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SEMIREGULAR MORPHISMS

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If M and N are modules, the concept of semiregularity (and regularity) of hom(M, N) is defined and studied, and the connection with the relative direct injective- and direct projective-properties is established. The relationship of semiregularity to the Jacobson radical of hom(M, N), to the singular and cosingular ideals of hom(M, N), and to the notion of lying over or under a direct summand, is described, and the basic results in the module case are extended.

Key Words: Jacobson radical; Lying over and under; Regular morphism; Semiregular morphisms.

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Throughout the article, R is an associative ring with identity, and by the term *regular ring* we mean regular in the sense of von Neumann. All modules are unitary left R-modules, and we write homomorphisms of modules on the right of their arguments. For a module M, we write $J(M)$ for the Jacobson radical of M, and we denote $J(R) = J$. For a submodule X of M, we use $X \subseteq^{\oplus} M$ to mean that X is a direct summand of M, and we write $X \trianglelefteq M$ and $X \ll M$ to indicate that X is an essential, respectively small, submodule of M. If ${}_RM$ and ${}_RN$ are modules, we use the following notation: $E_M = end_R(M)$, $U_M = aut_R(M)$, and $[M, N] = hom_R(M, N)$.

1. DIRECT *M*-INJECTIVE AND DIRECT *N*-PROJECTIVE MODULES

Following Hausen (1981) and Nicholson (1976), a module ${}_RM$ is called a *direct injective module* if every submodule K that is isomorphic to a direct summand of M is itself a direct summand. Dually, M is called a *direct projective module* if whenever a factor module M/K is isomorphic to a summand of M then K is a summand of M.

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These modules are studied in Mohamed and Müller (1990), where they are called *C2-modules* and *D2-modules*, respectively. The following "relative" versions of these concepts will be used frequently below. Let ${}_RM$ and ${}_RN$ be modules.

(1) *M is called a* direct N-injective *module if $K \cong P \subseteq^{\oplus} M$ with $K \subseteq N$ implies that $K \subseteq^{\oplus} N$.*
(2) *N is called a* direct M-projective *module if $M/K \cong P \subseteq^{\oplus} N$ implies that $K \subseteq^{\oplus} M$.*

We remark that a module M is direct injective if and only if M is a direct M-injective module, and that a module M is direct projective if and only if M is a direct M-projective. The following two lemmas are needed.

Lemma 1. *The following are equivalent for modules ${}_RM$ and ${}_RN$:*

(1) *M is a direct N-injective module; that is $K \cong P \subseteq^{\oplus} M$, $K \subseteq N$, implies $K \subseteq^{\oplus} N$.*
(2) *If $P \subseteq^{\oplus} M$ and $Q \subseteq^{\oplus} N$, then every monomorphism $P \overset{\alpha}{\to} Q$ splits.*
(3) *If $P \subseteq^{\oplus} M$, then every monomorphism $P \overset{\alpha}{\to} N$ splits.*
(4) *If $P \overset{\alpha}{\to} N$ is monic, $P \subseteq^{\oplus} M$, and $P \overset{\mu}{\to} M$, there exists $\beta : N \to M$ with $\alpha\beta = \mu$.*

$$\begin{array}{ccccc} 0 & \to & P & \overset{\alpha}{\to} & N \\ & & {\scriptstyle\mu}\downarrow & \swarrow{\scriptstyle\beta} & \\ & & M & & \end{array}$$

(5) *If $P \overset{\alpha}{\to} N$ is monic, $P \subseteq^{\oplus} M$, and $P \overset{\iota}{\to} M$ is the inclusion, there exists $\beta : N \to M$ with $\alpha\beta = \iota$.*

Proof. Note that (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are clear.

(1) $\Rightarrow$ (2) If α is as in (2), we have $P\alpha \cong P \subseteq M$ so $P\alpha \subseteq^{\oplus} N$ by (1). Hence $P\alpha \subseteq^{\oplus} Q$.

(3) $\Rightarrow$ (4) Given the situation in (4), $P \overset{\alpha}{\to} N$ splits by (3) so let $\psi : N \to P$ satisfy $\alpha\psi = 1_P$. If β is defined by $\beta = \psi\mu$, then $\alpha\beta = \alpha\psi\mu = 1_P\mu = \mu$.

(5) $\Rightarrow$ (1) Let $P \overset{\sigma}{\to} K \subseteq N$ be an isomorphism, $P \subseteq^{\oplus} M$. Then $P \overset{\sigma}{\to} N$ is monic so, by (5), let $\beta : N \to M$ satisfy $\sigma\beta = \iota$. If $\pi : M \to P$ is a projection, define $\psi = \beta\pi : N \to P$. Then $\sigma\psi = \iota\pi = 1_P$, so the monomorphism $\sigma : P \to N$ splits. This means $K = P\sigma \subseteq^{\oplus} N$. □

Note that $Q \subseteq^{\oplus} N$ in condition (2) can be replaced by $Q \subseteq N$ (but see condition (2) in the "dual" Lemma 2 below). Dually, we have

Lemma 2. *The following are equivalent for a module ${}_RM$:*

(1) *N is a direct M-projective module; that is $M/K \cong P \subseteq^{\oplus} N$ implies that $K \subseteq^{\oplus} M$.*
(2) *If $P \subseteq^{\oplus} N$ and $Q \subseteq^{\oplus} M$, then every epimorphism $Q \overset{\alpha}{\to} P$ splits.*
(3) *If $P \subseteq^{\oplus} N$, then every epimorphism $M \overset{\alpha}{\to} P$ splits.*

(4) *If* $M \overset{\alpha}{\to} P$ *is epic,* $P \subseteq^{\oplus} N$*, and* $N \overset{\lambda}{\to} P$*, there exists* $N \overset{\beta}{\to} M$ *with* $\beta\alpha = \lambda$*.*

$$\begin{array}{ccc} & & N \\ & \overset{\beta}{\swarrow} & \downarrow\lambda \\ M & \overset{\alpha}{\to} & P \to 0 \end{array}$$

(5) *If* $M \overset{\alpha}{\to} P$ *is epic,* $P \subseteq^{\oplus} N$*, and* $N \overset{\pi}{\to} P$ *is any projection, there exists* $\beta : N \to M$ *with* $\beta\alpha = \pi$*.*

Proof. As before, (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are clear.

(1) $\Rightarrow$ (2) We have $M = Q \oplus Q'$ so $M/(ker(\alpha) \oplus Q') \cong P$. Hence $ker(\alpha) \oplus Q' \subseteq^{\oplus} M$ by (1). Thus $ker(\alpha) \subseteq^{\oplus} M$, and so $ker(\alpha) \subseteq^{\oplus} Q$.

(3) $\Rightarrow$ (4) Given the situation in (4), $M \overset{\alpha}{\to} P$ splits by (3) so let $\psi : P \to M$ satisfy $\psi\alpha = 1_P$. If β is defined by $\beta = \lambda\psi$, then $\beta\alpha = \lambda\psi\alpha = \lambda 1_P = \lambda$.

(5) $\Rightarrow$ (1) Let $M \overset{\phi}{\to} M/K \overset{\sigma}{\to} P \subseteq^{\oplus} N$, where ϕ is the coset map and σ is an isomorphism. Then $M \overset{\phi\sigma}{\to} P$ is epic so, if $\pi : N \to P$ is a projection, (5) gives $\beta : N \to M$ such that $\beta\phi\sigma = \pi$. Then $(\beta|_P)\phi\sigma = 1_P$ so the epimorphism $M \overset{\phi\sigma}{\to} P$ splits. Hence $K = ker(\phi) = ker(\phi\sigma)$ is a summand of M. □

2. REGULAR MORPHISMS

Recall that we will use the following notation where ${}_RM$ and ${}_RN$ are modules:

$$E_M = end_R(M)$$
$$U_M = aut_R(M),$$
$$[M, N] = hom_R(M, N).$$

In particular, $E_M = [M, M]$. A morphism $\alpha \in [M, N]$ is called *regular* if there exists $\beta \in [N, M]$ such that $\alpha\beta\alpha = \alpha$ (see Kasch and Mader, 2004). Moreover, we may assume that $\beta\alpha\beta = \beta$ also holds (pass to $\beta' = \beta\alpha\beta$). We call $[M, N]$ *regular* if every $\alpha \in [M, N]$ is regular. Thus E_M is a regular ring if and only if $[M, M]$ is regular. Zelmanowitz (1972) calls a module ${}_RN$ *regular* if, for any $x \in N$, there exists $\beta \in [N, R]$ such that $x = (x\beta)x$. Our definition encompasses this in that ${}_RN$ is regular in the sense of Zelmanowitz if and only if $[R, N]$ is regular in our sense.

Lemma 3. *Let* $\alpha \in [M, N]$ *be regular, say* $\alpha = \alpha\beta\alpha$*, where* $\beta \in [N, M]$*. Then:*

(1) $M = ker(\alpha) \oplus M\phi$ *and* $ker(\alpha) = ker(\phi)$*, where* $\phi^2 = \phi = \alpha\beta \in E_M$*.*
(2) $N = M\alpha \oplus ker(\varepsilon)$ *and* $M\alpha = N\varepsilon$*, where* $\varepsilon^2 = \varepsilon = \beta\alpha \in E_N$*.*

Proof. Clearly, $\phi^2 = \phi$ and $\varepsilon^2 = \varepsilon$. If $x \in M$ then $x - x\phi \in ker(\phi)$, so $M = ker(\phi) \oplus M\phi$. But $ker(\phi) = ker(\alpha)$ because $\phi = \alpha\beta$ and $\alpha = \phi\alpha$, so (1) is proven. Similarly, $N = N\varepsilon \oplus ker(\varepsilon)$, and $N\varepsilon = M\alpha$, because $\varepsilon = \beta\alpha$ and $\alpha = \alpha\varepsilon$. This proves (2). □

Theorem 4. *The following are equivalent for modules $_RM$ and $_RN$:*

(1) $[M, N]$ *is regular.*
(2) $M\alpha \subseteq^{\oplus} N$ *for every* $\alpha \in [M, N]$, *and N is direct M-projective.*
(3) $\sum_{i=1}^{k} M\alpha_i \subseteq^{\oplus} N$ *for any finite set* $\{\alpha_1, \ldots, \alpha_k\} \subseteq [M, N]$, *and N is direct M-projective.*
(4) $ker(\alpha) \subseteq^{\oplus} M$ *for every* $\alpha \in [M, N]$, *and M is direct N-injective.*
(5) $\bigcap_{i=1}^{k} ker(\alpha_i) \subseteq^{\oplus} M$ *for any finite set* $\{\alpha_1, \ldots, \alpha_k\} \subseteq [M, N]$, *and M is direct N-injective.*
(6) $M\alpha \subseteq^{\oplus} N$ *and* $ker(\alpha) \subseteq^{\oplus} M$ *for every* $\alpha \in [M, N]$.

Proof. (1) $\Rightarrow$ (2) If $\alpha \in [M, N]$, then $M\alpha \subseteq^{\oplus} N$ by Lemma 3. To show that N is direct M-projective, we verify (5) of Lemma 2. So let $P \subseteq^{\oplus} N$ with projection $\pi : N \to P$, and let $M \overset{\alpha}{\to} P$ be epic. By (1), let $\gamma \in [N, M]$ satisfy $\alpha = \alpha\gamma\alpha$. If $\varepsilon = \gamma\alpha$, then $\varepsilon^2 = \varepsilon \in E_N$ and $M\alpha = N\varepsilon$ by Lemma 3. Thus $N\varepsilon = P = N\pi$ and so $\pi\varepsilon = \pi$. Let $\beta = \pi(\gamma|_P) \in [N, M]$. Then $\beta\alpha = \pi\gamma\alpha = \pi\varepsilon = \pi$, as required.

(2) $\Rightarrow$ (3) We prove (3) by induction on k; the case $k = 1$ being (2). Assume that $k > 1$ and $\sum_{i=1}^{k-1} M\alpha_i = N\varepsilon$, where $\varepsilon^2 = \varepsilon \in E_N$. By (2), there exists $\tau^2 = \tau \in E_N$ such that $M\alpha_k(1_N - \varepsilon) = N\tau$. Then $\tau\varepsilon = 0$ so, if $\gamma = \varepsilon + \tau - \varepsilon\tau$, we have $\gamma^2 = \gamma \in E_N$. Thus $\sum_{i=1}^{k} M\alpha_i = N\varepsilon + M\alpha_k = N\varepsilon + M\alpha_k(1_N - \varepsilon) = N\varepsilon + N\tau = N\gamma$, proving (3).

(3) $\Rightarrow$ (4) Let $\alpha \in [M, N]$. Then $M/ker(\alpha) \cong M\alpha \subseteq^{\oplus} N$ by (3), so $ker(\alpha) \subseteq^{\oplus} M$ because N is direct M-projective. To see that M is direct N-injective, let $K \cong P \subseteq^{\oplus} M$, where $K \subseteq N$. If $M \overset{\pi}{\to} P \overset{\sigma}{\to} K$, where π is a projection and σ is an isomorphism, then $K = M\pi\sigma$ so $K \subseteq^{\oplus} N$ by (3), as required.

(4) $\Rightarrow$ (5) We prove (5) by induction on k; the case $k = 1$ being (4). Assume that $k > 1$ and that $X = \bigcap_{i=1}^{k-1} ker(\alpha_i) \subseteq^{\oplus} M$, say $M = X \oplus X'$. If $M \overset{\pi}{\to} X$ is the projection, then $\pi\alpha_k \in [M, N]$ and $ker(\pi\alpha_k) = [X \cap ker(\alpha_k)] \oplus X'$. But $ker(\pi\alpha_k) \subseteq^{\oplus} M$ by (4), so $X \cap ker(\alpha_k) \subseteq^{\oplus} M$. Since $X \cap ker(\alpha_k) = \bigcap_{i=1}^{k} ker(\alpha_i)$, this proves (5).

(5) $\Rightarrow$ (6) If $\alpha \in [M, N]$, then $ker(\alpha) \subseteq^{\oplus} M$ by (5). If $M = ker(\alpha) \oplus P$ then $M\alpha \cong P \subseteq^{\oplus} M$, so $M\alpha \subseteq^{\oplus} N$ because M is direct N-injective.

(6) $\Rightarrow$ (1) Given $M \overset{\alpha}{\to} N$, let $N = Y \oplus M\alpha$ and $M = ker(\alpha) \oplus X$ by (6). Thus $M\alpha = X\alpha$ and so we define $\beta : N \to M$ by $(y + x\alpha)\beta = x$ for $x \in X$ and $y \in Y$. Then β is well defined and, if $k \in ker(\alpha)$ and $x \in X$, we have $(k + x)\alpha\beta\alpha = [x(\alpha\beta)]\alpha = x\alpha = (k + x)\alpha$. Since $M = ker(\alpha) \oplus X$, this gives $\alpha\beta\alpha = \alpha$, proving (1). □

Taking $N = M$ in Theorem 4 gives

Corollary 5. *The following are equivalent for a module M:*

(1) E_M *is a regular ring.*
(2) $M\alpha \subseteq^{\oplus} M$ *for every* $\alpha \in E_M$ *and M is direct projective.*
(3) $\sum_{i=1}^{k} M\alpha_i \subseteq^{\oplus} M$ *for any finite set* $\{\alpha_1, \ldots, \alpha_k\} \subseteq E_M$ *and M is direct projective.*
(4) $ker(\alpha) \subseteq^{\oplus} M$ *for every* $\alpha \in E_M$ *and M is direct injective.*
(5) $\bigcap_{i=1}^{k} ker(\alpha_i) \subseteq^{\oplus} M$ *for any finite set* $\{\alpha_1, \ldots, \alpha_k\} \subseteq E_M$ *and M is direct injective.*
(6) $M\alpha \subseteq^{\oplus} M$ *and* $ker(\alpha) \subseteq^{\oplus} M$ *for every* $\alpha \in E_M$.

The equivalence (1) $\Leftrightarrow$ (6) is due to Azumaya (1960) (see Wisbauer, 1991, 37.7). If $M = R$ is a ring, (1) $\Leftrightarrow$ (4) asserts that a ring R is regular if and only if it is both left PP and left direct injective, where a ring is called a left PP ring if every principal left ideal is projective.

The equivalence (1) $\Leftrightarrow$ (5) in the next result is due to Zelmanowitz (1972). The left annihilator of an element x is denoted $\mathtt{l}(x)$.

Corollary 6. *The following are equivalent for a module N:*

(1) *N is a regular module.*
(2) *$Rx \subseteq^{\oplus} N$ for every $x \in N$ and N is direct ${}_RR$-projective.*
(3) *$\mathtt{l}(x) \subseteq^{\oplus} {}_RR$ for every $x \in N$ and ${}_RR$ is direct N-injective.*
(4) *$Rx \subseteq^{\oplus} N$ and $\mathtt{l}(x) \subseteq^{\oplus} {}_RR$ for every $x \in N$.*
(5) *Rx is projective, and is a direct summand of N for every $x \in N$.*

Corollary 7. *The following hold for modules M and N:*

(1) *M is injective and semisimple if and only if $[M, N]$ is regular for all N.*
(2) *N is projective and semisimple if and only if $[M, N]$ is regular for all M.*
(3) *M is semisimple if and only if $[M, N]$ is regular for all factor modules N of M.*
(4) *N is semisimple if and only if $[M, N]$ is regular for all submodules M of N.*

Proof. (1) Condition (4) of Lemma 1 shows that M is injective if and only if N is direct M-injective for all N. Hence the forward implication follows from (4) of Theorem 4, and the converse by (2) and (4) of Theorem 4.

(2) Condition (4) of Lemma 2 shows that N is projective if and only if M is direct N-projective for all M. Hence the forward implication follows from (2) of Theorem 4, and the converse by (2) and (4) of Theorem 4.

(3) and (4) By Theorem 4. □

3. THE JACOBSON RADICAL

We are going to extend the notion of the Jacobson radical of a ring to $[M, N]$. The following elementary result will be needed.

Lemma 8. *If $\alpha \in [M, N]$ and $\beta \in [N, M]$, then $1_M - \alpha\beta$ has a left (respectively right) inverse in E_M if and only if the same is true of $1_N - \beta\alpha$ in E_N.*

Proof. If $\omega(1_M - \alpha\beta) = 1_M$, we have

$$\begin{aligned}(1_N + \beta\omega\alpha)(1_N - \beta\alpha) &= (1_N - \beta\alpha) + \beta\omega\alpha(1_N - \beta\alpha)\\ &= (1_N - \beta\alpha) + \beta\omega(1_M - \alpha\beta)\alpha = 1_N.\end{aligned}$$

The proof for right inverses is similar. □

Using Lemma 8, we can see that the two sets in the following definition are equal:

$$J[M, N] = \{\alpha \in [M, N] \mid 1_M - \alpha\beta \in U_M \text{ for all } \beta \in [N, M]\}$$
$$= \{\alpha \in [M, N] \mid 1_N - \beta\alpha \in U_N \text{ for all } \beta \in [N, M]\}.$$

We call $J[M, N]$ the *Jacobson radical* of $[M, N]$ (see Kasch and Mader, 2004). Thus $J[R, R] = J(R)$ is the usual Jacobson radical of the ring R.

Lemma 9. *The following hold for modules M and N:*

(1) *$J[M, N]$ is a sub-bimodule of the bimodule ${}_{E_M}[M, N]_{E_N}$.*
(2) *$[A, M]\, J[M, N]\, [N, B] \subseteq J[A, B]$ for all modules A and B.*
(3) *If $Q \subseteq N$ then $J[M, Q] \subseteq J[M, N]$.*
(4) *If $M = P \oplus P'$ and $\alpha \in J[P, N]$, extend α to $\bar{\alpha} \in [M, N]$ by taking $P'\bar{\alpha} = 0$. Then $\bar{\alpha} \in J[M, N]$.*

Proof. (1) By Lemma 8, we only need to show that $J[M, N]$ is closed under addition. Let $\alpha_1, \alpha_2 \in J[M, N]$ and $\beta \in [N, M]$. Then $1_M - (\alpha_1 + \alpha_2)\beta = \sigma - \alpha_2\beta$, where $\sigma = 1_M - \alpha_1\beta \in U_M$. Hence $1_M - (\alpha_1 + \alpha_2)\beta = (1_M - \alpha_2\beta\sigma^{-1})\sigma \in U_M$, as required.

(2) Let $\alpha \in J[M, N]$. If $\lambda \in [A, M]$ and $\mu \in [N, B]$ we must show that $\lambda\alpha\mu \in J[A, B]$; that is $1_A - (\lambda\alpha\mu)\omega \in U_A$ for all $\omega \in [B, A]$; that is $1_A - \lambda(\alpha\mu\omega) \in U_A$ for all ω. By Lemma 8, it suffices to show that $1_M - (\alpha\mu\omega)\lambda \in U_M$ for all ω. But this follows because $\alpha \in J[M, N]$ and $\mu\omega\lambda \in [N, M]$ for all ω.

(3) If $\alpha \in J[M, Q]$ and $\beta \in [N, M]$, then $\beta|_Q \in [Q, N]$ so $1_M - \alpha(\beta|_Q) \in U_M$. But $\alpha(\beta|_Q) = \alpha\beta$.

(4) Given the situation in (4), let $\pi : M \to P$ be a projection. If $\beta \in [N, M]$, we have $\beta\pi \in [N, P]$ so $1_N - \beta\pi\alpha \in U_N$ by hypothesis. Since $\beta\pi\alpha = \beta\bar{\alpha}$, this proves (4). □

If $M = \bigoplus_{i=1}^{s} M_i$ and $N = \bigoplus_{j=1}^{t} N_j$ are modules, then (using the canonical injections and projections) $[M, N]$ has a natural matrix representation as

$$[M, N] = \begin{bmatrix} [M_1, N_1] & [M_1, N_2] & \cdots & [M_1, N_t] \\ [M_2, N_1] & [M_2, N_2] & \cdots & [M_2, N_t] \\ \vdots & \vdots & & \vdots \\ [M_s, N_1] & [M_s, N_2] & \cdots & [M_s, N_t] \end{bmatrix} = [[M_i, N_j]],$$

where the elements of M and N are written as rows, and the matrix $[[M_i, N_j]]$ acts by right matrix multiplication.

Theorem 10. *If $M = \bigoplus_{i=1}^{s} M_i$ and $N = \bigoplus_{j=1}^{t} N_j$ are modules, then $J[M, N] = [J[M_i, N_j]]$.*

Proof. Let $\alpha = [\alpha_{ij}] \in J[[M_i, N_j]]$; we show that $\alpha_{kl} \in J[M_k, N_l]$ for any $1 \le k \le s$ and $1 \le l \le t$. Given $\beta \in [N_l, M_k]$, we must show that $1_{M_k} - \alpha_{kl}\beta$ is a unit in E_{M_k}. Write $E_{kl}(\beta)$ for the $t \times s$ matrix with (l, k)-entry β and all other entries zero. Then $1_M - \alpha\beta$ is the $s \times s$ block triangular matrix $\left[\begin{smallmatrix} A & X \\ 0 & B \end{smallmatrix}\right]$ where A and B are triangular with identity maps on the diagonal, except that the $(1, 1)$-entry of B is $1_{M_k} - \alpha_{kl}\beta$. Since $1_M - \alpha\beta$ is invertible (by hypothesis), it follows that $1_{M_k} - \alpha_{kl}\beta$ is also invertible. Hence $\alpha_{kl} \in J[M_k, N_l]$.

Conversely, let $\alpha = [\alpha_{ij}] \in J[M_i, N_j]$; we must show that $\alpha \in J[M, N]$. We may assume that the only nonzero entry of α is $\alpha_{kl} \neq 0$. Then if $\beta = [\beta_{ij}] \in [N, M]$ is arbitrary, $1_M - \alpha\beta$ is again a block triangular matrix $\left[\begin{smallmatrix} A & X \\ 0 & B \end{smallmatrix}\right]$ where A and B are triangular with identity maps on the diagonal, except that the $(1, 1)$-entry of B is $1_{M_k} - \alpha_{kl}\beta_{lk}$. It follows that $1_M - \alpha\beta$ is invertible, so $\alpha \in J[[M_i, N_j]]$. □

We need a special case of Theorem 10 later. If e and f are idempotents in a ring R, we identify $[Re, Rf] = eRf$ where elements of eRf act by right multiplication. Thus

$$\begin{aligned} J(eRf) &= \{a \in eRf \mid e - ab \in U(eRe) \text{ for all } b \in fRe\} \\ &= \{a \in eRf \mid f - ba \in U(fRf) \text{ for all } b \in fRe\}. \end{aligned}$$

Note that $eJ(R)f \subseteq J(eRf)$. Indeed, writing $J(R) = J$, if $a \in eJf$ and $b \in fRe$, then $ab \in eJfRe \subseteq eJe = J(eRe)$. We prove that $eJf = J(eRf)$ if $ef = 0$ or $fe = 0$.

Lemma 11. *Let e, f and g be idempotents in R with $e, f \in S = gRg$. If $J(eSf) \subseteq J(S)$, then $J(eRf) \subseteq J(R)$.*

Proof. We have $J(S) = gJ(R)g \subseteq J(R)$, so we show that $J(eRf) \subseteq J(eSf)$. If $a \in J(eRf)$, then $a \in eRf = (eg)R(gf) = eSf$. Hence, if $b \in fSe$ we must show that $e - ab \in U(eSe)$. But $b \in fRe$ so $e - ab \in U(eRe)$ because $a \in J(eRf)$. If v is the inverse of $e - ab$ in eRe, then $v = eve = (eg)v(ge) \in eSe$, as required. □

Theorem 12. *Let $e, f \in R$ be idempotents. If $ef = 0$, then $J(eRf) = eJ(R)f$ and $J(fRe) = fJ(R)e$.*

Proof. We must show that $J(eRf) \subseteq J(R)$ and $J(fRe) \subseteq J(R)$. Since $ef = 0$, $g = e + f - fe$ is an idempotent and $e, f \in S = gRg$. Hence, by Lemma 11, it suffices to show that $J(eSf) \subseteq J(S)$ and $J(fSe) \subseteq J(S)$. Write $h = f - fe$, so that $h^2 = h$, $g = e + h$, and $eh = 0 = he$. Then $s \mapsto \left[\begin{smallmatrix} ese & esh \\ hse & hsh \end{smallmatrix}\right]$ defines a ring isomorphism $S \to \left[\begin{smallmatrix} eSe & eSh \\ hSe & hSh \end{smallmatrix}\right]$, where $e \mapsto \left[\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix}\right]$, and $h \mapsto \left[\begin{smallmatrix} 0 & 0 \\ 0 & h \end{smallmatrix}\right]$ and $f \mapsto \left[\begin{smallmatrix} 0 & 0 \\ fe & h \end{smallmatrix}\right]$. (Note that $h(fe) = fe$ because $hf = f$.)

Thus we may assume that $S = \left[\begin{smallmatrix} A & V \\ W & B \end{smallmatrix}\right]$, $e = \left[\begin{smallmatrix} 1_A & 0 \\ 0 & 0 \end{smallmatrix}\right]$, and $f = \left[\begin{smallmatrix} 0 & 0 \\ w_0 & 1_B \end{smallmatrix}\right]$. Then $eSe = \left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $eSf = \left\{\left[\begin{smallmatrix} vw_0 & v \\ 0 & 0 \end{smallmatrix}\right] \mid v \in V\right\}$, and $fSe = \left\{\left[\begin{smallmatrix} 0 & 0 \\ w_0a+w & 0 \end{smallmatrix}\right] \mid a \in A \text{ and } w \in W\right\}$.

To see that $J(eSf) \subseteq J(S)$, let $x = \left[\begin{smallmatrix} v_0w_0 & v_0 \\ 0 & 0 \end{smallmatrix}\right] \in J(eSf)$ and $y = \left[\begin{smallmatrix} a & v \\ w & b \end{smallmatrix}\right] \in S$. We must show that $1_S - xy = \left[\begin{smallmatrix} 1_A - v_0(w_0a+w) & -v_0(w_0v+b) \\ 0 & 1_B \end{smallmatrix}\right]$ is a unit in S, that is $1_A - v_0(w_0a + w)$ is a unit in A. But if we write $z = \left[\begin{smallmatrix} 0 & 0 \\ w_0a+w & 0 \end{smallmatrix}\right] \in fSe$, then $\left[\begin{smallmatrix} 1_A - v_0(w_0a+w) & 0 \\ 0 & 0 \end{smallmatrix}\right] = e - xz \in U(eSe) = \left[\begin{smallmatrix} U(A) & 0 \\ 0 & 0 \end{smallmatrix}\right]$ because $x \in J(eSf)$. This proves that $J(eSf) \subseteq J(S)$.

Now let $p = \left[\begin{smallmatrix} 0 & 0 \\ w_0a_0+w_1 & 0 \end{smallmatrix}\right] \in J(fSe)$. Given $y = \left[\begin{smallmatrix} a & v \\ w & b \end{smallmatrix}\right] \in S$, we must show that $1_S - yp = \left[\begin{smallmatrix} 1_A - v(w_0a_0+w_1) & 0 \\ -b(w_0a_0+w_1) & 1_B \end{smallmatrix}\right] \in U(S)$; that is $1_A - v(w_0a_0 + w_1) \in U(A)$. To this end, let $q = \left[\begin{smallmatrix} vw_0 & v \\ 0 & 0 \end{smallmatrix}\right] \in eSf$. Then $\left[\begin{smallmatrix} 1_A - v(w_0a_0+w_1) & 0 \\ 0 & 0 \end{smallmatrix}\right] = e - qp \in U(eSe) = \left[\begin{smallmatrix} U(A) & 0 \\ 0 & 0 \end{smallmatrix}\right]$ because $p \in J(fSe)$, as required. This shows that $J(fSe) \subseteq J(S)$. □

There is a large class of rings R for which $J(eRf) = eJ(R)f$ for all idempotents $e, f \in R$. Call a ring R *semipotent* if every right (respectively left) ideal not contained in $J(R)$ contains a nonzero idempotent; equivalently, if $a \notin J(R)$ implies $cac = c$ for some $c \neq 0$ in R.

Theorem 13. *If R is semipotent, then $J(eRf) = eJ(R)f$ for all idempotents e, f in R.*

Proof. Let $a \in J(eRf)$; we must show that $a \in J(R)$. If $a \notin J(R)$ let $cac = c \neq 0$ by hypothesis. If $b = fce$, then $bab = fc(eaf)ce = fcace = b$. Moreover, $e - ab \in U(eRe)$ because $e \in J(eRf)$. Hence $ab = 0$, because ab is an idempotent in eRe (since $bab = b$). But then $c = cacac = c(af)c(ea)c = cabac = 0$, a contradiction. □

We do not have an example of a ring R and idempotents $e, f \in R$ with $J(eRf) \nsubseteq J(R)$.

4. SEMIREGULAR MORPHISMS

In this section, we define the concept of semiregularity for $[M, N]$ where ${}_RM$ and ${}_RN$ are modules, and extend many ring theoretic results. A ring R is called *semiregular* (Nicholson, 1976) if, for every $a \in R$, there exists $b \in R$ such that $bab = b$ and $a - aba \in J(R)$. Accordingly, we call a morphism $\alpha \in [M, N]$ *semiregular* if there exists $\beta \in [N, M]$ such that

$$\beta = \beta\alpha\beta \qquad \text{and} \qquad \alpha - \alpha\beta\alpha \in J[M, N].$$

We say that $[M, N]$ is *semiregular* if every $\alpha \in [M, N]$ is semiregular.

Thus E_N is a semiregular ring if and only if $[N, N]$ is semiregular. A module N is said to be *semiregular* (Nicholson, 1976) if for every $x \in N$ there exists $\beta \in [N, R]$ such that $(x\beta)^2 = x\beta$ and $x - (x\beta)x \in J(N)$. Thus, as in the regular case, a module N is semiregular if and only if $[R, N]$ is semiregular.

Lemma 14. *The following are equivalent for $\alpha \in [M, N]$:*

(1) *α is semiregular;*
(2) *There exists $\beta \in [N, M]$ such that $(\alpha\beta)^2 = \alpha\beta$ in E_M and $\alpha - \alpha\beta\alpha \in J[M, N]$;*
(3) *There exists a regular element $\gamma \in [M, N]$ such that $\alpha - \gamma \in J[M, N]$;*
(4) *There exists $\gamma^2 = \gamma \in \alpha[N, M] \subseteq E_M$ such that $\alpha - \gamma\alpha \in J[M, N]$.*

Proof. (1) ⇒ (2) If $\beta \in [N, M]$ satisfies $\beta = \beta\alpha\beta$ and $\alpha - \alpha\beta\alpha \in J[M, N]$, then $(\alpha\beta)^2 = \alpha\beta$.

(2) ⇒ (3) If $\beta \in [N, M]$ satisfies $(\alpha\beta)^2 = \alpha\beta$ and $\alpha - \alpha\beta\alpha \in J[M, N]$, and if $\gamma = \alpha\beta\alpha$, then $\gamma \in [M, N]$, $\alpha - \gamma \in J[M, N]$, and $\gamma\beta\gamma = \alpha\beta\alpha\beta\alpha\beta\alpha = \alpha\beta\alpha = \gamma$.

(3) $\Rightarrow$ (4) By (3) choose $\delta \in [M, N]$ such that $\alpha - \delta \in J[M, N]$ and $\delta\beta\delta = \delta$ where $\beta \in [N, M]$. Write $\varepsilon = \delta\beta$ so that $\varepsilon^2 = \varepsilon \in E_M$ and $\varepsilon\delta = \delta$.

Claim. $\varepsilon\alpha$ is regular and $\alpha - \varepsilon\alpha \in J[M, N]$.

Proof. We have $\alpha - \varepsilon\alpha = (1_M - \varepsilon)\alpha = (1_M - \varepsilon)(\alpha - \delta) \in J[M, N]$ by Lemma 9. Similarly, $\varepsilon - \alpha\beta = (\delta - \alpha)\beta \in J[M, M] = J(E_M)$, so there exists $\phi \in E_M$ such that $(1_M - \varepsilon + \alpha\beta)\phi = 1_M$. Hence $\varepsilon\alpha\beta\phi = \varepsilon$, whence $(\varepsilon\alpha)(\beta\phi)(\varepsilon\alpha) = \varepsilon\alpha$. This proves the Claim.

Write $\eta = \varepsilon\alpha$, choose $\mu \in [N, M]$ such that $\eta = \eta\mu\eta$ and $\mu\eta\mu = \mu$, and define $\gamma = \alpha\mu\varepsilon$. Then $\gamma^2 = \gamma \in \alpha[N, M]$ and $\alpha - \gamma\alpha = \alpha - \alpha\mu\eta = (\alpha - \eta)(1_N - \mu\eta) \in J[M, N]$ by the Claim. This proves (4).

(4) $\Rightarrow$ (1) Choose $\gamma = \gamma^2$ as in (4), write $\gamma = \alpha\rho$, where $\rho \in [N, M]$, and let $\beta = \rho\gamma$. Then $\beta \in [N, M]$ and $\alpha - \alpha\beta\alpha = \alpha - \gamma\alpha \in J[M, N]$ and $\beta\alpha\beta = \rho\gamma = \beta$. □

Corollary 15. *$[M, N]$ is regular if and only if $[M, N]$ is semiregular and $J[M, N] = 0$.*

Corollary 16. *Let $\alpha, \beta \in [M, N]$ with $\alpha - \beta \in J[M, N]$. If either α or β is semiregular so also is the other.*

Proof. If β is semiregular, Lemma 14 provides a regular $\gamma \in [M, N]$ such that $\beta - \gamma \in J[M, N]$. Thus, $\alpha - \gamma = (\alpha - \beta) + (\beta - \gamma) \in J[M, N]$. So α is semiregular by Lemma 14. If α is semiregular, use the fact that $\beta - \alpha = -(\alpha - \beta) \in J[M, N]$. □

Corollary 17. *If $\alpha \in [M, N]$ is semiregular so are $\sigma\alpha$ and $\alpha\tau$, where $\sigma \in U_M$ and $\tau \in U_N$.*

Proof. Write $\beta \backsim \alpha$ to signify that $\beta \in [N, M]$ satisfies $\beta = \beta\alpha\beta$ and $\alpha - \alpha\beta\alpha \in J[M, N]$. Then $\beta\sigma^{-1} \backsim \sigma\alpha$ and $\tau^{-1}\beta \backsim \alpha\tau$ using Lemma 9. □

We are going to characterize when $[M_1 \oplus \cdots \oplus M_n, N_1 \oplus \cdots \oplus N_m]$ is semiregular, and several technical lemmas are needed. We begin with

Lemma 18. *Let $A \subseteq^{\oplus} M$ and $B \subseteq^{\oplus} N$. If $[M, N]$ is semiregular, then $[A, B]$ is semiregular.*

Proof. Let $M \overset{\pi_1}{\to} A \overset{\iota_1}{\to} M$ and $N \overset{\pi_2}{\to} B \overset{\iota_2}{\to} N$ be the canonical projections and inclusions. If $\alpha \in [A, B]$, then $\bar{\alpha} = \pi_1\alpha\iota_2 \in [M, N]$ is semiregular, so there exists $\beta \in [N, M]$ such that $\beta = \beta\bar{\alpha}\beta$ and $\bar{\alpha} - \bar{\alpha}\beta\bar{\alpha} \in J[M, N]$. Let $\hat{\beta} = \iota_2\beta\pi_1 \in [B, A]$. Then $\hat{\beta}\alpha\hat{\beta} = \iota_2\beta\bar{\alpha}\beta\pi_1 = \iota_2\beta\pi_1 = \hat{\beta}$, and $\alpha - \alpha\hat{\beta}\alpha = \iota_1\pi_1\alpha\iota_2\pi_2 - (\iota_1\pi_1\alpha)(\iota_2\beta\pi_1)(\alpha\iota_2\pi_2) = \iota_1(\bar{\alpha} - \bar{\alpha}\beta\bar{\alpha})\pi_2 \in J[A, B]$ by Lemma 9. So α is semiregular. □

Lemma 19. *Let M and N be modules with $N = N_1 \oplus N_2$. Then $[M, N]$ is semiregular if and only if $[M, N_i]$ is semiregular for $i = 1, 2$.*

Proof. One direction is by Lemma 18. Suppose that each $[M, N_i]$ is semiregular, and let $\alpha \in [M, N]$. For $i = 1, 2$, let $N \overset{\pi_i}{\to} N_i \overset{\iota_i}{\to} N$ be the canonical maps, and write

$\alpha_i = \alpha\pi_i \in [M, N_i]$. Then $\alpha = \alpha_1 \iota_1 + \alpha_2 \iota_2$. Since α_1 is semiregular, Lemma 14 provides $\phi^2 = \phi \in \alpha_1[N_1, M] \subseteq E_M$ such that $\alpha_1 - \phi\alpha_1 \in J[M, N_1]$. Similarly, since $\alpha_2 - \phi\alpha_2 \in [M, N_2]$ is semiregular, there exists $\gamma^2 = \gamma \in (\alpha_2 - \phi\alpha_2)[N_2, M] \subseteq E_M$ such that $(\alpha_2 - \phi\alpha_2) - \gamma(\alpha_2 - \phi\alpha_2) \in J[M, N_2]$. This becomes $(1_M - \eta)\alpha_2 \in J[M, N_2]$, where $\eta = \phi + \gamma - \gamma\phi \in E_M$. We have $\eta^2 = \eta$ (since $\phi\gamma = 0$) so it suffices (again by Lemma 14) to show that $\eta \in \alpha[N, M]$ and $(1_M - \eta)\alpha \in J[M, N]$. We have $(1_M - \eta) = (1_M - \gamma) \times (1_M - \phi)$ so

$$(1_M - \eta)\alpha = (1_M - \eta)(\alpha_1 \iota_1 + \alpha_2 \iota_2) = (1_M - \gamma)(1_M - \phi)\alpha_1 \iota_1 + (1_M - \eta)\alpha_2 \iota_2.$$

By Lemma 9, this shows that $(1_M - \eta)\alpha \in J[M, N]$ because $(1_M - \phi)\alpha_1 \in J[M, N_1]$ and $(1_M - \eta)\alpha_2 \in J[M, N_2]$. Finally, to see that $\eta \in \alpha[N, M]$, write $\phi = \alpha_1\beta_1$ and $\gamma = (\alpha_2 - \phi\alpha_2)\beta_2$, where $\beta_i \in [N_i, M]$ for $i = 1, 2$. Since $\alpha_i\beta_i = (\alpha\pi_i)\beta_i = \alpha(\pi_i\beta_i) \in \alpha[N, M]$, it follows that $\phi, \gamma \in \alpha[N, M]$ for each i, and hence that $\eta \in \alpha[N, M]$. □

Lemma 20. *Let $\alpha \in [M, N]$. Suppose that $\alpha - \gamma\alpha$ is semiregular for some $\gamma^2 = \gamma \in \alpha[N, M] \subseteq E_M$. Then α is semiregular.*

Proof. Since $\alpha - \gamma\alpha$ is semiregular, Lemma 14 gives $\phi^2 = \phi \in (\alpha - \gamma\alpha)[N, M] \subseteq E_M$ such that $(\alpha - \gamma\alpha) - \phi(\alpha - \gamma\alpha) \in J[M, N]$. If $\eta = \gamma + \phi - \phi\gamma$, then $\eta^2 = \eta \in \alpha[N, M]$ because $\gamma\phi = 0$ and $\gamma \in \alpha[N, M]$. Moreover, $\alpha - \eta\alpha = \alpha - (\gamma + \phi - \phi\gamma)\alpha = (\alpha - \gamma\alpha) - \phi(\alpha - \gamma\alpha) \in J[M, N]$. So α is semiregular by Lemma 14. □

Lemma 21. *Let M and N be modules with $M = M_1 \oplus M_2$. Then $[M, N]$ is semiregular if and only if $[M_i, N]$ is semiregular for $i = 1, 2$.*

Proof. One direction is by Lemma 18. Suppose that $[M_i, N]$ is semiregular for $i = 1, 2$. Let $\alpha \in [M, N]$; we prove that α is semiregular. For $i = 1, 2$, let $M \xrightarrow{\pi_i} M_i \xrightarrow{\iota_i} M$ be canonical maps, and write $\alpha_i = \iota_i\alpha \in [M_i, N]$. Then $\alpha = \pi_1\alpha_1 + \pi_2\alpha_2$. Since α_1 is semiregular, there exists $\beta_1 \in [N, M_1]$ such that $\beta_1 = \beta_1\alpha_1\beta_1$ and $\alpha_1 - \alpha_1\beta_1\alpha_1 \in J[M_1, N]$. Set $\eta = \alpha\beta_1\iota_1$. Then $\eta^2 = \eta \in \alpha[N, M] \subseteq E_M$. To prove that α is semiregular it suffices, by Lemma 20, to show that $\alpha - \eta\alpha$ is semiregular.

To this end, write $\phi = \pi_1\alpha_1\beta_1\iota_1 \in E_M$ and $\gamma = \pi_2\alpha_2\beta_1\iota_1 \in E_M$, so that $\phi + \gamma = \alpha\beta_1\iota_1 = \eta$. We have $\gamma\pi_2 = 0$ and $\phi\pi_2 = 0$ so

$$\alpha - \eta\alpha = (\pi_1\alpha_1 + \pi_2\alpha_2) - (\phi + \gamma)(\pi_1\alpha_1 + \pi_2\alpha_2) = (\pi_1\alpha_1 - \phi\pi_1\alpha_1) + (\pi_2\alpha_2 - \gamma\pi_1\alpha_1). \qquad (*)$$

The first term in $(*)$ is in $J[M, N]$ by Lemma 9 because $\alpha_1 - \alpha_1\beta_1\alpha_1 \in J[M_1, N]$. Indeed:

$$\pi_1\alpha_1 - \phi\pi_1\alpha_1 = \pi_1\alpha_1 - (\pi_1\alpha_1\beta_1\iota_1)\pi_1\alpha_1 = \pi_1(\alpha_1 - \alpha_1\beta_1\alpha_1) \in J[M, N].$$

Hence to prove that $\alpha - \eta\alpha$ is semiregular, it suffices by Corollary 16 to show that the second term $\lambda = \pi_2\alpha_2 - \gamma\pi_1\alpha_1$ in $(*)$ is semiregular. Note that $\lambda \in [M, N]$.

Since $\iota_2\lambda \in [M_2, N]$ is semiregular, there exists $\upsilon \in [N, M_2]$ such that $\upsilon = \upsilon(\iota_2\lambda)\upsilon$ and $(\iota_2\lambda) - (\iota_2\lambda)\upsilon(\iota_2\lambda) \in J[M_2, N]$. Write $\omega = \upsilon\iota_2 \in [N, M]$. Then $\omega = \omega\lambda\omega$, so it suffices to show that $\lambda - \lambda\omega\lambda \in J[M, N]$. To see this, observe first that $\pi_2\iota_2\gamma = \gamma$, so $\lambda = \pi_2\iota_2\lambda$. Hence

$$\lambda - \lambda\omega\lambda = \pi_2\iota_2\lambda - (\pi_2\iota_2\lambda)(\upsilon\iota_2)\lambda = \pi_2(\iota_2\lambda - \iota_2\lambda\upsilon\iota_2\lambda) \in J[M, N]$$

by Lemma 9, as required. So λ is semiregular, and the proof is complete. □

The main result of this section is the following theorem.

Theorem 22. *Let $M = \bigoplus_{i=1}^{n} M_i$ and $N = \bigoplus_{j=1}^{m} N_j$ be modules. Then $[M, N]$ is semiregular if and only if $[M_i, N_j]$ is semiregular for all i and j.*

Proof. The necessity is by Lemma 18, and the sufficiency is by Lemmas 19 and 21, and by induction. □

Corollary 23. *Let $M = \bigoplus_{i=1}^{n} M_i$. Then $E_M = [M, M]$ is a semiregular ring if and only if $[M_i, M_j]$ is semiregular for all $1 \leq i, j \leq n$.*

An important special case gives a "local" condition for establishing that a ring is semiregular.

Corollary 24. *Let $1 = e_1 + \cdots + e_n$ in a ring R, where $e_1, \ldots, e_n$ are orthogonal idempotents. The following are equivalent*:

(1) *R is semiregular.*
(2) *For all i and j and all $a \in e_iRe_j$, there exists $b \in e_jRe_i$ such that $bab = b$ and $a - aba \in e_iJ(R)e_j$.*

Proof. (1) ⇒ (2) If $a \in e_iRe_j$, choose $c \in R$ such that $cac = c$ and $a - aca \in J(R)$. Then (2) holds with $b = e_jce_i \in e_jRe_i$.

(2) ⇒ (1) We have $R = Re_1 \oplus \cdots \oplus Re_n$. For all i and j, identify $[Re_i, Re_j] = e_iRe_j$, where elements of e_iRe_j act by right multiplication. Then $J(e_iRe_j) = e_iJ(R)e_j$ by Theorem 12, and the rest is by Theorem 22 because $R \cong [R, R]$. □

Corollary 25. *Let M be a finitely generated module and let $N = \bigoplus_{i\in I} N_i$. Then $[M, N]$ is semiregular if and only if $[M, N_i]$ is semiregular for all $i \in I$.*

Proof. One direction is clear by Lemma 18. Suppose that $[M, N_i]$ is semiregular for all $i \in I$. If $\alpha \in [M, N]$, then $M\alpha \subseteq \bigoplus_{i\in F} N_i = P$, where F is a finite subset of I. Since $\alpha \in [M, P]$ and $[M, P]$ is semiregular by Theorem 22, there exists $\gamma \in [P, M]$ such that $\gamma\alpha\gamma = \gamma$ and $\alpha - \alpha\gamma\alpha \in J[M, P]$. Define $\beta \in [N, P]$ by $\beta|_P = \gamma$ and $(\bigoplus_{i\in I-F} N_i)\beta = 0$. Then $\beta\alpha\beta = \beta$ and $\alpha - \alpha\beta\alpha = \alpha - \alpha\gamma\alpha \in J[M, P]$. But $J[M, P] \subseteq J[M, N]$ because $P \subseteq N$, so α is semiregular. □

Letting $M = R$ in Corollary 25 yields the following corollary.

Corollary 26 (Nicholson, 1976, Theorem 1.10)**.** *Let $N = \bigoplus_{i\in I} N_i$. Then N is a semiregular module if and only if N_i is a semiregular module for all $i \in I$.*

Remark 27. In general, the fact that all E_{M_i} are semiregular for $i = 1, 2, \ldots, n$ does not imply that E_M is semiregular where $M = \bigoplus_{i=1}^n M_i$.

Proof. Ware (1971, Example 3.4) gives an example of a regular ring R and a projective module $M = P \oplus Q$ such that $E_P \cong R \cong E_Q$ but E_M is not semiregular (see Nicholson, 1976, Example 3.8). □

Remark 28. Corollary 23 does not hold in general if the direct sum is infinite.

Proof. We use the example of Ware (1971, Example 3.4) again. Let I be an infinite index set, for each $i \in I$ let $K_i = \mathbb{Z}_2$, and let $R = \Pi_{i \in I} K_i$ be the direct product of rings. Let $P = R$ and $Q = \bigoplus_{i \in I} K_i \subseteq R$ and $M = P \oplus Q$. For each $i \in I$, let $Q_i = K_i$ be the ith component of R. Then each Q_i is a simple R-module, $\bigoplus_{i \in I} Q_i \cong Q$, and $M = P \oplus Q \cong P \oplus (\bigoplus_{i \in I} Q_i)$. So E_M is not semiregular as seen in Remark 27. But $[M_1, M_2]$ is indeed semiregular for any $M_1, M_2 \in \{Q_i : i \in I\} \cup \{P\}$. To see this, denote the nonzero identity element of Q_i by 1_i.

(1) $E_P \cong R$ is clearly regular, so is semiregular.
(2) For $i \in I$, E_{Q_i} is clearly semiregular since Q_i is a simple R-module.
(3) For $i, j \in I$ with $i \neq j$, $[Q_i, Q_j] = 0$, so $[Q_i, Q_j]$ is semiregular.
(4) For $0 \neq \alpha \in [P, Q_i]$, it must be that $\alpha(1) = 1_i$. Let $\beta \in [Q_i, P]$ with $\beta(1_i) = 1_i$. Then $\beta\alpha\beta = \beta$ and $\alpha\beta\alpha = \alpha$, so α is semiregular.
(5) For $0 \neq \alpha \in [Q_i, P]$, it must be that $\alpha(1_i) = 1_i$. Let $\beta \in [P, Q_i]$ with $\beta(1) = 1_i$. Then $\beta\alpha\beta = \beta$ and $\alpha\beta\alpha = \alpha$, so α is semiregular. □

Using arguments similar to and simpler than those in the proof of Theorem 22, we can prove the following theorem.

Theorem 29. *Let $M = \bigoplus_{i=1}^n M_i$, and $N = \bigoplus_{j=1}^m N_j$. Then $[M, N]$ is regular if and only if $[M_i, N_j]$ is regular for all $1 \leq i \leq n$ and $1 \leq j \leq m$.*

Corollary 30. *Let M and N be modules.*

(1) *If $M = \bigoplus_{i=1}^n M_i$, then E_M is regular if and only if $[M_i, M_j]$ is regular for all $1 \leq i$, $j \leq n$.*
(2) *If M is finitely generated and $N = \bigoplus_{i \in I} N_i$, then $[M, N]$ is regular if and only if $[M, N_i]$ is regular for all $i \in I$.*
(3) *(Zelmanowitz, 1972, Theorem 2.8) Let $N = \bigoplus_{i \in I} N_i$. Then N is a regular module if and only if N_i is a regular module for all $i \in I$.*

Proof. (1) is immediate, the proof of (2) is similar to that of Corollary 25, and (3) then follows letting $M = R$. □

Remark 31. The module $M = \bigoplus_{i=1}^n M_i$ in Remark 27 shows that E_M need not be regular if each E_{M_i} is regular. The module M in Remark 28 shows that (1) of Corollary 30 does not hold in general if the direct sum is infinite.

5. LYING OVER AND UNDER

For a submodule N of M, we write $N \trianglelefteq M$ (respectively $N \ll M$) to mean that N is an essential (respectively a small) submodule of M. Following Nicholson (1976, Theorem 3.1), a submodule N of a module M is said to *lie over a summand* of M if there exists a direct decomposition $M = P \oplus Q$ with $P \subseteq N$ and $Q \cap N \ll M$; and N is said to *lie under a summand* of M if there exists a direct decomposition $M = P \oplus Q$ with $N \subseteq P$ and $N \trianglelefteq P$.

Following Beidar and Kasch (2001), the *singular ideal* $\Delta[M, N]$ and the *cosingular ideal* $\nabla[M, N]$ of $[M, N]$ are defined by

$$\Delta[M, N] = \{\alpha \in [M, N] \mid ker(\alpha) \trianglelefteq M\}, \quad \text{and}$$
$$\nabla[M, N] = \{\alpha \in [M, N] \mid M\alpha \ll N\},$$

respectively. The following result appears in Beidar and Kasch (2001, Theorems 2.3 and 2.4), but we include a short proof for completeness.

Lemma 32. *Let M and N be modules.*

(1) *$\Delta[M, N] \subseteq J[M, N]$ for all N if every monomorphism $M \to M$ splits.*
(2) *$\nabla[M, M] \subseteq J[M, N]$ for all M if every epimorphism $M \to M$ splits.*

Proof. (1) Let $\alpha \in \Delta[M, N]$. We have $ker(\alpha) \cap ker(1_M - \alpha\beta) = 0$ for all $\beta \in [N, M]$, so $1_M - \alpha\beta$ is monic because $ker(\alpha) \trianglelefteq M$. Hence $M(1_M - \alpha\beta) \subseteq^{\oplus} M$ by hypothesis. But $ker(\alpha) \subseteq M(1_M - \alpha\beta)$, so $1_M - \alpha\beta$ is epic again because $ker(\alpha) \trianglelefteq M$. Thus $\alpha \in J[M, N]$.

(2) Let $\alpha \in \nabla[M, N]$. We have $M\alpha + N(1_N - \beta\alpha) = N$ for all $\beta \in [N, M]$, so $1_N - \beta\alpha$ is epic because $M\alpha \ll N$. Hence $ker(1_N - \beta\alpha) \subseteq^{\oplus} M$ by hypothesis. But $ker(1_N - \beta\alpha) \subseteq M\alpha$, so $1_M - \beta\alpha$ is monic again because $M\alpha \trianglelefteq M$. Thus $\alpha \in J[M, N]$. □

The results in Lemma 32 are well-known in case $M = N$, as is the fact that $\Delta[M, M] = J(E_M)$ when M is quasi-injective, and $\nabla[M, M] = J(E_M)$ when M is quasi-projective. The next result extends Nicholson and Yousif (2001, Theorem 2.4).

Theorem 33. *If the module M is both direct M-injective and direct N-injective, the following are equivalent:*

(1) *$[M, N]$ is semiregular and $\Delta[M, N] = J[M, N]$;*
(2) *$ker(\alpha)$ lies under a direct summand of M for any $\alpha \in [M, N]$.*

Proof. (1) $\Rightarrow$ (2) Let $\alpha \in [M, N]$. By (1), there exists $\beta \in [N, M]$ such that $\beta = \beta\alpha\beta$ and $\alpha - \alpha\beta\alpha \in J[M, N] = \Delta[M, N]$. Thus $(\alpha\beta)^2 = \alpha\beta \in E_M$, so $M = M(\alpha\beta) \oplus M(1_M - \alpha\beta)$. Clearly, $ker(\alpha) \subseteq M(1_M - \alpha\beta)$, so it remains to show that $ker(\alpha) \trianglelefteq M(1_M - \alpha\beta)$. Write $K = ker(\alpha - \alpha\beta\alpha)$, so that $K \trianglelefteq M$. Then $K \cap M(1_M - \alpha\beta) \subseteq ker(\alpha)$, and $K \cap M(1_M - \alpha\beta) \trianglelefteq M(1_M - \alpha\beta)$. Hence $ker(\alpha) \trianglelefteq M(1_M - \alpha\beta)$, proving (2).

(2) $\Rightarrow$ (1) Let $\alpha \in [M, N]$. By (2) write $M = P \oplus Q$ with $ker(\alpha) \trianglelefteq P$. Then $Q \cong Q\alpha \subseteq N$ so $Q\alpha$ is a direct summand of N because M is direct N-injective. Let $\beta = (\alpha|_Q)^{-1}$. Then $Q(\alpha - \alpha\beta\alpha) = Q(\alpha - 1_Q\alpha) = 0$, whence $ker(\alpha) + Q \subseteq ker(\alpha - \alpha\beta\alpha)$. Thus $ker(\alpha - \alpha\beta\alpha) \trianglelefteq M$ so $\alpha - \alpha\beta\alpha \in \Delta[M, N] \subseteq J[M, N]$ by Lemma 32 since M is direct M-injective. Moreover, $\beta - \beta\alpha\beta = \beta - \beta 1_Q = 0$, so $\beta = \beta\alpha\beta$ and we have proven that $[M, N]$ is semiregular. Finally, if $\alpha \in J[M, N]$, then $1_M - \alpha\beta \in U_M$, so the fact that $Q(1_M - \alpha\beta) = 0$ implies that $Q = 0$. Thus $ker(\alpha) \trianglelefteq P = M$, so $\alpha \in \Delta[M, N]$. With Lemma 32, this shows that $\Delta[M, N] = J[M, N]$, and (1) follows. □

Letting $M = N$ above yields the following corollary.

Corollary 34. *Let M be a module and write $A = \{\alpha \in E_M : ker(\alpha) \trianglelefteq M\}$.*

(1) *If M is direct-injective, then $A \subseteq J(E_M)$.*
(2) *Moreover, E_M is semiregular and $A = J(E_M)$ if and only if $ker(\alpha)$ lies under a direct summand of M for each $\alpha \in E_M$.*

There is a “dual” version of Theorem 33.

Theorem 35. *If the direct projective module M is direct N-projective, the following are equivalent*:

(1) *$[M, N]$ is semiregular and $\nabla[M, N] = J[M, N]$;*
(2) *$M\alpha$ lies over a direct summand of M for any $\alpha \in [M, N]$.*

Proof. (1) $\Rightarrow$ (2) Let $\alpha \in [M, N]$. By (1), there exists $\beta \in [N, M]$ such that $\beta = \beta\alpha\beta$ and $\alpha - \alpha\beta\alpha \in J[M, N] = \nabla[M, N]$. Then $(\beta\alpha)^2 = \beta\alpha \in E_N$ and so $N = N(\beta\alpha) \oplus N(1_N - \beta\alpha)$. Clearly, $N(\beta\alpha) \subseteq M\alpha$. Moreover, $M\alpha \cap N(1_N - \beta\alpha) \subseteq M(\alpha - \alpha\beta\alpha) \ll N$ (since $\alpha - \alpha\beta\alpha \in \nabla[M, N]$) and so $M\alpha$ lies over the summand $N(\beta\alpha)$ of N.

(2) $\Rightarrow$ (1) Let $\alpha \in [M, N]$. By (2) write $N = P \oplus Q$ with $P \subseteq M\alpha$ and $Q \cap M\alpha \ll N$. Choose $\pi^2 = \pi \in E_N$ such that $P = N\pi$ and $ker(\pi) = Q$. Then $\alpha\pi : M \to P$ is an epimorphism. Since $P \subseteq^{\oplus} N$ and N is direct M-projective, $ker(\alpha\pi)$ is a summand of M, say $M = ker(\alpha\pi) \oplus Y$. Then $(\alpha\pi)|_Y : Y \to P$ is an isomorphism. Write $\sigma = [(\alpha\pi)|_Y]^{-1} : P \to Y$ and define $\beta = \pi\sigma \in [N, M]$. Then $\beta = \pi\beta$ and $1_Y = (\alpha\pi)|_Y\beta$. Thus,

$$M(\alpha - \alpha\beta\alpha) = (ker(\alpha\pi) \oplus Y)(\alpha - \alpha\pi\beta\alpha) = ker(\alpha\pi)(\alpha - \alpha\pi\beta\alpha) = ker(\alpha\pi)\alpha,$$

and so $M(\alpha - \alpha\beta\alpha) = ker(\alpha\pi)\alpha \subseteq ker(\pi) \cap M\alpha = Q \cap M\alpha \ll N$. This shows that $\alpha - \alpha\beta\alpha \in \nabla[M, N] \subseteq J[M, N]$, using Lemma 32, because M is direct M-projective. Moreover, $N\beta = P\sigma = Y$, so the definition of σ gives $\beta\alpha\beta = \beta$. Thus we have proven that $[M, N]$ is semiregular. It remains to show that $J[M, N] \subseteq \nabla(M, N)$. Let $\alpha \in J[M, N]$. As above, $M\alpha(1_N - \beta\alpha) = M(\alpha - \alpha\beta\alpha) \subseteq Q$, so $M\alpha \subseteq Q$ because $1_N - \beta\alpha \in U_N$. Hence $M\alpha = Q \cap M\alpha \ll N$, so $\alpha \in \nabla(M, N)$. □

Letting $M = N$ above yields the following corollary.

Corollary 36. *Let M be a module and write $A = \{\alpha \in E_M : M\alpha \ll M\}$.*

(1) *If M is direct-projective, then $A \subseteq J(E_M)$.*
(2) *Moreover, E_M is semiregular and $A = J(E_M)$ if and only if $M\alpha$ lies over a direct summand of M for each $\alpha \in E_M$.*

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