$ forming an **o**-symmetric family.

Finally, recall that Theorem 1 of [12] implies in a straightforward way that

$$2 + N_{\mathbb{S}^{d-2}}\left(\frac{\pi}{4}\right) \leq 2 + \frac{1}{\Omega_{d-2}(\frac{\pi}{4})}\left(\frac{1}{2} + \frac{3\ln\ln(d-2)}{\ln(d-2)} + \frac{3}{\ln(d-2)}\right)(d-2)\ln(d-2), \tag{8}$$

where $\Omega_{d-2}(\frac{\pi}{4})$ is the fraction of the surface of $\mathbb{S}^{d-2}$ covered by a closed spherical cap of angular radius $\frac{\pi}{4}$. Hence, the estimate $\Omega_{d-2}(\frac{\pi}{4}) > \frac{1}{2^{\frac{d-2}{2}}\sqrt{2\pi(d-1)}}$ (see for example, Lemma 2.1 in [21]) combined with (8) yields that

$$2 + N_{\mathbb{S}^{d-2}}\left(\frac{\pi}{4}\right) \leq 2 + 2^{\frac{d-2}{2}}\sqrt{2\pi(d-1)}\left(\frac{1}{2} + \frac{3\ln\ln(d-2)}{\ln(d-2)} + \frac{3}{\ln(d-2)}\right)(d-2)\ln(d-2) \tag{9}$$

holds for all $d \geq 5$. Furthermore,

$$2 + 2^{\frac{d-2}{2}}\sqrt{2\pi(d-1)}\left(\frac{1}{2} + \frac{3\ln\ln(d-2)}{\ln(d-2)} + \frac{3}{\ln(d-2)}\right)(d-2)\ln(d-2) < 2^d \tag{10}$$

holds for all $d \geq 19$. Indeed, Fig. 5 shows that $\frac{g(x)}{h(x)} < 1$ holds for all $x \geq 19$, where

$$g(x) := 2 + 2^{\frac{x-2}{2}}\sqrt{2\pi(x-1)}\left(\frac{1}{2} + \frac{3\ln\ln(x-2)}{\ln(x-2)} + \frac{3}{\ln(x-2)}\right)(x-2)\ln(x-2)$$

and $h(x) := 2^x$. For more details on this see the Appendix. Thus, (9) (resp., (10)) combined with Theorem 6 finishes the proof of Corollary 7.

7. Proof of Theorem 9

Theorem 9 concerns illuminating the cap bodies $\mathrm{Sp}_{\mathbf{B}^d}[\pm\mathbf{x}_1, \ldots, \pm\mathbf{x}_n]$ that are symmetric about every coordinate hyperplane $H_j = \{\mathbf{x} \in \mathbb{E}^d | \langle \mathbf{x}, \mathbf{e}_j \rangle = 0\}$, $1 \leq j \leq d$ in $\mathbb{E}^d$. According to Lemma 12, we only need to show that the open spherical caps $C_{\mathbb{S}^{d-1}}(\mathbf{y}_i, \pi/2 - \alpha_i)$, $1 \leq i \leq n$ can be pierced by $4d$ points in $\mathbb{S}^{d-1}$ such that the positive hull of these $4d$ unit vectors is $\mathbb{E}^d$.

We start by trying to use the $2d$ points $\{\pm\mathbf{e}_j \mid 1 \leq j \leq d\}$. If all the above mentioned open spherical caps are pierced by these $2d$ points, then the cap body in question can be illuminated by $2d$ directions and we are done. So, suppose there is a vertex $\mathbf{x}_i$ such that the cap $C_{\mathbb{S}^{d-1}}(\mathbf{y}_i, \pi/2-\alpha_i)$ isn't pierced by any of the $2d$ points $\{\pm\mathbf{e}_j \mid 1 \le j \le d\}$. Suppose then that $k \ge 0$ of the points $\pm\mathbf{e}_j, 1 \le j \le d$ lie on the boundary of this cap, and the rest of these points are not in the cap's closure $C_{\mathbb{S}^{d-1}}[\mathbf{y}_i, \pi/2-\alpha_i]$. This leads us to

Definition 6. An open spherical cap $C_{\mathbb{S}^{d-1}}(\mathbf{y}_i, \pi/2-\alpha_i)$ is a k-spanning cap if exactly $k \ge 0$ points of the set $\{\pm\mathbf{e}_j \mid 1 \le j \le d\}$ lie on the boundary of $C_{\mathbb{S}^{d-1}}(\mathbf{y}_i, \pi/2-\alpha_i)$, and the other $2d-k$ points from the set $\{\pm\mathbf{e}_j \mid 1 \le j \le d\}$ do not belong to $C_{\mathbb{S}^{d-1}}[\mathbf{y}_i, \pi/2-\alpha_i]$. The images of a k-spanning cap under arbitrary composition of finitely many reflections about the coordinate hyperplanes of $\mathbb{E}^d$ are called a k-spanning family of caps.

To properly study these k-spanning families, we need the following fact: the underlying spherical caps $C_{\mathbb{S}^{d-1}}[\pm\mathbf{y}_i, \alpha_i]$, $1 \le i \le n$ of $\mathrm{Sp}_{\mathbf{B}^d}[\pm\mathbf{x}_1, \dots, \pm\mathbf{x}_n]$ form a packing in $\mathbb{S}^{d-1}$. This means $\alpha_i + \alpha_j \le l(\widehat{\mathbf{y}_i, \mathbf{y}_j})$ for any $i \ne j \in \{1, \dots, n\}$. For the piercing caps $C_{\mathbb{S}^{d-1}}(\mathbf{y}_i, \pi/2-\alpha_i)$ and $C_{\mathbb{S}^{d-1}}(\mathbf{y}_j, \pi/2-\alpha_j)$ we can rewrite this condition as

$$(\pi/2-\alpha_i) + (\pi/2-\alpha_j) \ge \pi - l(\widehat{\mathbf{y}_i, \mathbf{y}_j}). \tag{11}$$

Lemma 15. *The open spherical cap $C_{\mathbb{S}^{d-1}}(\mathbf{y}, \varphi)$ with $0 < \varphi < \frac{\pi}{2}$ belongs to some k-spanning family if and only if the coordinates of $\mathbf{y}$ form a permutation of the sequence $\underbrace{\pm 1/\sqrt{k}, \dots, \pm 1/\sqrt{k}}_{k}, \underbrace{0, \dots, 0}_{d-k}$ and φ is equal to $\arccos(1/k)$, where $2 \le k \le d$.*

Proof. The statement is trivial in one direction. Namely, it is clear that the open spherical cap with the center and radius as described is not pierced by any vectors from $\{\pm\mathbf{e}_j | 1 \le j \le d\}$. In particular, it is a k-spanning cap for $2 \le k \le d$. It is also clear, that its images under arbitrary composition of finitely many reflections about the coordinate hyperplanes of $\mathbb{E}^d$ are k-spanning caps as well, forming a k-spanning family.

So, we are left to prove the non-trivial direction. Since any k-spanning family is unconditionally symmetric, we may assume that the cap $C_{\mathbb{S}^{d-1}}(\mathbf{y}, \varphi)$ with center $\mathbf{y} = (x_1, \dots, x_d)$ is such that the x_j's are non-negative. Without loss of generality, suppose $\mathbf{e}_1, \dots, \mathbf{e}_k \in \mathrm{bd}\, C_{\mathbb{S}^{d-1}}[\mathbf{y}, \varphi]$ and $\mathbf{e}_{k+1}, \dots, \mathbf{e}_d$ are not in $C_{\mathbb{S}^{d-1}}[\mathbf{y}, \varphi]$, where $0 < \varphi < \pi/2$. We can rewrite this condition as follows:

$$\begin{cases} x_j = \cos\varphi, \text{ if } j \le k \\ x_j < \cos\varphi, \text{ if } j > k \end{cases} \tag{12}$$

To finish the proof we only need to show that $x_{k+1} = x_{k+2} = \dots = x_d = 0$. Suppose $x_{k+1} > 0$. Then let the cap $C_{\mathbb{S}^{d-1}}(\mathbf{y}', \varphi)$ be a reflection of $C_{\mathbb{S}^{d-1}}(\mathbf{y}, \varphi)$ about the coordinate hyperplane H_{k+1}, hence $\mathbf{y}' = (x_1, \dots, x_k, -x_{k+1}, x_{k+2}, \dots, x_d)$. Using the inequality (11) for the caps $C_{\mathbb{S}^{d-1}}(\mathbf{y}', \varphi)$ and the $C_{\mathbb{S}^{d-1}}(\mathbf{y}, \varphi)$, we get the following:

$$\begin{aligned} &\varphi + \varphi \ge \pi - l(\widehat{\mathbf{y}, \mathbf{y}'}), \\ &\cos 2\varphi \le -\cos(l(\widehat{\mathbf{y}, \mathbf{y}'})), \\ &\cos 2\varphi \le -(x_1^2 + \dots + x_k^2 - x_{k+1}^2 + x_{k+2}^2 + \dots + x_d^2), \\ &2\cos^2\varphi - 1 \le -1 + 2x_{k+1}^2, \\ &\cos\varphi \le x_{k+1}. \end{aligned}$$

That clearly contradicts the second part of (12) and so, $x_{k+1} = 0$, and the same goes for $x_{k+2}, \dots, x_d$. Finally, by the first part of (12) one obtains that $\varphi = \arccos(1/\sqrt{k})$. It follows via $0 < \varphi < \pi/2$ that $2 \le k \le d$, finishing the proof of Lemma 15. □

Claim 16. The open spherical cap $C_{\mathbb{S}^{d-1}}(\mathbf{x}, \alpha)$ is pierced by $\mathbf{u} \in \mathbb{S}^{d-1}$ or $-\mathbf{u} \in \mathbb{S}^{d-1}$ if and only if $\left|\frac{\langle \mathbf{x}, \mathbf{u} \rangle}{\cos\alpha}\right| > 1$.

Proof. Clearly, $\mathbf{u}$ pierces $C_{\mathbb{S}^{d-1}}(\mathbf{x}, \alpha)$ if and only if $l(\widehat{\mathbf{u}, \mathbf{x}}) < \alpha$, i.e., $\cos(l(\widehat{\mathbf{u}, \mathbf{x}})) > \cos\alpha$. As α ranges from 0 to $\frac{\pi}{2}$ it is equivalent to $\frac{\langle \mathbf{u}, \mathbf{x} \rangle}{\cos\alpha} > 1$. Similarly, $-\mathbf{u}$ piercing $C_{\mathbb{S}^{d-1}}(\mathbf{x}, \alpha)$ is equivalent to $\frac{\langle \mathbf{u}, \mathbf{x} \rangle}{\cos\alpha} < -1$. Bringing these statements together gives us the claim. □

Each non-k-spanning cap is pierced by some point from $\{\pm\mathbf{e}_j \mid 1 \le j \le d\}$. If we take our new piercing points close enough to $\{\pm\mathbf{e}_j \mid 1 \le j \le d\}$, we still pierce all the non-k-spanning caps. Thus, we only need to construct a set of $4d$ points on $\mathbb{S}^{d-1}$ such that there is a point in a sufficiently small neighborhood of every point from $\{\pm\mathbf{e}_j \mid 1 \le j \le d\}$ moreover, every k-spanning cap is pierced.

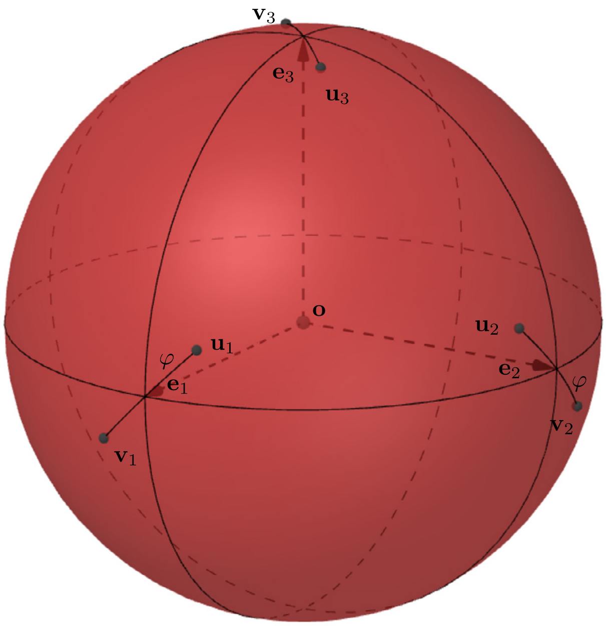


Fig. 6. Example of the points $\mathbf{u}_j, \mathbf{v}_j$ on $\mathbb{S}^2$.

Lemma 17. *Let φ be an angle in $(0, \pi/2)$, and let the points $\mathbf{u}_j, \mathbf{v}_j \in \mathbb{S}^{d-1}$, $1 \le j \le d$ be defined in the following way:*

$$\mathbf{u}_j = \left(\underbrace{\frac{\sin\varphi}{\sqrt{d-1}}, \frac{\sin\varphi}{\sqrt{d-1}}, \ldots, \frac{\sin\varphi}{\sqrt{d-1}}, \cos\varphi}_{j}, \frac{\sin\varphi}{\sqrt{d-1}}, \ldots, \frac{\sin\varphi}{\sqrt{d-1}} \right),$$

$$\mathbf{v}_j = \left(\underbrace{-\frac{\sin\varphi}{\sqrt{d-1}}, -\frac{\sin\varphi}{\sqrt{d-1}}, \ldots, -\frac{\sin\varphi}{\sqrt{d-1}}, \cos\varphi}_{j}, -\frac{\sin\varphi}{\sqrt{d-1}}, \ldots, -\frac{\sin\varphi}{\sqrt{d-1}} \right).$$

If φ is sufficiently small, then the $4d$ vectors $\left\{\pm\mathbf{u}_j, \pm\mathbf{v}_j \mid 1 \le j \le d\right\}$ pierce any k-spanning cap.

Proof. Essentially, as seen in the Fig. 6, we obtain $\mathbf{u}_j$ by rotating $\mathbf{e}_j$ with an angle φ towards the point $\left(1/\sqrt{k}, \ldots, 1/\sqrt{k}\right)$, and $\mathbf{v}_j$ we get by rotating $\mathbf{e}_j$ away from the same point.

Let $C_{\mathbb{S}^{d-1}}(\mathbf{y}, \alpha)$ be an open spherical cap of a k-spanning family. Lemma 15 implies that $\alpha = \arccos(1/\sqrt{k})$ and $\mathbf{y} = \frac{1}{\sqrt{k}}(s_1, \ldots, s_d)$ such that $s_j \in \{0, \pm1\}$ and $\sum_{j=1}^{d} s_j^2 = k$, where $2 \le k \le d$. We will need the parameter $s = \sum_{j=1}^{d} s_j$ as well. Next, we pick some $1 \le j \le d$ such that $s_j \ne 0$. According to Claim 16, $C_{\mathbb{S}^{d-1}}(\mathbf{y}, \alpha)$ is pierced by $\mathbf{u}_j$ or $-\mathbf{u}_j$ (resp., $\mathbf{v}_j$ or $-\mathbf{v}_j$) if and only if $\left|\langle \mathbf{u}_j, \sqrt{k}\mathbf{y}\rangle\right| > 1$ (resp., $\left|\langle \mathbf{v}_j, \sqrt{k}\mathbf{y}\rangle\right| > 1$). Now, observe that

$$\langle \mathbf{u}_j, \sqrt{k}\mathbf{y}\rangle = s_j\cos\varphi + (s - s_j)\frac{\sin\varphi}{\sqrt{d-1}} \text{ and } \langle \mathbf{v}_j, \sqrt{k}\mathbf{y}\rangle = s_j\cos\varphi - (s - s_j)\frac{\sin\varphi}{\sqrt{d-1}}. \tag{13}$$

If $s_j(s - s_j) > 0$, then (13) implies that for any sufficiently small φ one has $\left|\langle \mathbf{u}_j, \sqrt{k}\mathbf{y}\rangle\right| > 1$. Similarly, if $s_j(s - s_j) < 0$, then by (13) $\left|\langle \mathbf{v}_j, \sqrt{k}\mathbf{y}\rangle\right| > 1$ holds for any sufficiently small φ. So, we are left with the case when $s_j(s - s_j) = 0$. Since $s_j \ne 0$, that yields $s_j = s$. Thus, $s = \pm1$ and so, we just pick a different j so that $s \ne s_j$ and repeat the above process. Indeed, we can do that since $s = \pm1$, and that means we must have both 1's and -1's in the sequence $s_1, \ldots, s_d$. Otherwise, the sign of all the non-zero s_j's would be the same, and that would result in $s = \pm k$, a contradiction because $k \ge 2$. This completes the proof of Lemma 17. □

Clearly, the positive hull of the vectors $\left\{\pm\mathbf{u}_j, \pm\mathbf{v}_j \mid 1 \le j \le d\right\}$ is $\mathbb{E}^d$. Moreover, if φ is sufficiently small, then any cap $C_{\mathbb{S}^{d-1}}(\mathbf{y}_i, \pi/2 - \alpha_i)$ that isn't a k-spanning cap for some $1 \le i \le n$ and $2 \le k \le d$, is pierced by a point from

$\{\pm\mathbf{u}_j, \pm\mathbf{v}_j \mid 1 \le j \le d\}$. Finally, if $C_{\mathbb{S}^{d-1}}(\mathbf{y}_i, \pi/2 - \alpha_i)$ is a k-spanning cap for some $1 \le i \le n$ and $2 \le k \le d$, then Lemma 17 implies that it is pierced by the $4d$ vectors $\{\pm\mathbf{u}_j, \pm\mathbf{v}_j \mid 1 \le j \le d\}$. That concludes the proof of Theorem 9.

Declaration of competing interest

We claim no conflict of interest.

Data availability

No data was used for the research described in the article.

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Appendix A

A.1. More on the last inequality of (2)

We prove the following inequality in this section:

$$3 + 2^{d-2}\sqrt{2\pi(d-1)}\left(\frac{1}{2} + \frac{3\ln\ln(d-2)}{\ln(d-2)} + \frac{3}{\ln(d-2)}\right)(d-2)\ln(d-2) < 2^{d+1}d^{\frac{3}{2}}\ln d, \tag{14}$$

where $d \ge 4$. For the integer values $4 \le d \le 10$ one can check the inequality numerically, i.e., one can show that

$$f(x) := \frac{3 + 2^{x-2}\sqrt{2\pi(x-1)}\left(\frac{1}{2} + \frac{3\ln\ln(x-2)}{\ln(x-2)} + \frac{3}{\ln(x-2)}\right)(x-2)\ln(x-2)}{2^{x+1}x^{\frac{3}{2}}\ln x}$$

is less than 1 for any integer x chosen from the interval $3 < x \le 10$. Next, suppose that $d > 10$. Clearly, inequality (14) is equivalent to the following inequality:

$$\frac{3}{2^{d+1}d^{\frac{3}{2}}\ln d} + 2^{-3}\frac{\sqrt{2\pi(d-1)}}{\sqrt{d}}\left(\frac{1}{2} + \frac{3\ln\ln(d-2)}{\ln(d-2)} + \frac{3}{\ln(d-2)}\right)\frac{(d-2)}{d}\frac{\ln(d-2)}{\ln d} < 1.$$

First, observe that

$$\frac{3}{2^{d+1}d^{3/2}\ln d} + 2^{-3}\frac{\sqrt{2\pi(d-1)}}{\sqrt{d}}\left(\frac{1}{2} + \frac{3\ln\ln(d-2)}{\ln(d-2)} + \frac{3}{\ln(d-2)}\right)\frac{(d-2)}{d}\frac{\ln(d-2)}{\ln d}$$
$$< \frac{3}{2^d} + \frac{\sqrt{2\pi}}{8}\left(\frac{1}{2} + \frac{3\ln\ln(d-2)}{\ln(d-2)} + \frac{3}{\ln(d-2)}\right).$$

Second, consider the function $a(x) := \left(\frac{1}{2} + \frac{3\ln\ln(x-2)}{\ln(x-2)} + \frac{3}{\ln(x-2)}\right)$. Then $a'(x) = -\frac{\ln\ln(x-2)}{(x-2)\ln^4(x-2)}$. From this it follows that $a'(x) < 0$ for $x > e + 2$, hence $a(x)$ monotone decreasing over the interval $x > 10$. Finally, observe that $\frac{3}{2^{10}} + a(10) < 1$. As both $a(x)$ and $\frac{3}{10^x}$ are monotone decreasing, (14) holds for all $d > 10$.

A.2. More on inequality (10)

Here we prove the inequality

$$2 + 2^{\frac{d-2}{2}}\sqrt{2\pi(d-1)}\left(\frac{1}{2} + \frac{3\ln\ln(d-2)}{\ln(d-2)} + \frac{3}{\ln(d-2)}\right)(d-2)\ln(d-2) < 2^d$$

for $d \ge 19$. First, one can check numerically that the above inequality holds for any integer $19 \le d \le 50$. Second, suppose that $d > 50$. From the fact that the function $a(x) = \left(\frac{1}{2} + \frac{3\ln\ln(x-2)}{\ln(x-2)} + \frac{3}{\ln(x-2)}\right)$ is monotone decreasing over the interval $x > 5$, it follows that $a(d) < a(50) < 3$ holds for any $d > 50$. Thus, it follows that

$$\begin{aligned} & 2+2^{\frac{d-2}{2}}\sqrt{2\pi(d-1)}\left(\frac{1}{2}+\frac{3\ln\ln(d-2)}{\ln(d-2)}+\frac{3}{\ln(d-2)}\right)(d-2)\ln(d-2) \\ & \quad < 2^{\frac{d}{2}}\sqrt{2\pi(d-1)}\left(\frac{1}{2}+\frac{3\ln\ln(d-2)}{\ln(d-2)}+\frac{3}{\ln(d-2)}\right)(d-2)\ln(d-2) \\ & \qquad < 2^{d/2}(3\sqrt{d})3d\ln d \\ & \qquad < 2^{d/2}(9d^2) < 2^d \end{aligned}$$

holds for any $d > 50$.

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