Rigid Ball-Polyhedra in Euclidean 3-Space

Károly Bezdek · Márton Naszódi

Abstract A ball-polyhedron is the intersection with non-empty interior of finitely many (closed) unit balls in Euclidean 3-space. One can represent the boundary of a ball-polyhedron as the union of vertices, edges, and faces defined in a rather natural way. A ball-polyhedron is called a simple ball-polyhedron if at every vertex exactly three edges meet. Moreover, a ball-polyhedron is called a standard ball-polyhedron if its vertex–edge–face structure is a lattice (with respect to containment). To each edge of a ball-polyhedron, one can assign an inner dihedral angle and say that the given ball-polyhedron is locally rigid with respect to its inner dihedral angles if the vertex–edge–face structure of the ball-polyhedron and its inner dihedral angles determine the ball-polyhedron up to congruence locally. The main result of this paper is a Cauchy-type rigidity theorem for ball-polyhedra stating that any simple and standard ball-polyhedron is locally rigid with respect to its inner dihedral angles.

Keywords Ball-polyhedron · Dual ball-polyhedron · Truncated Delaunay complex · (Infinitesimally) Rigid polyhedron · Rigid ball-polyhedron

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1 Introduction

Let $\mathbb{E}^3$ denote the three-dimensional (3D) Euclidean space. As in [3,4] a *ball-polyhedron* is the intersection with non-empty interior of finitely many closed congruent balls in $\mathbb{E}^3$. In fact, one may assume that the closed congruent 3D balls in question are of unit radius, that is, they are unit balls of $\mathbb{E}^3$. Also, it is natural to assume that removing any of the unit balls defining the intersection in question yields the intersection of the remaining unit balls becoming a larger set. (Equivalently, using the terminology introduced in [4], whenever we take a ball-polyhedron, we always assume that it is generated by a *reduced family* of unit balls.) Furthermore, following [3,4], one can represent the boundary of a ball-polyhedron in $\mathbb{E}^3$ as the union of *vertices, edges*, and *faces* defined in a rather natural way. A *standard ball-polyhedron* is one boundary structure of which is not "pathological": its vertex–edge–face structure is a lattice (called the face lattice of the given standard ball-polyhedron). For the definitions of vertex, edge, face, and standardness, see Sect. 3. In order to get a more complete picture on ball-polyhedra, we refer the interested reader to [3,4] as well as [8].

One of the best-known results in the geometry of convex polyhedra is Cauchy's rigidity theorem: If two convex polyhedra P and Q in $\mathbb{E}^3$ are combinatorially equivalent with the corresponding faces being congruent, then the angles between the corresponding pairs of adjacent faces are also equal and thus, P is congruent to Q. Putting it somewhat differently, the combinatorics of an arbitrary convex polyhedron and its face angles completely determine its inner dihedral angles. For more details on Cauchy's rigidity theorem and on its extensions, we refer the interested reader to [5]. In our joint paper [3], we have been looking for analogs of Cauchy's rigidity theorem for ball-polyhedra. In order to quote properly, the relevant results from [3] we need to recall the following terminology. To each edge of a ball-polyhedron in $\mathbb{E}^3$ we can assign an *inner dihedral angle*. In other words, take any point p in the relative interior of the edge, and take the two unit balls that contain the two faces of the ball-polyhedron meeting along that edge. Now, the inner dihedral angle along this edge is the angular measure of the intersection of the two half-spaces supporting the two unit balls at p. The angle in question is obviously independent of the choice of p. Moreover, at each vertex of a face of a ball-polyhedron, there is a *face angle* formed by the two edges meeting at the given vertex (which is, in fact, the angle between the two tangent halflines of the two edges meeting at the given vertex). Finally, we say that the standard ball-polyhedron P in $\mathbb{E}^3$ is *globally rigid with respect to its face angles* (resp., *its inner dihedral angles*) if the following holds. If Q is another standard ball-polyhedron in $\mathbb{E}^3$ face lattice of which is isomorphic to that of P and face angles of which (resp., inner dihedral angles) are equal to the corresponding face angles (resp. inner dihedral angles) of P, then Q is congruent to P. We note that in [3], we used the word "rigid" for this notion. We change that terminology to "globally rigid" because in the present paper, we consider a local version of the problem using the term "locally rigid."

Furthermore, a ball-polyhedron of $\mathbb{E}^3$ is called *simplicial* if all its faces are bounded by three edges. It is not hard to see that any simplicial ball-polyhedron is, in fact, a standard one. Now, we are ready to state the main (rigidity) result of [3]: The face lattice and the face angles determine the inner dihedral angles of any standard

ball-polyhedron in $\mathbb{E}^3$. In particular, if P is a simplicial ball-polyhedron in $\mathbb{E}^3$, then P is globally rigid with respect to its face angles. The following fundamental analog question is still an open problem (see [2, p. 63]).

Problem 1.1 Prove or disprove that the face lattice and the inner dihedral angles determine the face angles of any standard ball-polyhedron in $\mathbb{E}^3$.

One can regard this problem as an extension of the (still unresolved) conjecture of Stoker [12] according to which for convex polyhedra, the face lattice and the inner dihedral angles determine the face angles. For an overview on the status of the Stoker conjecture and in particular, for the recent remarkable result of Mazzeo and Montcouquiol on proving the infinitesimal version of the Stoker conjecture, see [10]. The following special case of Problem 1.1 has already been put forward as a conjecture in [3]. For this, we need to recall that a ball-polyhedron is called a *simple ball-polyhedron*, if, at every vertex, exactly three edges meet. Now, based on our terminology introduced above, the conjecture in question [3, p. 257] can be phrased as follows:

Conjecture 1.2 *Let P be a simple and standard ball-polyhedron of $\mathbb{E}^3$. Then P is globally rigid with respect to its inner dihedral angles.*

We do not know whether the conditions of Conjecture 1.2 are necessary. However, if the ball-polyhedron Q fails to be a standard ball-polyhedron because it possesses a pair of faces sharing more than one edge, then Q is flexible (so, it is not globally rigid) as shown in Sect. 4 of [3].

The main result of the present paper, Theorem 2.1, is a local version of Conjecture 1.2.

2 Main Result

We say that the standard ball-polyhedron P of $\mathbb{E}^3$ is *locally rigid with respect to its inner dihedral angles*, if there is an $\varepsilon > 0$ with the following property. If Q is another standard ball-polyhedron of $\mathbb{E}^3$ face lattice of which is isomorphic to that of P and inner dihedral angles of which are equal to the corresponding inner dihedral angles of P such that the corresponding faces of P and Q lie at Hausdorff distance at most ε from each other, then P and Q are congruent.

Now, we are ready to state the main result of this paper.

Theorem 2.1 *Let P be a simple and standard ball-polyhedron of $\mathbb{E}^3$. Then P is locally rigid with respect to its inner dihedral angles.*

Also, it is natural to say that the standard ball-polyhedron P of $\mathbb{E}^3$ is *locally rigid with respect to its face angles*, if there is an $\varepsilon > 0$ with the following property. If Q is another standard ball-polyhedron of $\mathbb{E}^3$ face lattice of which is isomorphic to that of P and face angles of which are equal to the corresponding face angles of P such that the corresponding faces of P and Q lie at Hausdorff distance at most ε from each other, then P and Q are congruent. As according to [3] the face lattice and the face angles determine the inner dihedral angles of any standard ball-polyhedron in $\mathbb{E}^3$, Theorem 2.1 implies the following claim in a straightforward way.

Corollary 2.2 *Let P be a simple and standard ball-polyhedron of $\mathbb{E}^3$. Then P is locally rigid with respect to its face angles.*

In the rest of this paper, we give a proof of Theorem 2.1.

3 The Combinatorial Structure of a Ball-Polyhedron

Let P be a ball-polyhedron in $\mathbb{E}^3$ given (as throughout the paper) by a reduced family of generating balls. A boundary point is called a *vertex* if it belongs to at least three of the closed unit balls defining the ball-polyhedron. A *face* of the ball-polyhedron is the intersection of one of the generating closed unit balls with the boundary of the ball-polyhedron. We say that the face of P corresponds to the center of the generating ball. Finally, if the intersection of two faces is non-empty, then it is the union of (possibly degenerate) circular arcs. The non-degenerate arcs are called *edges* of the ball-polyhedron. Obviously, if a ball-polyhedron in $\mathbb{E}^3$ is generated by at least three unit balls, then it possesses vertices, edges, and faces. Clearly, the vertices, edges, and faces of a ball-polyhedron (including the empty set and the ball-polyhedron itself) are partially ordered by inclusion forming the *vertex–edge–face structure* of the given ball-polyhedron.

We note that in [3], the vertex–edge–face structure of an arbitrary ball-polyhedron is incorrectly referred to as a face lattice. Indeed, Figure 4.1 of [3] shows an example of a ball-polyhedron vertex–edge–face structure of which is not a lattice (with respect to inclusion). Thus, it is natural to define the following fundamental family of ball-polyhedra: a ball-polyhedron in $\mathbb{E}^3$ is a *standard ball-polyhedron* if its vertex–edge–face structure is a lattice (with respect to inclusion). This is the case if, and only if, the intersection of any two faces is either empty, or one vertex or one edge, and every two edges share at most one vertex. In this case, we simply call the vertex–edge–face structure in question the *face lattice* of the standard ball-polyhedron. This definition implies that any standard ball-polyhedron of $\mathbb{E}^3$ is generated by at least four unit balls.

In connection with the above definition, we note that the family of standard ball-polyhedra was introduced and investigated in the more general, *n*-dimensional setting in [4]. The 3D case of that definition (Definition 6.4 in [4]) coincides with the definition given above. (See also Remark 9.1 and the paragraph preceding it in [4].) For more insight on the vertex–edge–face structure of ball-polyhedra in $\mathbb{E}^3$, we refer the interested reader to [8].

4 Infinitesimally Rigid Polyhedra, Dual Ball-Polyhedron, Truncated Delaunay Complex

In this section, we introduce the notations and the main tools that are needed for our proof of Theorem 2.1.

Recall that a *convex polyhedron* of $\mathbb{E}^3$ is a bounded intersection of finitely many closed halfspaces in $\mathbb{E}^3$. A *polyhedral complex* in $\mathbb{E}^3$ is a finite family of convex polyhedra such that any vertex, edge, and face of a member of the family is again a member of the family, and the intersection of any two members is empty or a vertex or

an edge or a face of both members. In this paper, a *polyhedron* of $\mathbb{E}^3$ means the union of all members of a 3D polyhedral complex in $\mathbb{E}^3$ possessing the additional property that its (topological) boundary in $\mathbb{E}^3$ is a surface in $\mathbb{E}^3$ (i.e., a 2D topological manifold embedded in $\mathbb{E}^3$).

We denote the convex hull of a set C by $[C]$. Following [7], we call a polyhedron Q in $\mathbb{E}^3$

- *weakly convex* if its vertices are in convex position (i.e., if its vertices are the vertices of a convex polyhedron);
- *co-decomposable* if its complement in $[Q]$ can be triangulated (i.e., obtained as a simplicial complex) without adding new vertices;
- *weakly co-decomposable* if it is contained in a convex polyhedron $\tilde{Q}$ such that all vertices of Q are vertices of $\tilde{Q}$, and the complement of Q in $\tilde{Q}$ can be triangulated without adding new vertices.

The boundary of every polyhedron in $\mathbb{E}^3$ is the disjoint union of planar convex polygons and hence, it can be triangulated without adding new vertices. Now, let P be a polyhedron in $\mathbb{E}^3$ and let T be a triangulation of its boundary without adding new vertices. We call the 1-skeleton $G(T)$ of T the *edge graph* of T. By an *infinitesimal flex* of the edge graph $G(T)$ in $\mathbb{E}^3$, we mean an assignment of vectors to the vertices of $G(T)$ (i.e., to the vertices of P) such that the displacements of the vertices in the assigned directions induce a zero first-order change of the edge lengths: $(p_i - p_j) \cdot (q_i - q_j) = 0$ for every edge $p_i p_j$ of $G(T)$, where q_i is the vector assigned to the vertex p_i. An infinitesimal flex is called trivial if it is the restriction of an infinitesimal rigid motion of $\mathbb{E}^3$. Finally, we say that the polyhedron P is *infinitesimally rigid* if every infinitesimal flex of the edge graph $G(T)$ of T is trivial. (It is not hard to see that the infinitesimal rigidity of a polyhedron is a well-defined notion i.e., independent of the triangulation T. For more details on this as well as for an overview on the theory of rigidity we refer the interested reader to [5].) We need the following remarkable rigidity theorem of Izmestiev and Schlenker [7] for the proof of Theorem 2.1.

Theorem 4.1 (Izmestiev–Schlenker, [7]) *Every weakly co-decomposable polyhedron of $\mathbb{E}^3$ is infinitesimally rigid.*

We note that Izmestiev and Schlenker [7] give a different definition of a polyhedron than ours, which yields a somewhat wider class of sets in $\mathbb{E}^3$. Their theorem in its original form contains the additional restriction that the polyhedron is "decomposable" (i.e., it can be triangulated without new vertices), which automatically holds for sets satisfying our narrower definition of a polyhedron. Last but not least, one of the referees of our paper noted that by definition every weakly co-decomposable polyhedron is, in fact, a weakly convex one, and therefore it is natural to state Theorem 4.1 in the above form (i.e., not mentioning weakly convexity among the conditions).

The closed ball of radius ρ centered at p in $\mathbb{E}^3$ is denoted by $\mathbf{B}(p, \rho)$. Also, it is convenient to use the notation $\mathbf{B}(p) := \mathbf{B}(p, 1)$. For a set $C \subseteq \mathbb{E}^3$, we denote the intersection of closed unit balls with centers in C by $\mathbf{B}(C) := \cap\{\mathbf{B}(c) : c \in C\}$. Recall that every ball-polyhedron $P = \mathbf{B}(C)$ can be generated such that $\mathbf{B}(C \setminus \{c\}) \neq \mathbf{B}(C)$ holds for any $c \in C$. Therefore, whenever we take a ball-polyhedron $P = \mathbf{B}(C)$, we

always assume the above mentioned reduced property of C. The following duality theorem has been proved in [3], which is also needed for our proof of Theorem 2.1.

Theorem 4.2 (Bezdek–Naszódi, [3]) *Let P be a standard ball-polyhedron of $\mathbb{E}^3$. Then the intersection P^* of the closed unit balls centered at the vertices of P is another standard ball-polyhedron face lattice of which is dual to that of P (i.e., there exists an order reversing bijection between the face lattices of P and P^*).*

For a more recent discussion on the above duality theorem and its generalizations, we refer the interested reader to [8].

Let us give a detailed construction of the so-called truncated Delaunay complex of an arbitrary ball-polyhedron, which is going to be the underlying polyhedral complex of the given ball-polyhedron playing a central role in our proof of Theorem 2.1. We leave some of the proofs of the claims mentioned in the rest of this section to the reader partly because they are straightforward and partly because they are also well known (see [1,11], and in particular, [6]).

The *farthest-point Voronoi tiling* corresponding to a finite set $C := \{c_1, \ldots, c_n\}$ in $\mathbb{E}^3$ is the family $\mathcal{V} := \{V_1, \ldots, V_n\}$ of closed *convex polyhedral sets* $V_i := \{x \in \mathbb{E}^3 : |x - c_i| \geq |x - c_j| \text{ for all } j \neq i, 1 \leq j \leq n\}$, $1 \leq i \leq n$. (Here, a closed convex polyhedral set means a not necessarily bounded intersection of finitely many closed halfspaces in $\mathbb{E}^3$.) We call the elements of $\mathcal{V}$ *farthest-point Voronoi cells*. In the sequel, we omit the words "farthest-point" as we do not use the other (more popular) Voronoi tiling: the one capturing closest points.

It is known that $\mathcal{V}$ is a tiling of $\mathbb{E}^3$. We call the vertices, (possibly unbounded) edges and (possibly unbounded) faces of the Voronoi cells of $\mathcal{V}$ simply the *vertices*, *edges* and *faces* of $\mathcal{V}$.

The *truncated Voronoi tiling* corresponding to C is the family $\mathcal{V}^t$ of closed convex sets $\{V_1 \cap \mathbf{B}(c_1), \ldots, V_n \cap \mathbf{B}(c_n)\}$. From the definition, it follows that $\mathcal{V}^t = \{V_1 \cap P, \ldots, V_n \cap P\}$ where $P = \mathbf{B}(C)$. We call elements of $\mathcal{V}^t$ *truncated Voronoi cells*.

Next, we define the (farthest-point) *Delaunay complex* $\mathcal{D}$ assigned to the finite set $C = \{c_1, \ldots, c_n\} \subset \mathbb{E}^3$. It is a polyhedral complex on the vertex set C. For an index set $I \subseteq \{1, \ldots, n\}$, the convex polyhedron $[c_i : i \in I]$ is a member of $\mathcal{D}$ if, and only if, there is a point p in $\cap\{V_i : i \in I\}$ which is not contained in any other Voronoi cell. In other words, $[c_i : i \in I] \in \mathcal{D}$ if, and only if, there is a point $p \in \mathbb{E}^3$ and a radius $\rho \geq 0$ such that $\{c_i : i \in I\} \subset \text{bd } \mathbf{B}(p, \rho)$ and $\{c_i : i \notin I\} \subset \text{int } \mathbf{B}(p, \rho)$. It is known that $\mathcal{D}$ is a *polyhedral complex*; in fact, it is a tiling of $[C]$ by convex polyhedra.

Lemma 4.3 *Let $C = \{c_1, \ldots, c_n\} \subset \mathbb{E}^3$ be a finite set, and $\mathcal{V} = \{V_1, \ldots, V_n\}$ be the corresponding Voronoi tiling of $\mathbb{E}^3$. Then*

(V) *For any vertex p of $\mathcal{V}$, there is an index set $I \subseteq \{1, \ldots, n\}$ with $\dim[c_i : i \in I] = 3$ such that $[c_i : i \in I] \in \mathcal{D}$ and $p = \cap\{V_i : i \in I\}$.*
    And vice versa: if $I \subseteq \{1, \ldots, n\}$ with $\dim[c_i : i \in I] = 3$ is such that $[c_i : i \in I] \in \mathcal{D}$, then $\cap\{V_i : i \in I\}$ is a vertex of $\mathcal{V}$.

(E) *For any edge ℓ of $\mathcal{V}$, there is an index set $I \subseteq \{1, \ldots, n\}$ with $\dim[c_i : i \in I] = 2$ such that $[c_i : i \in I] \in \mathcal{D}$ and $\ell = \cap\{V_i : i \in I\}$.*

And vica versa*: if $I \subseteq \{1, \ldots, n\}$ with $\dim[c_i : i \in I] = 2$ is such that $[c_i : i \in I] \in \mathcal{D}$, then $\cap\{V_i : i \in I\}$ is an edge of $\mathcal{V}$.*

(F) *For any face f of $\mathcal{V}$, there is an index set $I \subseteq \{1, \ldots, n\}$ with $|I| = 2$ such that $[c_i : i \in I] \in \mathcal{D}$ and $f = \cap\{V_i : i \in I\}$.*
And vica versa*: if $I \subseteq \{1, \ldots, n\}$ with $|I| = 2$ is such that $[c_i : i \in I] \in \mathcal{D}$, then $\cap\{V_i : i \in I\}$ is a face of $\mathcal{V}$.*

Proof We outline the proof of (V) as the rest follows the same argument. Let p be a vertex of $\mathcal{V}$, and let $I = \{i \in \{1, \ldots, n\} : p \in V_i\}$. Now p lies on the boundary of some Voronoi cells. The centers corresponding to these Voronoi cells are $\{c_i : i \in I\}$. As p is shared by their Voronoi cells, these centers are at an equal distance from p; in other words, they lie on a sphere around p. Now, suppose that these centers are co-planar. Then they lie on a circle such that the line, through the center of the circle and perpendicular to the plane of the circle, passes through p. Then all the Voronoi cells $\{V_i : i \in I\}$ contain a relative neighborhood of p within this line. Thus, p is not a vertex—a contradiction.

For the reverse statement: Let I be such that $[c_i : i \in I] \in \mathcal{D}$ and $\dim[c_i : i \in I] = 3$. It follows from the first condition on I that $\cap\{V_i : i \in I\} \neq \emptyset$, and from that second condition that $\cap\{V_i : i \in I\}$ is a singleton, say $\{p\}$. Clearly, p is a vertex of $\mathcal{V}$. $\square$

We define the *truncated Delaunay complex* $\mathcal{D}^t$ corresponding to C similarly to $\mathcal{D}$: For an index set $I \subseteq \{1, \ldots, n\}$, the convex polyhedron $[c_i : i \in I]$ is a member of $\mathcal{D}^t$ if, and only if, there is a point p in $\cap\{V_i \cap \mathbf{B}(c_i) : i \in I\}$ which is not contained in any other truncated Voronoi cell. Note that the truncated Voronoi cells are contained in the ball-polyhedron $\mathbf{B}(C)$. Thus, $[c_i : i \in I] \in \mathcal{D}^t$ if, and only if, there is a point $p \in \mathbf{B}(C)$ and a radius $\rho \geq 0$ such that $\{c_i : i \in I\} \subset \mathrm{bd}\, \mathbf{B}(p, \rho)$ and $\{c_i : i \notin I\} \subset \mathrm{int}\, \mathbf{B}(p, \rho)$ (Fig. 1).

5 Proof of Theorem 2.1

Lemma 5.1 *Let $P = \mathbf{B}(C)$ be a simple ball-polyhedron in $\mathbb{E}^3$. Then no vertex of the Voronoi tiling $\mathcal{V}$ corresponding to C is on* $\mathrm{bd}\, P$, *and no edge of $\mathcal{V}$ is tangent to P.*

Proof By (V) of Lemma 4.3, at least four Voronoi cells meet in any vertex of $\mathcal{V}$. Moreover, the intersection of each Voronoi cell with $\mathrm{bd}\, P$ is a face of P, as P is generated by a reduced set of centers. Hence, if a vertex of $\mathcal{V}$ were on $\mathrm{bd}\, P$, then at least four faces of P would meet at a point, contradicting the assumption that P is simple.

Let ℓ be en edge of $\mathcal{V}$, and assume that it contains a point $p \in \mathrm{bd}\, P$. By the previous paragraph, $p \in \mathrm{relint}\, \ell$. From Lemma 4.3 (E) it follows that p is in the intersection of some Voronoi cells $\{V_i : i \in I\}$ with $\dim[c_i : i \in I] = 2$. Clearly, ℓ is orthogonal to the plane $\mathrm{aff}\{c_i : i \in I\}$. Finally, there is an $\varepsilon > 0$ such that $P \cap \mathbf{B}(p, \varepsilon) = \mathbf{B}(\{c_i : i \in I\}) \cap \mathbf{B}(p, \varepsilon)$ and hence, ℓ must intersect $\mathrm{int}\, P$, as ℓ intersects $\mathrm{int}\,\big(\mathbf{B}(\{c_i : i \in I\}) \cap \mathbf{B}(p, \varepsilon)\big)$. $\square$

Lemma 5.2 *Let $P = \mathbf{B}(C)$ be a simple ball-polyhedron in $\mathbb{E}^3$. Then $\mathcal{D}^t$ is a sub-polyhedral complex of $\mathcal{D}$, that is, $\mathcal{D}^t \subseteq \mathcal{D}$, and faces, edges, and vertices of members of $\mathcal{D}^t$ are again members of $\mathcal{D}^t$.*
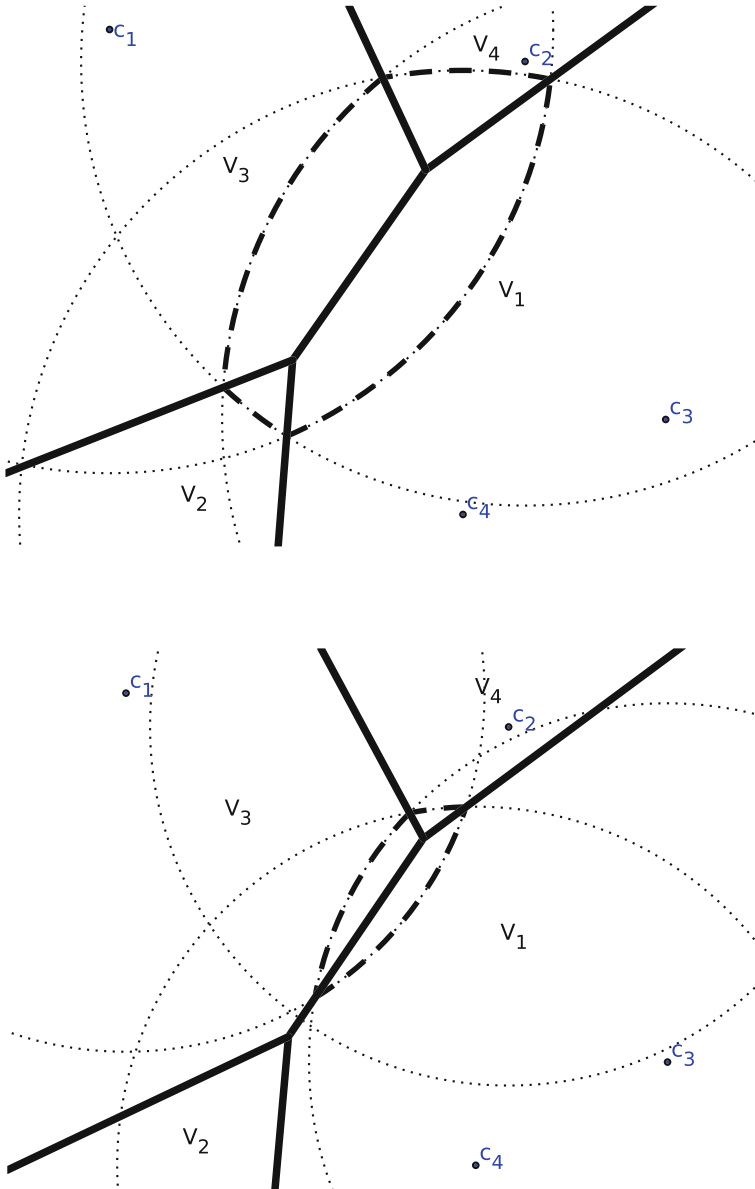
Fig. 1 Given four points, $c_1, \ldots, c_4$. The *bold solid lines* bound the four Voronoi cells, $V_1, \ldots, V_4$. The *bold dashed circular arcs* bound the planar ball-polyhedron—a disk-polygon. The part of each Voronoi cell inside the disk-polygon is the corresponding truncated Voronoi cell. On the first example, $[c_1, c_3, c_4]$ and $[c_1, c_3, c_2]$ are the 2D Delaunay cells, and $[c_1, c_2]$, $[c_1, c_3]$, $[c_1, c_4]$, $[c_2, c_3]$, $[c_3, c_4]$ are the 1D Delaunay cells. The truncated Delaunay complex coincides with the non-truncated one. On the second example, the Voronoi and the Delaunay complexes are the same as on the first, but the truncated Delaunay complex is different. The only 2D-truncated Delaunay cell is $[c_1, c_3, c_4]$. The 1D-truncated Delaunay cells are $[c_1, c_3]$, $[c_1, c_4]$, $[c_3, c_4]$

Proof Clearly, $\mathcal{D}^t \subseteq \mathcal{D}$, and their vertex sets are identical (both are C).

First, we show that a (two-dimensional—2D) face of a 3D member of $\mathcal{D}^t$ is again a member of $\mathcal{D}^t$. Let $[c_i : i \in I] \in \mathcal{D}^t$ be a 3D member of $\mathcal{D}^t$. Then, the corresponding vertex (Lemma 4.3 (V)) v of $\mathcal{V}$ is in int P by Lemma 5.1. For a given face of $[c_i : i \in I]$, there is a corresponding edge (Lemma 4.3 (E)) ℓ of $\mathcal{V}$. Clearly, v is an endpoint of ℓ. Now, relint $\ell \cap P \neq \emptyset$, and thus, the face $[c_i : i \in I]$ of $\mathcal{V}$ corresponding to ℓ is in $\mathcal{D}^t$.

Next, let $[c_i : i \in I] \in \mathcal{D}^t$ be a 2D member of $\mathcal{D}^t$ and let $[c_i, c_j]$ be one of its edges. Then, for the corresponding edge ℓ of $\mathcal{V}$ we have relint $\ell \cap P \neq \emptyset$. By Lemma 5.1, ℓ is not tangent to P, thus relint $\ell \cap$ int $P \neq \emptyset$. Now, $[c_i, c_j]$ corresponds to a face (Lemma 4.3 (F)) f of $\mathcal{V}$. Clearly, ℓ is an edge of f. As an edge of f intersects int P, we have relint $f \cap P \neq \emptyset$ and hence, $f \cap P$ is a 2D face of the truncated Voronoi tiling. It follows that $[c_i, c_j]$ is in $\mathcal{D}^t$.   □

The following lemma helps us to understand the 2D members of $\mathcal{D}^t$.

Let $P = \mathbf{B}(C)$ be a simple and standard ball-polyhedron in $\mathbb{E}^3$. Denote by Q the polyhedral complex formed by the 3D members of $\mathcal{D}^t$ and all of their faces, edges and vertices (i.e., we drop "hanging" faces/edges/vertices of $\mathcal{D}^t$, that is, those faces/edges/vertices that do not belong to a 3D member). Clearly, $\cup Q$ is a subset of $\mathbb{E}^3$ and thus, its boundary is defined. We equip this boundary with a polyhedral complex structure in the obvious way as follows: we define the boundary of Q as the collection of those faces, edges and vertices of Q that lie on the boundary of $\cup Q$. We denote this polyhedral complex by bd Q.

Lemma 5.3 *Let $P = \mathbf{B}(C)$ be a simple and standard ball-polyhedron in $\mathbb{E}^3$, and Q be defined as above. Then the 2D members of* bd Q *are triangles, and a triangle $[c_1, c_2, c_3]$ is in* bd Q *if, and only if, the corresponding faces F_1, F_2, F_3 of P meet (at a vertex of P).*

Proof By Lemma 5.2, the 2D members of bd Q are 2D members of $\mathcal{D}^t$. Let $[c_i : i \in I] \in \mathcal{D}^t$ with $\dim[c_i : i \in I] = 2$. Then clearly, $[c_i : i \in I] \in \mathcal{D}$ and, by Lemma 4.3 (E), it corresponds to an edge ℓ of $\mathcal{V}$ which intersects P. Now, ℓ is a closed line segment, or a closed ray, or a line. By Lemma 5.1, ℓ is not tangent to P, and (by Lemma 5.1) ℓ has no endpoint on bd P. Thus, ℓ intersects the interior of P. We claim that ℓ has at least one endpoint in int P. Suppose, it does not. Then $\ell \cap$ bd P is a pair of points and so, the faces of P corresponding to indices in I meet at more than one point. As $|I| \geq 3$, it contradicts the assumption that P is standard. We remark that this is a crucial point where we used the standardness of P. So, ℓ has either one or two endpoints in int P. If it has two, then the two distinct 3D Delaunay cells corresponding to those endpoints (as in Lemma 4.3 (V)) are both members of $\mathcal{D}^t$ and contain the planar convex polygon $[c_I : i \in I]$, and thus, $[c_I : i \in I]$ is not on the boundary of Q. If ℓ has one endpoint in int P, then there is a unique 3D polyhedron in $\mathcal{D}^t$ (the one corresponding to that endpoint of ℓ) that contains the planar convex polygon $[c_i : i \in I]$. Moreover, in this case ℓ intersects bd P at a vertex of P. As P is simple, that vertex is contained in exactly three faces of P, and hence, $[c_i : i \in I]$ is a triangle.

Next, working in the reverse direction, assume that F_1, F_2, and F_3 are faces of P that meet at a vertex v of P. Then v is in exactly three Voronoi cells, V_1, V_2 and V_3. Thus, $[c_1, c_2, c_3] \in \mathcal{D}$, and $\ell := V_1 \cap V_2 \cap V_3$ is an edge of $\mathcal{V}$. By the above argument,

ℓ has one endpoint in P and so, $[c_1, c_2, c_3]$ is a member of $\mathcal{D}^t$, and has the property that exactly one 3D member of $\mathcal{D}^t$ contains it. It follows that $[c_1, c_2, c_3]$ is in bd Q. □

From the last paragraph of the proof and the fact that P has at least one vertex, we can deduce the following

Remark 5.4 With the notations and the assumptions of Lemma 5.3, $\mathcal{D}^t$ contains at least one 3D cell, and the vertex set of Q is C.

We recall that the *nerve* of a set family $\mathcal{G}$ is the abstract simplicial complex

$$\mathcal{N}(\mathcal{G}) := \big\{ \{G_i \in \mathcal{G} : i \in I\} : \underset{i \in I}{\cap} G_i \neq \emptyset \big\}.$$

Now, let $P = \mathbf{B}(C)$ be a simple and standard ball-polyhedron in $\mathbb{E}^3$ and let $\mathcal{F}$ denote the set of its faces. Let $\mathcal{S}$ be the abstract simplicial complex on the vertex set C generated by the 2D members of bd Q (for the definition of bd Q, see the paragraph preceding Lemma 5.3), which are, according to Lemma 5.3, certain triples of points in C. Both $\mathcal{S}$ and the nerve $\mathcal{N}(\mathcal{F})$ of $\mathcal{F}$ are 2D abstract simplicial complexes. For the definition of an abstract simplicial complex and its geometric realization, see [9].

We claim that they both have the following "edge property": any edge is contained in a 2D simplex. Indeed, $\mathcal{S}$ has this property by definition, as it is a simplicial complex generated by a family of 2D simplices. On the other hand, $\mathcal{N}(\mathcal{F})$ also has this property, because P is simple and standard, and hence any edge of P has a vertex as an endpoint which is a point of intersection of three faces of P.

Consider the mapping $\phi : c_i \mapsto F_i$ that maps each center point in C to the corresponding face of P. This is a bijection between the 0-dimensional members of $\mathcal{S}$ and the 0-dimensional members of $\mathcal{N}(\mathcal{F})$. By Lemma 5.3, the 2D members of $\mathcal{S}$ correspond via ϕ to the 2D members of $\mathcal{N}(\mathcal{F})$. By the "edge property" in the previous paragraph, it follows that ϕ is an isomorphism of the two abstract simplicial complexes, $\mathcal{S}$ and $\mathcal{N}(\mathcal{F})$.

By Theorem 4.2, $\mathcal{N}(\mathcal{F})$ is isomorphic to the face-lattice of another standard ball-polyhedron: P^*. As P^* is a convex body in $\mathbb{E}^3$ (i.e., a compact convex set with non-empty interior in $\mathbb{E}^3$), the union of its faces is homeomorphic to the 2-sphere. Thus, $\mathcal{S}$ as an abstract simplicial complex is homeomorphic to the 2-sphere. On the other hand, bd Q is a geometric realization of $\mathcal{S}$. Thus, we have obtained that bd Q is a geometric simplicial complex which is homeomorphic to the 2-sphere. It follows that Q is homeomorphic to the 3-ball. So, we have that Q is a polyhedron (the point being: it is topologically nice, that is, its boundary is a surface, as required by the definition of a polyhedron in Sect. 4).

Clearly, Q is a weakly convex polyhedron as C is in convex position. Furthermore, Q is weakly co-decomposable, as $\mathcal{D}^t$ is a sub-polyhedral complex of $\mathcal{D}$ (by Lemma 5.2), which is a family of convex polyhedra the union of which is $[Q] = [C]$.

So far, we have proved that Q is a weakly co-decomposable polyhedron with triangular faces in $\mathbb{E}^3$. By Theorem 4.1, Q is infinitesimally rigid. As bd Q itself is a geometric simplicial complex, its edge graph is rigid because infinitesimal rigidity implies rigidity (for more details on this, see [5]). Finally, we recall that the edges of the polyhedron Q correspond to the edges of the ball-polyhedron P, and the lengths of

the edges of Q determine (via a one-to-one mapping) the corresponding inner dihedral angles of P. It follows that P is locally rigid with respect to its inner dihedral angles.

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