Mathematica Pannonica *New Series* 29 /NS 3/ (2023) 2, 149–152
DOI: 10.1556/314.2023.00014

AKADÉMIAI KIADÓ

ON BLASCHKE–SANTALÓ-TYPE INEQUALITIES FOR r-BALL BODIES

Károly Bezdek[1,2,*]

[1] Department of Mathematics and Statistics, University of Calgary, Canada
[2] Department of Mathematics, University of Pannonia, Veszprém, Hungary

Communicated by György Dósa

Original Research Paper
Received: Mar 22, 2023 • Accepted: May 24, 2023
First published online: Jun 5, 2023

ABSTRACT

Let $\mathbb{E}^d$ denote the d-dimensional Euclidean space. The r-ball body generated by a given set in $\mathbb{E}^d$ is the intersection of balls of radius r centered at the points of the given set. The author [Discrete Optimization 44/1 (2022), Paper No. 100539] proved the following Blaschke–Santaló-type inequalities for r-ball bodies: for all $0 < k < d$ and for any set of given d-dimensional volume in $\mathbb{E}^d$ the k-th intrinsic volume of the r-ball body generated by the set becomes maximal if the set is a ball. In this note we give a new proof showing also the uniqueness of the maximizer. Some applications and related questions are mentioned as well.

KEYWORDS

Euclidean space, intrinsic volume, isoperimetric inequality, isodiametric inequality, Brunn–Minkowski inequality, intersections of congruent balls, r-ball body, Blaschke–Santaló-type inequalities.

MATHEMATICS SUBJECT CLASSIFICATION (2020)

Primary 52A20; Secondary 52A22

1. INTRODUCTION

We denote the Euclidean norm of a vector $\mathbf{p}$ in the d-dimensional Euclidean space $\mathbb{E}^d$, $d > 1$ by $|\mathbf{p}| := \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle}$, where $\langle \cdot, \cdot \rangle$ is the standard inner product. Let $A \subset \mathbb{E}^d$ be a compact convex set, and $1 \leq k \leq d$. We denote the k-th intrinsic volume of A by $\mathrm{V}_k(A)$. It is well known that $\mathrm{V}_d(A)$ is the d-dimensional volume of A, $2\mathrm{V}_{d-1}(A)$ is the surface area of A, and $\frac{2\omega_{d-1}}{d\omega_d}\mathrm{V}_1(A)$ is equal to the mean width of A, where ω_d stands for the volume of a d-dimensional unit ball, that is, $\omega_d = \frac{\pi^{\frac{d}{2}}}{\Gamma(1+\frac{d}{2})}$. (For more details on intrinsic volumes see for example, [8]). In this note, for simplicity $\mathrm{V}_k(\emptyset) = 0$ for all $1 \leq k \leq d$. The closed Euclidean ball of radius r centered at $\mathbf{p} \in \mathbb{E}^d$ is denoted by $\mathbf{B}^d[\mathbf{p}, r] := \{\mathbf{q} \in \mathbb{E}^d \mid |\mathbf{p} - \mathbf{q}| \leq r\}$.

DEFINITION 1.1. For a set $\emptyset \neq X \subseteq \mathbb{E}^d$, $d > 1$ and $r > 0$ let the *r-ball body* X^r generated by X be defined by $X^r := \bigcap_{\mathbf{x} \in X} \mathbf{B}^d[\mathbf{x}, r]$.

* Corresponding author. E-mail: kbezdek@ucalgary.ca

We note that either $X^r = \emptyset$, or X^r is a point, or $\mathrm{int}(X^r) \neq \emptyset$. Perhaps not surprisingly, r-ball bodies of $\mathbb{E}^d$ have already been investigated in a number of papers however, under various names such as "überkonvexe Menge" ([12]), "r-convex domain" ([6]), "spindle convex set" ([1], [10]), "ball convex set" ([11]), "hyperconvex set" ([7]), and "r-dual set" ([2]). r-ball bodies satisfy some basic identities such as $\left((X^r)^r\right)^r = X^r$ and $(X \cup Y)^r = X^r \cap Y^r$, which hold for any $X \subseteq \mathbb{E}^d$ and $Y \subseteq \mathbb{E}^d$. Clearly, the operation is order-reversing namely, $X \subseteq Y \subseteq \mathbb{M}^d$ implies $Y^r \subseteq X^r$. In this note we investigate volumetric relations between X^r and X in $\mathbb{E}^d$. First, recall the theorem of Gao, Hug, and Schneider [8] stating that for any convex body of given volume in $\mathbb{S}^d$ the volume of the spherical polar body is maximal if the convex body is a ball. The author has proved the following Euclidean analogue of their theorem in [2]. Let $\mathbf{A} \subset \mathbb{E}^d$, $d > 1$ be a compact set of volume $V_d(\mathbf{A}) > 0$ and $r > 0$. If $\mathbf{B} \subseteq \mathbb{E}^d$ is a ball with $V_d(\mathbf{A}) = V_d(\mathbf{B})$, then

$$V_d(\mathbf{A}^r) \leq V_d(\mathbf{B}^r). \tag{1.1}$$

As the theorem of Gao, Hug, and Schneider [8] is often called a spherical counterpart of the Blaschke–Santaló inequality, therefore one may refer to (1.1) as a Blaschke–Santaló-type inequality for r-ball bodies in $\mathbb{E}^d$. Next recall that (1.1) has been extended by the author to spherical as well as hyperbolic spaces ([2]) and then to intrinsic volumes ([4]) proving the following Blaschke–Santaló-type inequalities for intrinsic volumes of r-ball bodies in $\mathbb{E}^d$ without precise equality condition, which we included here. (See [3] for the core ideas behind Theorem 1.2 and [13, Theorem 3.1] for a randomized version.)

THEOREM 1.2. Let $\mathbf{A} \subset \mathbb{E}^d$, $d > 1$ be a compact set of volume $V_d(\mathbf{A}) > 0$ and $r > 0$. If $\mathbf{B} \subset \mathbb{E}^d$ is a ball with $V_d(\mathbf{A}) = V_d(\mathbf{B})$, then

$$V_k(\mathbf{A}^r) \leq V_k(\mathbf{B}^r) \tag{1.2}$$

holds for all $1 \leq k \leq d$ with equality if and only if $\mathbf{A}$ is congruent (i.e., isometric) to $\mathbf{B}$.

Fodor, Kurusa, and Vígh [7] have proved the following inequality for $k = d$, which we extended to other intrinsic volumes as well. Corollary 1.3 follows from Theorem 1.2, the homogeneity (of degree k) of k-th intrinsic volume, and from the observation that $f(x) = x^k(r-x)^k$, $0 \leq x \leq r$ has a unique maximum value at $x = \frac{r}{2}$ for any $d > 1$, $1 \leq k \leq d$, and $r > 0$.

COROLLARY 1.3. Let $d > 1$, $1 \leq k \leq d$, $r > 0$, and $\mathbf{A} \subset \mathbb{E}^d$ be an r-ball body. Set $P_k(\mathbf{A}) := V_k(\mathbf{A})V_k(\mathbf{A}^r)$. Then

$$P_k(\mathbf{A}) \leq P_k\left(\mathbf{B}^d\left[\mathbf{o}, \frac{r}{2}\right]\right)$$

with equality if and only if $\mathbf{A}$ is congruent to $\mathbf{B}^d[\mathbf{o}, \frac{r}{2}]$.

As a further application we mention that Theorem 1.2 has been used in [4] (see also [2]) to prove the long-standing Kneser–Poulsen conjecture for uniform contractions of intersections of sufficiently many congruent balls. We close this section with the following complementary question to Theorem 1.2, which seems to be new and open, and can be regarded as a Mahler-type problem for r-ball bodies.

OPEN PROBLEM 1. Let $d > 2$, $1 \leq k \leq d$, and $0 < v < r^d\omega_d = V_d(\mathbf{B}^d[\mathbf{o}, r])$. Find the minimum of $V_k(\mathbf{A}^r)$ for all r-ball bodies $\mathbf{A} \subset \mathbb{E}^d$ of given volume $v = V_d(\mathbf{A})$.

REMARK 1.4. Problem 1 for $d = 2$ can be answered as follows. Let $0 < v < \pi r^2$. Then the minimum of $V_1(\mathbf{A}^r)$ (resp., $V_2(\mathbf{A}^r)$) for all r-disk domains $\mathbf{A} \subset \mathbb{E}^2$ of given area $v = V_2(\mathbf{A})$ is attained only for r-lenses, which are intersections of two disks of radius r.

In the rest of this note we give a short proof for Theorem 1.2 (which uses the Brunn–Minkowski inequality and the isoperimetric inequality instead of the Alexandrov-Fenchel inequality applied in [4]) and derive Remark 1.4.

2. PROOF OF THEOREM 1.2

Clearly, if $\mathbf{B}^r = \emptyset$, then $\mathbf{A}^r = \emptyset$ and (1.2) follows. Similarly, it is easy to see that if $\mathbf{B}^r$ is a point in $\mathbb{E}^d$, then (1.2) follows. Hence, we may assume that $\mathbf{B}^r = \mathbf{B}^d[\mathbf{o}, R]$ and $\mathbf{B} = \mathbf{B}^d[\mathbf{o}, r - R]$ with $0 < R < r$, where $\mathbf{o}$ denotes the origin in $\mathbb{E}^d$.

DEFINITION 2.1. Let $\emptyset \neq K \subset \mathbb{E}^d$, $d > 1$ and $r > 0$. Then the *r-ball convex hull* $\operatorname{conv}_r K$ of K is defined by

$$\operatorname{conv}_r K := \bigcap \{\mathbf{B}^d[\mathbf{x}, r] \mid K \subseteq \mathbf{B}^d[\mathbf{x}, r]\}.$$

Moreover, let the r-ball convex hull of $\mathbb{E}^d$ be $\mathbb{E}^d$. Furthermore, we say that $K \subseteq \mathbb{E}^d$ is *r-ball convex* if $K = \operatorname{conv}_r K$.

REMARK 2.2. We note that clearly, $\operatorname{conv}_r K = \emptyset$ if and only if $K^r = \emptyset$. Moreover, $\emptyset \neq K \subset \mathbb{E}^d$ is r-ball convex if and only if K is an r-ball body.

We need [2, Lemma 5] stated as

LEMMA 2.3. If $\emptyset \neq K \subseteq \mathbb{E}^d$, $d > 1$ and $r > 0$, then $K^r = (\operatorname{conv}_r K)^r$.

Now, via Lemma 2.3 we may assume that $\mathbf{A} \subset \mathbb{E}^d$ is an r-ball body of volume $V_d(\mathbf{A}) > 0$ and $\mathbf{B} = \mathbf{B}^d[\mathbf{o}, r - R]$ with $0 < R < r$ such that $V_d(\mathbf{A}) = V_d(\mathbf{B})$. Next, recall [3, Proposition 2.5] which we state as

LEMMA 2.4. Let $d > 1$ and $r > 0$. If $\mathbf{A} \subset \mathbb{E}^d$ is an r-ball body, then $\mathbf{A} + (-\mathbf{A}^r) = \mathbf{B}^d[\mathbf{o}, r]$, where $+$ denotes the Minkowski sum.

Thus, the Brunn–Minkowski inequality for intrinsic volumes ([9], Eq. (74)) and Lemma 2.4 imply

$$V_k(\mathbf{A})^{\frac{1}{k}} + V_k(\mathbf{A}^r)^{\frac{1}{k}} = V_k(\mathbf{A})^{\frac{1}{k}} + V_k(-\mathbf{A}^r)^{\frac{1}{k}} \leq V_k(\mathbf{A} + (-\mathbf{A}^r))^{\frac{1}{k}} = V_k(\mathbf{B}^d[\mathbf{o}, r])^{\frac{1}{k}} \tag{2.1}$$

with equality if and only if ($\mathbf{A}$ and $-\mathbf{A}^r$ are homothetic, i.e.,) $\mathbf{A}$ is congruent to $\mathbf{B}$, where $1 \leq k \leq d$. Finally, (2.1), the isoperimetric inequality for intrinsic volumes stating that among convex bodies of given volume the balls have the smallest k-th intrinsic volume ([14, Section 7.4]), and the homogeneity (of degree k) of k-th intrinsic volume imply

$$\begin{aligned} V_k(\mathbf{A}^r)^{\frac{1}{k}} &\leq V_k(\mathbf{B}^d[\mathbf{o}, r])^{\frac{1}{k}} - V_k(\mathbf{A})^{\frac{1}{k}} \leq V_k(\mathbf{B}^d[\mathbf{o}, r])^{\frac{1}{k}} - V_k(\mathbf{B}^d[\mathbf{o}, r - R])^{\frac{1}{k}} \\ &= V_k(\mathbf{B}^d[\mathbf{o}, R])^{\frac{1}{k}} = V_k(\mathbf{B}^r)^{\frac{1}{k}}, \end{aligned}$$

with $V_k(\mathbf{A}^r)^{\frac{1}{k}} = V_k(\mathbf{B}^r)^{\frac{1}{k}}$ if and only if $\mathbf{A}$ is congruent to $\mathbf{B}$, where $1 \leq k \leq d$. This completes the proof of Theorem 1.2.

3. PROOF OF REMARK 1.4

Let $0 < v < \pi r^2$. Let $\mathbf{A} \subset \mathbb{E}^2$ be an r-disk domain of area $v = V_2(\mathbf{A})$. Lemma 2.4 implies that

$$V_1(\mathbf{A}) + V_1(\mathbf{A}^r) = \pi r. \tag{3.1}$$

Next, according to the reverse isoperimetric inequality of r-disk domains, which has been proved by Borisenko and Drach [5] (see [7] for a different proof without uniqueness of the extremal set), we have that

$$V_1(\mathbf{A}) \leq V_1(\mathbf{L}), \tag{3.2}$$

where $\mathbf{L}$ denotes the r-lens of area v in $\mathbb{E}^2$. Moreover, equality holds in (3.2) if and only if $\mathbf{A}$ is conguent to $\mathbf{L}$. Thus, (3.1) and (3.2) imply that $\pi r - V_1(\mathbf{L}) \leq V_1(\mathbf{A}^r)$. It follows via (3.1) that

$$V_1(\mathbf{L}^r) = \pi r - V_1(\mathbf{L}) \leq V_1(\mathbf{A}^r) \tag{3.3}$$

with equality if and only if $\mathbf{A}$ is congruent to $\mathbf{L}$. Finally, observe that (3.2) is equivalent to the statement that if $\mathbf{A}'$ is an r-disk domain and $\mathbf{L}'$ is an r-lens with $V_1(\mathbf{A}') = V_1(\mathbf{L}')$, then

$$V_2(\mathbf{L}') \leq V_2(\mathbf{A}') \tag{3.4}$$

with equality if and only if $\mathbf{A}'$ is conguent to $\mathbf{L}'$. Hence, (3.3) combined with (3.4) yields

$$V_2(\mathbf{L}^r) \leq V_2(\mathbf{A}^r)$$

with equality if and only if **A** is congruent to **L**. This completes the proof of Remark 1.4.

ACKNOWLEDGEMENTS

The author was partially supported by a Natural Sciences and Engineering Research Council of Canada Discovery Grant.

REFERENCES

[1] Bezdek, K., Lángi, Zs., Naszódi, M., and Papez, P. Ball-polyhedra. *Discrete Comput. Geom. 38*, 2 (2007), 201–230.
[2] Bezdek, K. From r-dual sets to uniform contractions. *Aequationes Math. 92*, 1 (2018), 123–134.
[3] Bezdek, K. and Naszódi, M. The Kneser–Poulsen conjecture for special contractions. *Discrete Comput. Geom. 60*, 4 (2018), 967–980.
[4] Bezdek, K. On the intrinsic volumes of intersections of congruent balls. *Discrete Optim. 44*, 1 (2022), Paper No. 100539 (7 pages).
[5] Borisenko, A. A. and Drach, K. D. Isoperimetric inequality for curves with curvature bounded below. *Matematicheskie Mat. Zametki 95*, 5 (2014), 656–665; English translation: *Math. Notes 95*, 5–6 (2014), 590–598.
[6] Fejes Tóth, L. Packing of r-convex discs. *Studia Sci. Math. Hungar. 17*, 1–4 (1982), 449–452.
[7] Fodor, F., Kurusa, Á., and Vígh, V. Inequalities for hyperconvex sets. *Adv. Geom. 16*, 3 (2016), 337–348.
[8] Gao, F., Hug, D., and Schneider, R. Intrinsic volumes and polar sets in spherical space. *Math. Notae 41* (2003), 159–176.
[9] Gardner, R. J. The Brunn–Minkowski inequality. *Bull. Am. Math. Soc. 39*, 3 (2002), 355–405.
[10] Kupitz, Y. S., Martini, H., and Perles, M. A. Ball polytopes and the Vázsonyi problem. *Acta Math. Hungar. 126*, 1–2 (2010), 99–163.
[11] Lángi, Zs., Naszódi, M., and Talata, I. Ball and spindle convexity with respect to a convex body. *Aequationes Math. 85*, 1–2 (2013), 41–67.
[12] Mayer, A. E. Eine Überkonvexität. *Math. Z. 39*, 1 (1935), 511–531.
[13] Paouris, G. and Pivovarov, P. Random ball-polyhedra and inequalities for intrinsic volumes. *Monatsh. Math. 182*, 3 (2017), 709–729.
[14] Schneider, R. *Convex bodies: the Brunn–Minkowski theory.* Encyclopedia of Mathematics and its Applications, Vol. 44. Cambridge University Press, Cambridge, 1993.