Illuminating spiky balls and cap bodies

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A B S T R A C T

The convex hull of a ball with an exterior point is called a spike (or cap). A union of finitely many spikes of a ball is called a spiky ball. If a spiky ball is convex, then we call it a cap body. In this note we upper bound the illumination numbers of 2-illuminable spiky balls as well as centrally symmetric cap bodies. In particular, we prove the Illumination Conjecture for centrally symmetric cap bodies in sufficiently large dimensions. In fact, we do a bit more by showing that any $d$-dimensional centrally symmetric cap body can be illuminated by $<2^d$ directions in Euclidean $d$-space for $d=3,4,9$ and $d\geq 19$. Furthermore, we strengthen the latter result for unconditionally symmetric cap bodies.

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1. Introduction

Let $\mathbb{E}^d$ denote the $d$-dimensional Euclidean vector space, with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ and let $e_1, \ldots, e_d$ be its standard basis. Its unit sphere centered at the origin is $S^{d-1} := \{x \in \mathbb{E}^d \mid \|x\| = 1\}$. A greatcircle of $S^{d-1}$ is an intersection of $S^{d-1}$ with a plane of $\mathbb{E}^d$ passing through $o$. Two points are called antipodes if they can be obtained as an intersection of $S^{d-1}$ with a line through $o$ in $\mathbb{E}^d$. If $a, b \in S^{d-1}$ are two points that are not antipodes, then we label the (uniquely determined) shortest geodesic arc of $S^{d-1}$ connecting $a$ and $b$ by $ab$. In other words, $ab$ is the shorter circular arc with endpoints $a$ and $b$ of the greatcircle $ab$ that passes through $a$ and $b$. The length of $ab$ is called the spherical (or angular) distance between $a$ and $b$ and it is labeled by $l(ab)$, where $0 < l(ab) < \pi$. The set $C_{S^{d-1}}(\alpha, \alpha) := \{y \in S^{d-1} \mid l(y|y) < \alpha\} = \{y \in S^{d-1}|\langle x, y \rangle > \cos \alpha\}$ (resp., $C_{S^{d-1}}(\alpha, \alpha) := \{y \in S^{d-1} \mid l(y|y) < \alpha\} = \{y \in S^{d-1}|\langle x, y \rangle > \cos \alpha\}$) is called the closed (resp., open) spherical cap of angular radius $\alpha$ centered at $x \in S^{d-1}$ for $0 < \alpha \leq \frac{\pi}{2}$. The closed Euclidean ball of radius $r$ centered at $p \in \mathbb{E}^d$ is denoted by $B^d(p, r) := \{q \in \mathbb{E}^d \mid \|p - q\| \leq r\}$. A $d$-dimensional convex body $K$ is a compact convex subset of $\mathbb{E}^d$ with non-empty interior. Then $K$ is said to be $o$-symmetric if $-K = K$ and $K$ is called centrally symmetric if some translate of $K$ is $o$-symmetric. A light source at a point $p$ outside a convex body $K \subset \mathbb{E}^d$, illuminates a point $x$ on the boundary of $K$ if the halfline originating from $p$ and passing through $x$ intersects the interior of $K$ at a point not lying between $p$ and $x$. The set of points $\{p_i : i = 1, \ldots, n\}$ in the exterior of $K$ is said to illuminate $K$ if every boundary point of $K$ is illuminated by some $p_i$. The illumination number $\Gamma(K)$ of $K$ is the smallest $n$ for which $K$ can be illuminated by $n$ point light sources. One can also consider illumination of $K \subset \mathbb{E}^d$ by directions instead of by exterior points. We say that a point $x$ on the boundary of $K$ is...
illuminated in the direction $v \in S^{d-1}$ if the halfline originating from $x$ and with direction vector $v$ intersects the interior of $K$. The former notion of illumination was introduced by Hadwiger [15], while the latter notion is due to Boltzynski [5]. It may not come as a surprise that the two concepts are equivalent in the sense that a convex body $K$ can be illuminated by $n$ point sources if and only if it can be illuminated by $n$ directions. The following conjecture of Boltzynski [5] and Hadwiger [15] has become a central problem of convex and discrete geometry and inspired a significant body of research.

**Conjecture 1 (Illumination Conjecture).** The illumination number $I(K)$ of any $d$-dimensional convex body $K$, $d \geq 2$, is at most $2^d$ and $I(K) = 2^d$ only if $K$ is an affine $d$-cube.

While Conjecture 1 has been proved in the plane ([5], [14], [15], and [19]), it is open for dimensions larger than 2. On the other hand, there are numerous partial results supporting Conjecture 1 in dimensions greater than 2. For details we refer the interested reader to the recent survey article [4] and the references mentioned there. Here we highlight only the following results. Let $K$ be an arbitrary $d$-dimensional convex body with $d > 1$. Rogers [25] (see also [26]) has proved that $I(K) \leq (2^d_d) d(\ln d + \ln \ln d + 5) = O(4^d \sqrt{d} \ln d)$. Huang, Sloomka, Tkocz, and Vritsiou [16] improved this bound of Rogers for sufficiently large values of $d$ to $c_1 d e^{-c_2 \sqrt{d}}$, where $c_1, c_2 > 0$ are universal constants. Lassak [18] improved the upper bound of Rogers for some small values of $d$ to $(d + 1) d^{-1} - (d - 1)(d - 2)d^{-1}$. In fact, the best upper bounds for the illumination numbers of convex bodies in dimensions 3, 4, 5, 6 are 14 ([23]), 96, 1091, 15373 ([24]). The best upper bound for the illumination numbers of centrally symmetric convex bodies of $\mathbb{E}^d$, $d > 1$ is $2^d (d \ln d + \ln \ln d + 5)$ proved by Rogers ([25] and [26]). In connection with this upper bound we note that [28] proves Conjecture 1 for unit balls of $1$-symmetric norms in $\mathbb{E}^d$ provided that $d$ is sufficiently large. We also mention in passing that Conjecture 1 has been confirmed for certain classes of convex bodies such as wide ball-bodies including convex bodies of constant width ([11], [2], [3], [8], [27]), convex bodies of Helly dimension 2 ([7]), and belt-bodies including zonoids and zonotopes ([6]). The present article has been motivated by the investigations in [21] and it aims at proving Conjecture 1 for sufficiently high dimensional centrally symmetric cap bodies studied under the name centrally symmetric spiky balls in [21]. Actually, we do a bit more. The details are as follows.

**Definition 1.** Let $B^d := B^d[0, 1]$ and let $x_1, \ldots, x_n \in \mathbb{E}^d \setminus B^d$. Then $\text{Sp}_B^d[x_1, \ldots, x_n] := \bigcup_{i=1}^n \text{conv}(B^d \cup \{x_i\})$ is called a spiky (unit) ball, where $\text{conv}(\cdot)$ refers to the convex hull of the corresponding set. If $x_i \notin \bigcup_{j \in \mathbb{N}, j \neq i} \text{conv}(B^d \cup \{x_j\})$ holds for some $1 \leq i \leq n$, then $x_i$ is called a vertex of $\text{Sp}_B^d[x_1, \ldots, x_n]$. A point $x$ on the boundary of the spiky ball $\text{Sp}_B^d[x_1, \ldots, x_n]$ is illuminated in the direction $v \in S^{d-1}$ if the halfline originating from $x$ and with direction vector $v$ intersects the interior of $\text{Sp}_B^d[x_1, \ldots, x_n]$ in points arbitrarily close to $x$. Furthermore, the set of directions $\{v_i : i = 1, \ldots, m\} \subset S^{d-1}$ is said to illuminate $\text{Sp}_B^d[x_1, \ldots, x_n]$ if every boundary point of $\text{Sp}_B^d[x_1, \ldots, x_n]$ is illuminated by some $v_i$. The illumination number $I(\text{Sp}_B^d[x_1, \ldots, x_n])$ of $\text{Sp}_B^d[x_1, \ldots, x_n]$ is the smallest $m$ for which $\text{Sp}_B^d[x_1, \ldots, x_n]$ can be illuminated by $m$ directions. Moreover, we say that the spiky ball $\text{Sp}_B^d[x_1, \ldots, x_n]$ with vertices $x_1, \ldots, x_n$ is $2$-illuminable if any two of its vertices can be simultaneously illuminated by a direction in $\mathbb{E}^d$. Finally, $\text{Sp}_B^d[x_1, \ldots, x_n]$ is called a cap body if it is a convex body in $\mathbb{E}^d$. (See Fig. 1.)

We note that cap bodies were first studied by Minkowski [20]. On the other hand, the family of $2$-illuminable spiky balls seems to be a new family of spiky balls that have not been investigated before.

**Definition 2.** If $0 < \alpha \leq \frac{\pi}{2}$, then let $N_{S^{d-1}}(\alpha)$ denote the minimum number of closed spherical caps of angular radius $\alpha$ that can cover $S^{d−1}$.

Our first result upper bounds the illumination numbers of $2$-illuminable spiky balls. We note that spiky balls without being $2$-illuminable can have arbitrarily large illumination numbers.

**Theorem 2.** Suppose that $\text{Sp}_B^d[x_1, \ldots, x_n]$ is a $2$-illuminable spiky ball with vertices $x_1, \ldots, x_n$ in $\mathbb{E}^d$.

(i) If $d = 2$, then $I(\text{Sp}_B^d[x_1, \ldots, x_n]) = 3$.
(ii) If $d = 3$, then $I(\text{Sp}_B^d[x_1, \ldots, x_n]) \leq 5$.
(iii) If $d \geq 4$, then $I(\text{Sp}_B^d[x_1, \ldots, x_n]) \leq 3 + N_{S^{d-2}}(\frac{\pi}{2})$.

**Corollary 3.** Let $\text{Sp}_B^d[x_1, \ldots, x_n]$ be a $2$-illuminable spiky ball with vertices $x_1, \ldots, x_n$ in $\mathbb{E}^d$, $d \geq 4$. If $d = 4$, then $I(\text{Sp}_B^d[x_1, \ldots, x_n]) \leq 23$. If $d \geq 5$, then

$$I(\text{Sp}_B^d[x_1, \ldots, x_n]) \leq 3 + 2^{d-2} \sqrt{2\pi(d-1)} \left( \frac{1}{2} + \frac{3 \ln(d-2)}{\ln(d-2)} + \frac{3}{\ln(d-2)} \right) (d-2) \ln(d-2) < 2^{d+1} d^2 \ln d.$$
Definition 1. A cap body is a 3-dimensional centrally symmetric spiky body with vertices $\pm x_1, \ldots, \pm x_n$ in $\mathbb{R}^d$, $d \geq 3$. Then
\[ I(\text{Sp}_{\mathbb{B}}[\pm x_1, \ldots, \pm x_n]) \leq 2 + N_{5d-2} \left( \frac{\pi}{4} \right). \]

Corollary 7. Any 3-dimensional centrally symmetric cap body can be illuminated by 6 ($< 2^3$) directions in $\mathbb{R}^3$. (This is not a new result. It was proved via a dual method in [17].) On the other hand, any 4-dimensional centrally symmetric cap body can be illuminated by 12 ($< 2^4$) directions in $\mathbb{R}^4$. Moreover, if $\text{Sp}_{\mathbb{B}}[\pm x_1, \ldots, \pm x_n]$ is an o-symmetric cap body with vertices $\pm x_1, \ldots, \pm x_n$ in $\mathbb{R}^d$, $d \geq 5$, then
\[ I(\text{Sp}_{\mathbb{B}}[\pm x_1, \ldots, \pm x_n]) \leq 2 + 2 \frac{d^2}{d^2} \sqrt{2 \pi (d-1)} \left( \frac{1}{2} + \frac{3 \ln \ln(d-2)}{\ln(d-2)} + \frac{3}{\ln(d-2)} \right) (d-2) \ln(d-2), \]
where $2 + 2 \frac{d^2}{d^2} \sqrt{2 \pi (d-1)} \left( \frac{1}{2} + \frac{3 \ln \ln(d-2)}{\ln(d-2)} + \frac{3}{\ln(d-2)} \right) (d-2) \ln(d-2) < 2^d$ holds for all $d \geq 19$.

Remark 8. Clearly, Corollary 7 proves the Illumination Conjecture for centrally symmetric cap bodies of dimension $d$ for $d = 3, 4$ and $d \geq 19$. We note that based on Theorem 6, in order to prove the Illumination Conjecture for centrally symmetric cap bodies of dimension $d$ for $5 \leq d \leq 18$, it is sufficient to show that $N_{5d-2} \left( \frac{\pi}{4} \right) \leq 2^d - 2$ holds for all $5 \leq d \leq 18$. It seems that the method of the recent paper [8] has the potential to achieve this goal. Indeed, this goal has already been achieved for $d = 9$ in [8] by showing that $N_{57} \left( \frac{\pi}{4} \right) \leq 240 < 2^9 - 2 = 510$.

Definition 3. The cap body $K \subset \mathbb{R}^d$ is called 1-unconditionally symmetric if it symmetric about each coordinate hyperplane of $\mathbb{R}^d$.

We close this section with a strengthening of Corollary 7 for 1-unconditionally symmetric cap bodies. Recall that according to [17] if $K$ is a 1-unconditionally symmetric cap body in $\mathbb{R}^d$, then $I(K) \leq 8$.
**Theorem 9.** Let $K$ be a 1-unconditionally symmetric cap body in $\mathbb{E}^d$, $d \geq 5$. Then $I(K) \leq 4d$.

While this proves Conjecture 1 for 1-unconditionally symmetric cap bodies in dimensions $d \geq 5$, the $4d$ estimate does not seem to be sharp, and, in fact, we propose

**Conjecture 10.** Every 1-unconditionally symmetric cap body of $\mathbb{E}^d$ can be illuminated by 2d directions for all $d \geq 5$.

In the rest of the paper we prove Theorems 2, 6, and 9, Corollaries 3 and 7, and Remark 5.

2. Proof of Theorem 2

We start with

**Definition 4.** If $S_{B^d}(x_1, \ldots, x_n)$ is a spiky ball with vertices $x_1, \ldots, x_n$ in $\mathbb{E}^d$, then let $y_i$ and $0 < \alpha_i < \frac{1}{2}$ be defined for $1 \leq i \leq n$ by $C_{\gamma d-1}(y_i, \alpha_i) = \text{int} \left( \text{conv} \left( B^d \cup \{x_i\} \right) \right) \cap S^{d-1}$, where $\text{int}(\cdot)$ refers to the interior of the corresponding set in $\mathbb{E}^d$. We are going to refer to $C_{\gamma d-1}(y_i, \alpha_i)$ as the open spherical cap assigned to the vertex $x_i$ of $S_{B^d}(x_1, \ldots, x_n)$.

It is easy to see that the direction $v \in S^{d-1}$ illuminates the vertex $x_i$ of the spiky ball $S_{B^d}(x_1, \ldots, x_n)$ if and only if $v \in C_{\gamma d-1}(y_i, \alpha_i)$ and $v \in \text{int} \left( \text{conv} \left( B^d \cup \{x_i\} \right) \right) \cap S^{d-1}$ of directions whose positive hull pos($\{v_i : 1 \leq k \leq m\}$) $\subset S^{d-1}$, illuminates the spiky ball $S_{B^d}(x_1, \ldots, x_n)$ if and only if it illuminates the vertices $x_1, \ldots, x_n$ of $S_{B^d}(x_1, \ldots, x_n)$, the following statement is immediate.

**Lemma 11.** Let $S_{B^d}(x_1, \ldots, x_n)$ be a spiky (unit) ball with vertices $x_1, \ldots, x_n$ in $\mathbb{E}^d$. Then

(a) $S_{B^d}(x_1, \ldots, x_n)$ is 2-illuminable if and only if $C_{\gamma d-1}(\mathbf{y}, \mathbf{a}) \cap C_{\gamma d-1}(\mathbf{y}, \mathbf{a}) \neq \emptyset$ holds for all $1 \leq i < j \leq n$ moreover, (b) $\{v_i : 1 \leq k \leq m\} \subset S^{d-1}$ with pos($\{v_i : 1 \leq k \leq m\}$) $\subset S^{d-1}$ illuminates $S_{B^d}(x_1, \ldots, x_n)$ if and only if $C_{\gamma d-1}(\mathbf{y}, \mathbf{a}) \cap \{v_i : 1 \leq k \leq m\} \neq \emptyset$ holds for all $1 \leq i \leq n$.

Now, we are set to prove Theorem 2. Part (i): Let $S_{B^d}(x_1, \ldots, x_n)$ be a 2-illuminable spiky (unit) disk with vertices $x_1, \ldots, x_n$ in $\mathbb{E}^2$. Let $C := \{C_{\gamma d-1}(\mathbf{y}, \mathbf{a}) : 1 \leq i \leq n\}$ be the family of open circular arcs of length $< \pi$ assigned to the vertices of $S_{B^d}(x_1, \ldots, x_n)$. Without loss of generality we may assume that $C_{\gamma d-1}(\mathbf{y}, \mathbf{a})$ contains no other open circular arc of $C$. As by Part (a) of Lemma 11 $C_{\gamma d-1}(\mathbf{y}, \mathbf{a}) \cap C_{\gamma d-1}(\mathbf{y}, \mathbf{a}) = \emptyset$ holds for all $1 \leq i < j \leq n$ therefore, there exist $v_1, v_2 \in C_{\gamma d-1}(\mathbf{y}, \mathbf{a})$ with each of them lying sufficiently close to one of the two endpoints of $C_{\gamma d-1}(\mathbf{y}, \mathbf{a})$ such that $C_{\gamma d-1}(\mathbf{y}, \mathbf{a}) \cap \{v_1, v_2\} \neq \emptyset$ holds for all $1 \leq i \leq n$. Clearly, $v_1 \neq -v_2$ and so, one can choose $v_3 \in S^1$ such that pos($\{v_i : 1 \leq k \leq 3\}$) $\subset S^2$. Hence, by Part (b) of Lemma 11 $v_{-1} : 1 \leq k \leq 3\} \subset S^1$ illuminates $S_{B^d}(x_1, \ldots, x_n)$, implying $I(S_{B^d}(x_1, \ldots, x_n)) = 3$ in a straightforward way.

Part (ii): Let $S_{B^d}(x_1, \ldots, x_n)$ be a 2-illuminable spiky (unit) ball with vertices $x_1, \ldots, x_n$ in $\mathbb{E}^3$. Let $C := \{C_{\gamma d-1}(\mathbf{y}, \mathbf{a}) : 1 \leq i \leq n\}$ be the family of open spherical caps assigned to the vertices of $S_{B^d}(x_1, \ldots, x_n)$. By Part (a) of Lemma 11 any two members of $C$ intersect. Next, recall the following theorem of Danzer [10]: If $F$ is a family of finitely many closed spherical caps on $S^2$ such that every two members of $F$ intersect, then there exist 4 points on $S^2$ such that each member of $F$ contains at least one of them (i.e., 4 needles are always sufficient to pierce all members of $F$). Now, applying Danzer’s theorem to $C$ (or rather to the corresponding family of closed spherical caps with each closed spherical cap being somewhat smaller and concentric to an open spherical cap of $C$) one obtains the existence of $v_1, v_2, v_3, v_4 \in S^2$ with the property that $v_1, v_2, v_3$ are mutually independent and $C_{\gamma d-1}(\mathbf{y}, \mathbf{a}) \cap \{v_1, v_2, v_3, v_4\} \neq \emptyset$ holds for all $1 \leq i \leq n$. Finally, let us choose $v_0 \in S^2$ such that pos($\{v_i : 1 \leq k \leq 5\}$) $\subset S^2$. Thus, Part (b) of Lemma 11 implies in a straightforward way that $v_{-1} : 1 \leq k \leq 5\} \subset S^2$ illuminates $S_{B^d}(x_1, \ldots, x_n)$ and therefore $I(S_{B^d}(x_1, \ldots, x_n)) \leq 5$.

Part (iii): Let $S_{B^d}(x_1, \ldots, x_n)$ be a 2-illuminable spiky (unit) ball with vertices $x_1, \ldots, x_n$ in $\mathbb{E}^d$, $d \geq 4$. Let $C := \{C_{\gamma d-1}(\mathbf{y}, \mathbf{a}) : 1 \leq i \leq n\}$ be the family of open spherical caps assigned to the vertices of $S_{B^d}(x_1, \ldots, x_n)$. By Part (a) of Lemma 11 any two members of $C$ intersect. We need

**Definition 5.** Let $G_2(B^d)$ denote the smallest positive integer $k$ such that any finite family of pairwise intersecting $d$-dimensional closed balls in $\mathbb{E}^d$ is $k$-pierceable (i.e., the finite family of balls can be partitioned into $k$ subfamilies each having a non-empty intersection).

Now, recall Danzer’s estimate (see page 361 in [13]) according to which $G_2(B^d) \leq 1 + N_{\gamma d-1}(\pi)$. Let $s \in S^{d-1}$ be a point which is not a boundary point of any member of $C$. If $C'$ (resp., $C''$) consists of those members of $C$ that contain $s$ as an interior (resp., exterior) point, then clearly $C = C' \cup C''$. Let $H$ be the hyperplane tangent to $S^{d-1}$ at $-s$ in $\mathbb{E}^d$. If we take
3. Lemma 11 we get by the stereographic projection with center $s$ that maps $S^{d-1} \setminus s$ onto $H$, then applying Danzer’s estimate to the images of $C''$ in $H$ we get that there are $1 + N_{S^{d-2}}(\frac{\pi}{6})$ points of $S^{d-1}$ piercing the members of $C''$ in $S^{d-1}$. Hence, $C$ is pierceable by $2 + N_{S^{d-2}}(\frac{\pi}{6})$ points (including $s$) in $S^{d-1}$. As members of $C$ are open spherical caps of $S^{d-1}$ therefore there are $3 + N_{S^{d-2}}(\frac{\pi}{6})$ points in $S^{d-1}$ whose positive hull is $\mathbb{E}^d$ such that they pierce the members of $C$. Thus, by Part (b) of Lemma 11 we get that $I(S_{\mathcal{P}^2}[x_1, \ldots, x_n]) \leq 3 + N_{S^{d-2}}(\frac{\pi}{6})$. This completes the proof of Theorem 2.

3. Proof of Corollary 3

First, we recall that according to [29] there exists a covering of $S^2$ using 20 (closed) spherical caps of angular radius $\frac{\pi}{6}$. Thus, by Part (iii) of Theorem 2 if $\mathcal{S}_{\mathcal{P}^2}[x_1, \ldots, x_n]$ is a 2-illuminable spiky ball with vertices $x_1, \ldots, x_n$ in $\mathbb{E}^d$, then $I(\mathcal{S}_{\mathcal{P}^2}[x_1, \ldots, x_n]) \leq 23$.

Second, recall that Theorem 1 of [12] implies in a straightforward way that

$$3 + N_{S^{d-2}}\left(\frac{\pi}{6}\right) \leq 3 + \frac{1}{\Omega_{d-2}(\frac{\pi}{6})} \left(\frac{1}{2} + \frac{3 \ln(d-2)}{\ln(d-2)} + \frac{3}{\ln(d-2)}\right)(d-2) \ln(d-2), \tag{1}$$

where $\Omega_{d-2}(\frac{\pi}{6})$ is the fraction of the surface of $S^{d-2}$ covered by a closed spherical cap of angular radius $\frac{\pi}{6}$. Next, the estimate $\Omega_{d-2}(\frac{\pi}{6}) > \frac{1}{2d-2} \frac{1}{\sqrt{2\pi(d-1)}}$ (see for example, Lemma 2.1 in [21]) combined with (1) yields that

$$3 + N_{S^{d-2}}\left(\frac{\pi}{6}\right) \leq 3 + 2d-2 \sqrt{2\pi(d-1)} \left(\frac{1}{2} + \frac{3 \ln(d-2)}{\ln(d-2)} + \frac{3}{\ln(d-2)}\right)(d-2) \ln(d-2) < 2d+1 \sqrt{d^2 \ln d} \tag{2}$$

holds for all $d \geq 5$. Indeed, see Fig. 3 for the graph of the function

$$f(x) := \frac{3 + 2x-2 \sqrt{2\pi(x-1)} \left(\frac{1}{2} + \frac{3 \ln(x-2)}{\ln(x-2)} + \frac{3}{\ln(x-2)}\right)(x-2) \ln(x-2)}{2x^2+1x \ln x}, \quad x > 3$$
which clearly implies the last inequality of (2). For more details on this see the Appendix. Finally, if \( \text{Sp}_l[x_1, \ldots, x_n] \) is a 2-illuminable spiky (unit) ball with vertices \( x_1, \ldots, x_n \) in \( \mathbb{E}^d \), \( d \geq 5 \), then (2) combined with Part (iii) of Theorem 2 finishes the proof of Corollary 3.

4. Proof of Remark 5

Recall the following construction of Danzer [10]: there exist 10 closed circular disks in \( \mathbb{E}^2 \) such that any two of them intersect and it is impossible to pierce them by 3 needles. It follows in a straightforward way that there exists a family \( C \) of 10 open circular disks in \( \mathbb{E}^2 \) (each being somewhat larger and concentric to a closed circular disk of the previous family) such that any two of them intersect and it is impossible to pierce them by 3 needles. Now, let \( H \) be the plane tangent to \( S^2 \) at the point say, \( -s \) with \( C \) lying in \( H \). If we take the stereographic projection with center \( s \) that maps \( H \) onto \( S^2 \setminus s \) and label the image of the family \( C \) by \( C' \), then \( C' \) is a family of 10 open spherical caps in \( \mathbb{S}^2 \) such that any two of them intersect and it is impossible to pierce them by 3 needles. By choosing \( C' \) within a small neighborhood \( B_H(-s) \) of \( -s \) in \( H \), we get that each member of \( C' \) is an open spherical cap of angular radius \( < \frac{\pi}{2} \). Next, let us take the spiky unit ball \( \text{Sp}_l[x_1, \ldots, x_{10}] \) with \( \{C_{S^2}(-y_i, \frac{\pi}{2} - \alpha_i) \mid 1 \leq i \leq 10 \} = C' \). Clearly, due to Part (a) of Lemma 11, \( \text{Sp}_l[x_1, \ldots, x_{10}] \) is 2-illuminable. Finally, if we choose \( B_H(-s) \) sufficiently small, such that the spherical caps of \( C' \) all lie in a hemisphere, then Part (b) of Lemma 11 and Part (ii) of Theorem 2 yield that \( l(\text{Sp}_l[x_1, \ldots, x_{10}]) = 5 \).

Next, we recall the following construction of Bourgain and Lindenstrauss [9]: there exists \( d^* \) such that for any \( d \geq d^* \) one possesses a finite point set \( P \) of diameter 1 in \( \mathbb{E}^d \) whose any covering by unit diameter closed balls requires at least \( 1.0645^d \) balls. Hence, if we take the unit diameter closed balls centered at the points of \( P \) in \( \mathbb{E}^d \), then any two balls intersect and it is impossible to pierce them by fewer than \( \lfloor 1.0645^d \rfloor \) needles. It follows in a straightforward way that for any \( d \geq d^* \) there exists a family \( C_{d'} \) of open balls centered at the points of \( P \) in \( \mathbb{E}^d \) (each being somewhat larger and concentric to a unit diameter closed ball of the previous family) such that any two of them intersect and it is impossible to pierce them by fewer than \( \lfloor 1.0645^{d-1} \rfloor \) needles. Now, let \( H \) be the hyperplane tangent to \( S^{d-1} \) at the point say, \( -s \) with \( C_{d-1} \) lying in \( H \). If we take the stereographic projection with center \( s \) that maps \( H \) onto \( S^{d-1} \setminus s \) and label the image of the family \( C_{d-1} \) by \( C'_{d-1} \), then \( C'_{d-1} \) is a family of open spherical caps in \( \mathbb{S}^{d-1} \) such that any two of them intersect and it is impossible to pierce them by fewer than \( \lfloor 1.0645^{d-1} \rfloor \) needles. By choosing \( C_{d-1} \) within a small neighborhood \( B_H(-s) \) of \( -s \) in \( H \), we get that each member of \( C'_{d-1} \) is an open spherical cap of angular radius \( < \frac{\pi}{2} \). Next, let us take the spiky unit ball \( \text{Sp}_l[x_1, \ldots, x_n] \) with \( \{C_{S^2}(-y_i, \frac{\pi}{2} - \alpha_i) \mid 1 \leq i \leq n \} = C'_{d-1} \). Clearly, due to Part (a) of Lemma 11, \( \text{Sp}_l[x_1, \ldots, x_n] \) is 2-illuminable. Finally, if we choose \( B_H(-s) \) sufficiently small, such that the spherical caps of \( C'_{d-1} \) all lie in a hemisphere, then Part (b) of Lemma 11 yields that \( l(\text{Sp}_l[x_1, \ldots, x_n]) \geq 1 + \lfloor 1.0645^{d-1} \rfloor \), where \( d \geq d^* + 1 \).

5. Proof of Theorem 6

First, using Definition 4 we prove

**Lemma 12.** Let \( \text{Sp}_l[\pm x_1, \ldots, \pm x_n] \) be an \( o \)-symmetric cap body with vertices \( \pm x_1, \ldots, \pm x_n \) in \( \mathbb{E}^d \), \( d \geq 3 \). Then

(a) \( C_{S^d-1}[\pm y_i, \frac{\pi}{2} - \alpha_i] \cap C_{S^d-1}[\pm y_j, \frac{\pi}{2} - \alpha_j] \neq \emptyset \) holds for all \( 1 \leq i < j \leq n \) moreover,

(b) \( \{v_k : 1 \leq k \leq m \} \subseteq \mathbb{S}^{d-1} \) with \( \text{pos}(v_k : 1 \leq k \leq m) = \mathbb{E}^d \) illuminates \( \text{Sp}_l[\pm x_1, \ldots, \pm x_n] \) if and only if \( C_{S^d-1}(\pm y_i, \frac{\pi}{2} - \alpha_i) \cap \{v_k : 1 \leq k \leq m \} \neq \emptyset \) holds for all \( 1 \leq i \leq n \).

**Proof.** Due to convexity and symmetry of \( \text{Sp}_l[\pm x_1, \ldots, \pm x_n] \), the underlying spherical caps \( C_{S^d-1}[\pm y_i, \alpha_i] \), \( 1 \leq i \leq n \) form a packing in \( \mathbb{S}^{d-1} \) (see the Fig. 4 for the examples of the spiky ball cap configurations). Now, let \( 1 \leq i < j \leq n \). For Part (a) it is sufficient to show that

\[
C_{S^d-1}[\pm y_i, \frac{\pi}{2} - \alpha_i] \cap C_{S^d-1}[\pm y_j, \frac{\pi}{2} - \alpha_j] \neq \emptyset.
\]  

(Namely, the same argument and symmetry will imply that \( C_{S^d-1}(\pm y_i, \frac{\pi}{2} - \alpha_i) \cap C_{S^d-1}(\pm y_j, \frac{\pi}{2} - \alpha_j) \neq \emptyset. \) Let \( H_{ij} \) be a hyperplane passing through \( o \) and separating \( C_{S^d-1}(y_i, \alpha_i) \) and \( C_{S^d-1}(y_j, \alpha_j) \). Furthermore, let \( n_{ij} \in \mathbb{S}^{d-1} \) (resp., \( -n_{ij} \in \mathbb{S}^{d-1} \)) be the same side of \( H_{ij} \) as \( C_{S^d-1}(y_i, \alpha_i) \) (resp., \( C_{S^d-1}(y_j, \alpha_j) \)) such that \( \langle \pm n_{ij}, z \rangle = 0 \) for all \( z \in H_{ij} \). Clearly \( -n_{ij} \in C_{S^d-1}[\pm y_i, \frac{\pi}{2} - \alpha_i] \) Moreover, \( n_{ij} \in C_{S^d-1}[\pm y_j, \frac{\pi}{2} - \alpha_j] \) implying \( -n_{ij} \in C_{S^d-1}[y_j, \frac{\pi}{2} - \alpha_j] \). Thus, (3) follows, finishing the proof of Part (a). Finally, Part (b) follows from Part (b) of Lemma 11 in a straightforward way.

Second, based on Lemma 12, in order to prove Theorem 6 it is sufficient to show

**Theorem 13.** Let \( \{C_{S^d-1}[\pm z_i, \beta_i] \mid 1 \leq i \leq n \} \subseteq \mathbb{S}^{d-1} \) be an \( o \)-symmetric family of \( 2n \) closed spherical caps with \( d \geq 3 \) and \( 0 < \beta_i < \frac{\pi}{2} \), \( 1 \leq i \leq n \) such that
Proof. Without loss of generality we may assume that the points \(\{\pm z_i \mid 1 \leq i \leq n\}\) are pairwise distinct and
\[
0 < \beta_1 \leq \beta_2 \leq \cdots \leq \beta_n < \frac{\pi}{2}.
\]
Let \(H\) be the hyperplane of \(\mathbb{E}^d\) with normal vectors \(\pm z_1\) passing through \(o\), and let \(S^{d-2} := H \cap S^{d-1}\).

Sublemma 1. \(S^{d-2} \cap C_{S^{d-1}}[\pm z_i, \beta_i]\) is a \((d-2)\)-dimensional closed spherical cap of angular radius at least \(\frac{\pi}{4}\) for all \(2 \leq i \leq n\).

Proof. Let \(H^+\) be the closed halfspace of \(\mathbb{E}^d\) bounded by \(H\) that contains \(z_1\). Let \(i\) be fixed with \(2 \leq i \leq n\). Without loss of generality we may assume that \(z_i \in H^+\) and our goal is to show that \(S^{d-2} \cap C_{S^{d-1}}[z_i, \beta_i]\) is a \((d-2)\)-dimensional closed spherical cap of angular radius at least \(\frac{\pi}{4}\). Let \(\beta\) be the smallest positive real such that
\[
\beta_1 \leq \beta \leq \beta_i \quad \text{and} \quad C_{S^{d-1}}[z_i, \beta] \cap C_{S^{d-1}}[-z_1, \beta_1] \neq \emptyset
\]
and therefore also \(C_{S^{d-1}}[z_i, \beta] \cap C_{S^{d-1}}[z_1, \beta_1] \neq \emptyset\).

Thus, either \(C_{S^{d-1}}[z_i, \beta]\) is tangent to \(C_{S^{d-1}}[-z_1, \beta_1]\) at some point of \(z_i(-z_1)\) (Case 1) or \(\beta_1 = \beta\) (Case 2).

Case 1: Let \(b_i := \overrightarrow{z_i(-z_1)} \cap S^{d-2}\) and \(a_i \in \text{bd} \left( C_{S^{d-1}}[z_i, \beta] \right) \cap S^{d-2}\), where \(\text{bd}(\cdot)\) denotes the boundary of the corresponding set in \(S^{d-1}\). If \(z_i \in H\), then \(z_i = b_i\) and \(\beta = l(a_i, b_i)\) and therefore, (6) yields \(\frac{\pi}{4} = \frac{2\beta_1 + 2\beta}{4} \leq \beta\), finishing the proof of Sublemma 1. So, we are left with the case when \(a_i, b_i\), and \(z_i\) are pairwise distinct points on \(S^{d-1}\) and the spherical triangle with vertices \(a_i, b_i,\) and \(z_i\) has a right angle at \(b_i\). Clearly, \(l(a_i, z_i) = \beta\) and \(l(b_i, z_i) = \beta_1 + \beta - \frac{\pi}{2}\). Let \(\gamma := l(a_i, b_i)\). According to Napier’s trigonometric rule for the side lengths of a spherical right triangle we have \(\cos \beta = \cos \left( \beta_1 + \beta - \frac{\pi}{2} \right) \cos \gamma\). As \(\frac{\pi}{2} < \beta_1 + \beta < \pi\) and \(\beta_1 \leq \beta < \frac{\pi}{2}\), it follows that
\[
\cos \gamma = \frac{\cos \beta}{\sin(\beta_1 + \beta)} \leq \frac{\cos \beta}{\sin(2\beta)} \leq \frac{1}{2 \sin \beta} \leq \frac{1}{2 \sin \frac{\pi}{4}} = \frac{\sqrt{2}}{2}.
\]
Thus, \(\gamma > \frac{\pi}{4}\), implying that the angular radius of \(S^{d-2} \cap C_{S^{d-1}}[z_i, \beta_i]\) is \(\frac{\pi}{4}\). This completes the proof of Sublemma 1 in Case 1.

Case 2: Move \(C_{S^{d-1}}[z_i, \beta]\) without changing its radius such that \(z_i\) moves along \(\overrightarrow{z_i(-z_1)}\) and arrives at \(z_i^* \in z_i(-z_1)\) with the property that \(C_{S^{d-1}}[z_i^*, \beta_i]\) is tangent to \(C_{S^{d-1}}[-z_1, \beta_1]\) at some point of \(z_i^*(-z_1)\). Clearly,
\[
S^{d-2} \cap C_{S^{d-1}}[z_i, \beta] \subset S^{d-2} \cap C_{S^{d-1}}[z_i^*, \beta_i].
\]
Thus, the proof of Case 1 applied to \(C_{S^{d-1}}[z_i^*, \beta_i]\) finishes the proof of Sublemma 1.
Now, let $\mathbf{u}_1, \ldots, \mathbf{u}_{N_{S^d-2}(\pi/4)} \in S^{d-2}$ such that the $(d - 2)$-dimensional closed spherical caps $C_{S^{d-2}}[\mathbf{u}_j, \pi/4], \; 1 \leq j \leq N_{S^d-2}(\pi/4)$ cover $S^{d-2}$. It follows via Sublemma 1 that $\{\mathbf{u}_1, \ldots, \mathbf{u}_{N_{S^d-2}(\pi/4)}\} \cap (S^{d-2} \cap C_{S^d-1}(\pm \mathbf{z}_i, \beta_i)) \neq \emptyset$ holds for all $2 \leq i \leq n$. If necessary one can reposition the points $\mathbf{u}_1, \ldots, \mathbf{u}_{N_{S^d-2}(\pi/4)}$ by a properly chosen isometry in $S^{d-2}$ such that the stronger condition $\{\mathbf{u}_1, \ldots, \mathbf{u}_{N_{S^d-2}(\pi/4)}\} \cap (S^{d-2} \cap C_{S^d-1}(\pm \mathbf{z}_i, \beta_i)) \neq \emptyset$ holds as well for all $2 \leq i \leq n$. Finally, adding the points $\pm \mathbf{z}_1$ to $\{\mathbf{u}_1, \ldots, \mathbf{u}_{N_{S^d-2}(\pi/4)}\}$ completes the proof of Theorem 13. □

6. Proof of Corollary 7

Using $N_{S^1}(\pi/4) = 4$ and Theorem 6 one obtains in a straightforward way that any 3-dimensional centrally symmetric cap body can be illuminated by 6 directions in $E^3$. Next, recall that $9 \leq N_{S^2}(\pi/4) \leq 10$ ([29]). This statement combined with Theorem 6 yields that any 4-dimensional centrally symmetric cap body can be illuminated by 12 directions in $E^4$.

Remark 14. We note that the proof of Theorem 6 combined with the observation that $S^1$ can be covered by 4 closed circular arcs of length $\pi/4$ forming an o-symmetric family implies that any 3-dimensional o-symmetric cap body can be illuminated by 6 o-symmetric directions in $E^3$. However, a similar argument is not likely to work for the 4-dimensional setting because on the one hand, $9 \leq N_{S^2}(\pi/4) \leq 10$ ([29]) on the other hand, it does not seem to be possible to cover $S^2$ neither with 9 nor with 10 closed spherical caps of angular radius $\pi/4$ forming an o-symmetric family.

Finally, recall that Theorem 1 of [12] implies in a straightforward way that

$$2 + N_{S^{d-2}}\left(\frac{\pi}{4}\right) \leq 2 + \frac{1}{\Omega_{d-2}(\pi/4)} \left(\frac{1}{2} + \frac{3 \ln(d-2)}{\ln(d-2)} + \frac{3}{\ln(d-2)}\right)(d-2) \ln(d-2),$$

where $\Omega_{d-2}(\pi/4)$ is the fraction of the surface of $S^{d-2}$ covered by a closed spherical cap of angular radius $\pi/4$. Hence, the estimate $\Omega_{d-2}(\pi/4) > \frac{1}{2^{d-2}} \frac{1}{2\pi(d-1)}$ (see for example, Lemma 2.1 in [21]) combined with (8) yields that

$$2 + N_{S^{d-2}}\left(\frac{\pi}{4}\right) \leq 2 + 2^\frac{d-2}{2}\left(\frac{1}{2} + \frac{3 \ln(d-2)}{\ln(d-2)} + \frac{3}{\ln(d-2)}\right)(d-2) \ln(d-2)$$

holds for all $d \geq 5$. Furthermore,

$$2 + 2^\frac{d-2}{2}\left(\frac{1}{2} + \frac{3 \ln(d-2)}{\ln(d-2)} + \frac{3}{\ln(d-2)}\right)(d-2) \ln(d-2) < 2^d$$

holds for all $d \geq 19$. Indeed, Fig. 5 shows that $\frac{g(x)}{h(x)} < 1$ holds for all $x \geq 19$, where

$$g(x) := 2 + 2^\frac{x-2}{2}\left(\frac{1}{2} + \frac{3 \ln(x-2)}{\ln(x-2)} + \frac{3}{\ln(x-2)}\right)(x-2) \ln(x-2)$$

and $h(x) := 2^x$. For more details on this see the Appendix. Thus, (9) (resp., (10)) combined with Theorem 6 finishes the proof of Corollary 7.

7. Proof of Theorem 9

Theorem 9 concerns illuminating the cap bodies $S_{Pd\{\pm x_1, \ldots, \pm x_n\}}$ that are symmetric about every coordinate hyperplane $H_j = \{x \in E^d | (x, e_j) = 0\}, \; 1 \leq j \leq d$ in $E^d$. According to Lemma 12, we only need to show that the open spherical caps $C_{S^{d-1}}(y, \pi/2 - \alpha_i), \; 1 \leq i \leq n$ can be pierced by 4d points in $S^{d-1}$ such that the positive hull of these 4d unit vectors is $E^d$.

We start by trying to use the 2d points $\{\pm e_j | 1 \leq j \leq d\}$. If all the above mentioned open spherical caps are pierced by these 2d points, then the cap body in question can be illuminated by 2d directions and we are done. So, suppose there is
a vertex \( x \), such that the cap \( C_{S^{d-1}}(y, \pi/2 - \alpha) \) isn’t pierced by any of the \( 2d \) points \( \{ \pm e_j \mid 1 \leq j \leq d \} \). Suppose then that \( k \geq 0 \) of the points \( \pm e_j, 1 \leq j \leq d \) lie on the boundary of this cap, and the rest of these points are not in the cap’s closure \( C_{S^{d-1}}[y, \pi/2 - \alpha] \). This leads us to

**Definition 6.** An open spherical cap \( C_{S^{d-1}}(y, \pi/2 - \alpha) \) is a \( k \)-spanning cap if exactly \( k \geq 0 \) points of the set \( \{ \pm e_j \mid 1 \leq j \leq d \} \) lie on the boundary of \( C_{S^{d-1}}(y, \pi/2 - \alpha) \), and the other \( 2d - k \) points from the set \( \{ \pm e_j \mid 1 \leq j \leq d \} \) do not belong to \( C_{S^{d-1}}[y, \pi/2 - \alpha] \). The images of a \( k \)-spanning cap under arbitrary composition of finitely many reflections about the coordinate hyperplanes of \( \mathbb{E}^d \) are called a \( k \)-spanning family of caps.

To properly study these \( k \)-spanning families, we need the following fact: the underlying spherical caps \( C_{S^{d-1}}[\pm y, \alpha] \), \( 1 \leq i \leq n \) of \( S^d \{ \pm x_1, \ldots, \pm x_n \} \) form a packing in \( S^{d-1} \). This means \( \alpha_i + \alpha_j \leq l(y, y) \) for any \( i \neq j \in \{1, \ldots, n\} \). For the piercing caps \( C_{S^{d-1}}(y, \pi/2 - \alpha) \) and \( C_{S^{d-1}}(y, \pi - \alpha) \) we can rewrite this condition as

\[
(\pi/2 - \alpha_k) + (\pi/2 - \alpha_j) \geq \pi - l(y, y).
\]  

(11)

**Lemma 15.** The open spherical cap \( C_{S^{d-1}}(y, \varphi) \) with \( 0 < \varphi < \pi/2 \) belongs to some \( k \)-spanning family if and only if the coordinates of \( y \) form a permutation of the sequence \( \frac{1}{\sqrt{k}}, \ldots, \pm \frac{1}{\sqrt{k}}, 0, \ldots, 0 \) and \( \varphi \) is equal to \( \arccos(1/k) \), where \( 2 \leq k \leq d \).

**Proof.** The statement is trivial in one direction. Namely, it is clear that the open spherical cap with the center and radius as described is not pierced by any vectors from \( \{ \pm e_j \mid 1 \leq j \leq d \} \). In particular, it is a \( k \)-spanning cap for \( 2 \leq k \leq d \). It is also clear, that its images under arbitrary composition of finitely many reflections about the coordinate hyperplanes of \( \mathbb{E}^d \) are \( k \)-spanning caps as well, forming a \( k \)-spanning family.

So, we are left to prove the non-trivial direction. Since any \( k \)-spanning family is unconditionally symmetric, we may assume that the cap \( C_{S^{d-1}}(y, \varphi) \) with center \( y = (x_1, \ldots, x_d) \) is such that the \( x_j \)'s are non-negative. Without loss of generality, suppose \( e_1, \ldots, e_k \in bd C_{S^{d-1}}[y, \varphi] \) and \( e_{k+1}, \ldots, e_d \) are not in \( C_{S^{d-1}}[y, \varphi] \), where \( 0 < \varphi < \pi/2 \). We can rewrite this condition as follows:

\[
\begin{aligned}
x_j &= \cos \varphi, \text{ if } j \leq k \\
x_j &< \cos \varphi, \text{ if } j > k
\end{aligned}
\]

(12)

To finish the proof we only need to show that \( x_{k+1} = x_{k+2} = \cdots = x_d = 0 \). Suppose \( x_{k+1} > 0 \). Then let the cap \( C_{S^{d-1}}(y', \varphi) \) be a reflection of \( C_{S^{d-1}}(y, \varphi) \) about the coordinate hyperplane \( H_{k+1} \), hence \( y' = (x_1, \ldots, x_k, -x_{k+1}, x_{k+2}, \ldots, x_d) \). Using the inequality (11) for the caps \( C_{S^{d-1}}(y', \varphi) \) and the \( C_{S^{d-1}}(y, \varphi) \), we get the following:

\[
\varphi + \varphi \geq \pi - l(y, y),
\]

\[
\cos 2\varphi \leq -\cos(l(y, y)),
\]

\[
\cos 2\varphi \leq -(x_1^2 + \cdots + x_k^2 - x_{k+1}^2 + x_{k+2}^2 + \cdots + x_d^2),
\]

\[
2 \cos^2 \varphi - 1 \leq -1 + 2x_{k+1}^2,
\]

\[
\cos \varphi \leq x_{k+1}.
\]

That clearly contradicts the second part of (12) and so, \( x_{k+1} = 0 \), and the same goes for \( x_{k+2}, \ldots, x_d \). Finally, by the first part of (12) one obtains that \( \varphi = \arccos(1/\sqrt{k}) \). It follows via \( 0 < \varphi < \pi/2 \) that \( 2 \leq k \leq d \), finishing the proof of Lemma 15. \( \Box \)

**Claim 16.** The open spherical cap \( C_{S^{d-1}}(x, \alpha) \) is pierced by \( u \in S^{d-1} \) or \( -u \in S^{d-1} \) if and only if \( \frac{\langle x, u \rangle}{\cos \alpha} > 1 \).

**Proof.** Clearly, \( u \) pierces \( C_{S^{d-1}}(x, \alpha) \) if and only if \( l(u, x) < \alpha \), i.e., \( \cos(l(u, x)) > \cos \alpha \). As \( \alpha \) ranges from 0 to \( \pi/2 \) it is equivalent to \( \frac{\langle u, x \rangle}{\cos \alpha} > 1 \). Similarly, \( -u \) piercing \( C_{S^{d-1}}(x, \alpha) \) is equivalent to \( \frac{\langle u, x \rangle}{\cos \alpha} < -1 \). Bringing these statements together gives us the claim. \( \Box \)

Each non-\( k \)-spanning cap is pierced by some point from \( \{ \pm e_j \mid 1 \leq j \leq d \} \). If we take our new piercing points close enough to \( \{ \pm e_j \mid 1 \leq j \leq d \} \), we will pierce all the non-\( k \)-spanning caps. Thus, we only need to construct a set of \( 4d \) points on \( S^{d-1} \) such that there is a point in a sufficiently small neighborhood of every point from \( \{ \pm e_j \mid 1 \leq j \leq d \} \) moreover, every \( k \)-spanning cap is pierced.
Lemma 17. Let $\varphi$ be an angle in $(0, \pi/2)$, and let the points $\mathbf{u}_j, \mathbf{v}_j \in S^{d-1}, 1 \leq j \leq d$ be defined in the following way:

$$\mathbf{u}_j = \left( \frac{\sin \varphi}{\sqrt{d-1}}, \frac{\sin \varphi}{\sqrt{d-1}}, \ldots, \frac{\sin \varphi}{\sqrt{d-1}}, \cos \varphi, \frac{\sin \varphi}{\sqrt{d-1}}, \ldots, \frac{\sin \varphi}{\sqrt{d-1}} \right),$$

$$\mathbf{v}_j = \left( -\frac{\sin \varphi}{\sqrt{d-1}}, -\frac{\sin \varphi}{\sqrt{d-1}}, \ldots, -\frac{\sin \varphi}{\sqrt{d-1}}, \cos \varphi, -\frac{\sin \varphi}{\sqrt{d-1}}, \ldots, -\frac{\sin \varphi}{\sqrt{d-1}} \right).$$

If $\varphi$ is sufficiently small, then the 4d vectors $\{\pm \mathbf{u}_j, \pm \mathbf{v}_j | 1 \leq j \leq d\}$ pierce any k-spanning cap.

Proof. Essentially, as seen in the Fig. 6, we obtain $\mathbf{u}_j$ by rotating $\mathbf{e}_j$ with an angle $\varphi$ towards the point $\left(1/\sqrt{k}, \ldots, 1/\sqrt{k}\right)$. And $\mathbf{v}_j$ we get by rotating $\mathbf{e}_j$ away from the same point.

Let $C_{S^{d-1}}(\mathbf{y}, \alpha)$ be an open spherical cap of a $k$-spanning family. Lemma 15 implies that $\alpha = \arccos(1/\sqrt{k})$ and $\mathbf{y} = \frac{1}{\sqrt{k}}(s_1, \ldots, s_d)$ such that $s_j \in [0, \pm 1]$ and $\sum_{j=1}^{d} s_j^2 = k$, where $2 \leq k \leq d$. We will need the parameter $s = \sum_{j=1}^{d} s_j$ as well.

Next, we pick some $1 \leq j \leq d$ such that $s_j \neq 0$. According to Claim 16, $C_{S^{d-1}}(\mathbf{y}, \alpha)$ is pierced by $\mathbf{u}_j$ or $-\mathbf{u}_j$ (resp., $\mathbf{v}_j$ or $-\mathbf{v}_j$) if and only if $|\langle \mathbf{u}_j, \sqrt{k}\mathbf{y} \rangle| > 1$ (resp., $|\langle \mathbf{v}_j, \sqrt{k}\mathbf{y} \rangle| > 1$). Now, observe that

$$|\langle \mathbf{u}_j, \sqrt{k}\mathbf{y} \rangle| = s_j \cos \varphi + (s - s_j) \frac{\sin \varphi}{\sqrt{d-1}} \quad \text{and} \quad |\langle \mathbf{v}_j, \sqrt{k}\mathbf{y} \rangle| = s_j \cos \varphi - (s - s_j) \frac{\sin \varphi}{\sqrt{d-1}}.$$  \hspace{1cm} (13)

If $s_j(s - s_j) > 0$, then (13) implies that for any sufficiently small $\varphi$ one has $|\langle \mathbf{u}_j, \sqrt{k}\mathbf{y} \rangle| > 1$. Similarly, if $s_j(s - s_j) < 0$, then by (13) $|\langle \mathbf{v}_j, \sqrt{k}\mathbf{y} \rangle| > 1$ holds for any sufficiently small $\varphi$. So, we are left with the case when $s_j(s - s_j) = 0$. Since $s_j \neq 0$, that yields $s_j = s$. Thus, $s = \pm 1$ and so, we just pick a different $j$ so that $s \neq s_j$ and repeat the above process. Indeed, we can do that since $s = \pm 1$, and that means we must have both 1’s and -1’s in the sequence $s_1, \ldots, s_d$. Otherwise, the sign of all the non-zero $s_j$’s would be the same, and that would result in $s = \pm k$, a contradiction because $k \geq 2$. This completes the proof of Lemma 17. \hfill $\square$

Clearly, the positive hull of the vectors $\{\pm \mathbf{u}_j, \pm \mathbf{v}_j | 1 \leq j \leq d\}$ is $E^d$. Moreover, if $\varphi$ is sufficiently small, then any cap $C_{S^{d-1}}(\mathbf{y}, \pi/2 - \alpha_1)$ that isn’t a $k$-spanning cap for some $1 \leq i \leq n$ and $2 \leq k \leq d$, is pierced by a point from
\( \{ \pm u_j, \pm v_j \mid 1 \leq j \leq d \} \). Finally, if \( C_{S^{d-1}}(y_i, \pi/2 - \alpha_i) \) is a \( k \)-spanning cap for some \( 1 \leq i \leq n \) and \( 2 \leq k \leq d \), then Lemma 17 implies that it is pierced by the 4d vectors \( \{ \pm u_j, \pm v_j \mid 1 \leq j \leq d \} \). That concludes the proof of Theorem 9.

**Declaration of competing interest**

We claim no conflict of interest.

**Data availability**

No data was used for the research described in the article.

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**Appendix A**

**A.1. More on the last inequality of (2)**

We prove the following inequality in this section:

\[
3 + 2^{d-2} \sqrt{2 \pi (d-1)} \left( \frac{1}{2} + \frac{3 \ln(d-2)}{\ln(d-2)} + \frac{3}{\ln(d-2)} \right) (d-2) \ln(d-2) < 2^{d+1} d^3 \ln d,
\]

where \( d \geq 4 \). For the integer values \( 4 \leq d \leq 10 \) one can check the inequality numerically, i.e., one can show that

\[
f(x) := \frac{3 + 2^{x-2} \sqrt{2 \pi (x-1)} \left( \frac{1}{2} + \frac{3 \ln(x-2)}{\ln(x-2)} + \frac{3}{\ln(x-2)} \right) (x-2) \ln(x-2)}{2^{x+1} x \ln x}\]

is less than 1 for any integer \( x \) chosen from the interval \( 3 < x \leq 10 \). Next, suppose that \( d > 10 \). Clearly, inequality (14) is equivalent to the following inequality:

\[
\frac{3}{2^{d+1} d^3 \ln d} + 2^{-3} \sqrt{2 \pi (d-1)} \left( \frac{1}{2} + \frac{3 \ln(d-2)}{\ln(d-2)} + \frac{3}{\ln(d-2)} \right) (d-2) \ln(d-2) < 1.
\]

First, observe that

\[
\frac{3}{2^{d+1} d^3 \ln d} + 2^{-3} \sqrt{2 \pi (d-1)} \left( \frac{1}{2} + \frac{3 \ln(d-2)}{\ln(d-2)} + \frac{3}{\ln(d-2)} \right) (d-2) \ln(d-2)
\]

\[
< \frac{3}{2^d} + \frac{\sqrt{2 \pi}}{d} \left( \frac{1}{2} + \frac{3 \ln(d-2)}{\ln(d-2)} + \frac{3}{\ln(d-2)} \right).
\]

Second, consider the function \( a(x) := \left( \frac{1}{2} + \frac{3 \ln(x-2)}{\ln(x-2)} + \frac{3}{\ln(x-2)} \right) \). Then \( a'(x) = -\frac{\ln(x-2)}{x-2} \ln^2(x-2) \). From this it follows that \( a'(x) < 0 \) for \( x > e + 2 \), hence \( a(x) \) monotone decreasing over the interval \( x > 10 \). Finally, observe that \( \frac{3}{2^d} + a(10) < 1 \). As both \( a(x) \) and \( \frac{1}{10} \) are monotone decreasing, (14) holds for all \( d > 10 \).

**A.2. More on inequality (10)**

Here we prove the inequality

\[
2 + 2^{d-2} \sqrt{2 \pi (d-1)} \left( \frac{1}{2} + \frac{3 \ln(d-2)}{\ln(d-2)} + \frac{3}{\ln(d-2)} \right) (d-2) \ln(d-2) < 2^d
\]

for \( d \geq 19 \). First, one can check numerically that the above inequality holds for any integer \( 19 \leq d \leq 50 \). Second, suppose that \( d > 50 \). From the fact that the function \( a(x) = \left( \frac{1}{2} + \frac{3 \ln(x-2)}{\ln(x-2)} + \frac{3}{\ln(x-2)} \right) \) is monotone decreasing over the interval \( x > 5 \), it follows that \( a(d) < a(50) < 3 \) holds for any \( d > 50 \). Thus, it follows that
2 + 2 \frac{\ln 2 - 1}{2} \sqrt{2\pi (d - 1)} \left( \frac{1}{2} + \frac{3 \ln \ln (d - 2)}{\ln (d - 2)} + \frac{3}{\ln (d - 2)} \right) (d - 2) \ln (d - 2)
< 2^\frac{d}{2} \left( 3 \sqrt{d} \right) d \ln d
< 2^\frac{d}{2} (9d^2) < 2^d

holds for any \( d > 50 \).

References