ANNIHILATOR-SMALL RIGHT IDEALS

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Abstract

A right ideal A of a ring R is called annihilator-small if $A + T = R$, T a right ideal, implies that $\mathtt{l}(T) = 0$, where $\mathtt{l}(\ )$ indicates the left annihilator. The sum A_r of all such right ideals turns out to be a two-sided ideal that contains the Jacobson radical and the left singular ideal, and is contained in the ideal generated by the total of the ring. The ideal A_r is studied, conditions when it is annihilator small are given, its relationship to the total of the ring is examined, and its connection with related rings is investigated.

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1. Introduction

It is well known that if a, b are elements of a ring R, then $1 - ab$ is a unit if and only if $1 - ba$ is a unit. The starting point in this paper is that the analogue for left annihilators is also true: $\mathtt{l}(1 - ab) = 0$ if and only if $\mathtt{l}(1 - ba) = 0$. This leads to the concept of an "annihilator-small" right ideal. In fact, the set K_r of all elements k such that kR is annihilator small turns out to be a semi-ideal that contains both the Jacobson radical and the left singular ideal and is contained in the total of the ring. The ideal A_r generated by K_r is an "annihilator-small" analogue of the Jacobson radical that contains every annihilator small right ideal, contains both the Jacobson radical and the left singular ideal, and is contained in the ideal generated by the total. Moreover, A_r is quite well behaved when passing to rings related to R (corners, matrix rings and power series rings). On the other hand, an example is given showing that A_r may not be annihilator small, and conditions when this happens are investigated. Along the way, a number of examples are given to rule out possible extensions of theorems.

Throughout the paper every ring R is associative with unity $1 \neq 0$, and all modules are unitary. We abbreviate the Jacobson radical as $J(R) = J$, and we write S_r, S_l, Z_r and Z_l for the right and left socles, and the right and left singular ideals of R, respectively. We write $\mathbb{Z}$ for the ring of integers and $\mathbb{Z}_n$ for the ring of integers modulo n. The notations $N \subseteq^{ess} M$ and $N \subseteq^{max} M$ denote respectively that a submodule $N \subseteq M$ is essential and maximal in the module M, and we write $N \subseteq^{\oplus} M$ to mean that N is a direct summand of M. Left and right annihilators of a subset $X \subseteq R$ are denoted by $\mathtt{l}(X)$ and $\mathtt{r}(X)$ respectively.

2. Annihilator-Small Right Ideals

Recall that a right ideal K of a ring R is called *small* in R if the only right ideal X of R such that $K + X = R$ is $X = R$. By analogy we say that a right ideal K of R is **annihilator-small** (**a-small**) if

$$K + X = R,\ X \text{ a right ideal of } R, \text{ implies that } \mathtt{l}(X) = 0.$$

We write $K \subseteq^{as} R_R$ in this case, and we define **a-small left** ideals analogously. The following observation is clear from the definition, and will be used repeatedly.

Lemma 1. *If $T \subseteq K \subseteq^{as} R_R$, where T is a right ideal of R, then $T \subseteq^{as} R_R$.*

The ring R is never a-small as a right ideal; if R is a domain, a right ideal K is a-small if and only if $K \neq R$. Small right ideals are clearly a-small (but the converse is false as we shall see); in particular the Jacobson radical J is a-small as a right ideal. The *left singular ideal* Z_l, defined by $Z_l = \{z \in R \mid \mathtt{l}(z) \subseteq^{ess} {}_RR\}$, is also a-small as a right ideal. In fact we have:

Proposition 2. *If K is an a-small right ideal of R_R, so also is $K + J + Z_l$.*

Proof. Let $(K + J + Z_l) + X = R$ where X is a right ideal. Then $K + Z_l + X = R$ because J is small in R_R, say $k + z + x = 1$ where $k \in K$, $z \in Z_l$ and $x \in X$. Hence $K + zR + X = R$ so $0 = \mathtt{l}(zR + X) = \mathtt{l}(z) \cap \mathtt{l}(X)$ because $K \subseteq^{as} R_R$. Thus $\mathtt{l}(X) = 0$ because $\mathtt{l}(z) \subseteq^{ess} {}_RR$. □

Lemma 3. *Let T be a right ideal of R and assume $\mathtt{l}(T) \subseteq^{ess} {}_RR$. Then $\mathtt{rl}(T) \subseteq^{as} R_R$; in particular, we obtain $T \subseteq^{as} R_R$.*

Proof. Let $\mathtt{rl}(T) + X = R$. Then $0 = \mathtt{l}(R) = \mathtt{lrl}(T) \cap \mathtt{l}(X) = \mathtt{l}(T) \cap \mathtt{l}(X)$, so $\mathtt{l}(X) = 0$ by hypothesis. The last observation is by Lemma 1 because $T \subseteq \mathtt{rl}(T)$ always holds. □

Note that the converse to Lemma 3 is true if $\mathtt{r}[(\mathtt{l}(T) \cap Rb] = \mathtt{rl}(T) + \mathtt{r}(b)$ holds for all right ideals T of R and all $b \in R$. In fact, if $\mathtt{l}(T) \cap Rb = 0$ this gives $\mathtt{rl}(T) + \mathtt{r}(b) = R$ so $\mathtt{lr}(b) = 0$ by hypothesis. Hence $b = 0$ because $Rb \subseteq \mathtt{lr}(b)$, proving that $\mathtt{l}(T) \subseteq^{ess} {}_RR$. In particular, the converse holds if R is a right *Ikeda-Nakayama* ring [2], that is $\mathtt{r}[L \cap K] = \mathtt{r}(L) + \mathtt{r}(K)$ for any left ideals L and K.

3. The ideal A_r.

In characterizing the Jacobson radical of a ring R one needs the observation that $1 - ab$ is a unit in R if and only if $1 - ba$ is a unit. The analogue for left annihilators is true: $\mathtt{l}(1 - ab) = 0$ if and only if $\mathtt{l}(1 - ba) = 0$. In fact we have:

Lemma 4. *If R is a ring, the following are equivalent for $k \in R$:*

(1) kR is a-small in R.
(2) $bR \supset bkR$ for all $0 \neq b \in R$.
(3) $\mathtt{l}(1 - kr) = 0$ for all $r \in R$.
(4) $\mathtt{l}(1 - rk) = 0$ for all $r \in R$.
(5) $\mathtt{l}(k - krk) = \mathtt{l}(k)$ for all $r \in R$.

Proof. (1)⇒(2). If $b \in bkR$ write $b = bkr$, $r \in R$, so that $b \in \mathtt{l}(1-kr)$. But $kR+(1-kr)R = R$, so $\mathtt{l}(1-kr) = 0$ by (1). Thus $b = 0$.

(2)⇒(3). If $b \in \mathtt{l}(1-kr)$, $r \in R$, then $b = bkr \in bkR$. Hence $b = 0$ by (2).

(3)⇒(4). If $b(1-rk) = 0$ then $br(1-kr) = b(r-rkr) = b(1-rk)r = 0$. Hence $br = 0$ by (3), and so $b = brk = 0$.

(4)⇒(5). If $b(k-krk) = 0$ then $bk(1-rk) = 0$ so $bk = 0$ by (4). This shows that $\mathtt{l}(k-krk) \subseteq \mathtt{l}(k)$; the other inclusion always holds.

(5)⇒(1). If $kR + X = R$, X a right ideal, write $1 = kr + x$, $r \in R$, $x \in X$. If $b \in \mathtt{l}(X)$ then $b = bkr$, so $b(k-krk) = b(1-kr)k = bxk = 0$. Hence $bk = 0$ by (5), and so $b = bkr = 0$. □

We observe in passing that in fact $\mathtt{l}(k) \oplus \mathtt{l}(1-kr) = \mathtt{l}(k-krk)$ and $\mathtt{l}(1-kr) \cong \mathtt{l}(1-rk)$ hold for all $k, r \in R$, proving again that (3)⇔(4)⇔(5) in Lemma 4. Note that condition (2) in Lemma 4 implies that if $k \in R$ is right a-small and not nilpotent then $kR \supset k^2R \supset k^3R \supset \cdots$ is strictly decreasing.

An element $k \in R$ is called **right a-small** if $kR \subseteq^{as} R_R$, that is k satisfies the conditions in Lemma 4; **left a-small** elements are defined analogously. A unit is never right a-small; if R is a domain an element k is right a-small if and only if $kR \neq R$, that is if and only if k is a non-unit. However Example 15 below shows that an element in a ring can be a-small on one side but not the other.

For convenience, define

$$K_r = K_r(R) = \{k \in R \mid k \text{ is right a-small in } R\} = \{k \in R \mid kR \subseteq^{as} R_R\}.$$

Note that $Z_l \subseteq K_r$ and $J \subseteq K_r$ by Proposition 2, but K_r may not be closed under addition (consider -2 and 3 in $R = \mathbb{Z}$). In general K_r is contained in the set of nonunits (with equality in a domain or a local ring). Conditions (3) and (4) in Lemma 4 show that:

Corollary 5. *If $k \in K_r$ then $kR \subseteq K_r$ and $Rk \subseteq K_r$.*

By virtue of the property in Corollary 5, $J = K_r$ is called a **semi-ideal** of the ring R. Note that we do not insist that a semi-ideal be closed under addition, but when this holds the semi-ideal is an ideal.

Corollary 6. *If $e = e^2 \in K_r$ then $e = 0$.*

Proposition 7. *The following are equivalent for a right ideal K of a ring R :*

(1) K is a-small in R.

(2) $K \subseteq K_r$; that is every element of K is right a-small.

(3) $\mathtt{l}(1-k) = 0$ for every $k \in K$.

Proof. (1)⇒(2) by Lemma 1, and (2)⇒(3) by Lemma 4. Given (3), let $K + X = R$, X a right ideal of R. If $1 = k + x$, $k \in K$, $x \in X$, then $\mathtt{l}(X) \subseteq \mathtt{l}(1-k) = 0$ by (3). Hence (3)⇒(1). □

As we have seen, the sum of a-small right ideals need not be a-small (consider $3\mathbb{Z} + (-2)\mathbb{Z}$ in $\mathbb{Z}$). With this in mind we define the **right AS-ideal** $A_r = A_r(R)$ of R to be the sum of all the a-small right ideals of R :

$$A_r = A_r(R) = \Sigma\{K \mid K \subseteq^{as} R_R\}.$$

We define the **left AS-ideal** $A_l = A_l(R)$ analogously. Clearly $K_r \subseteq A_r$ in every ring, but this may not be equality (consider $\mathbb{Z}$).

Theorem 8. *If R is any ring, then:*

(1) A_r is an ideal (two sided) of R that contains every a-small right ideal of R.

(2) $A_r = \{k_1 + k_2 + \cdots + k_n \mid k_i \in K_r$ for each i; $n \geq 1\}$.

(3) $A_r = K_r R = R K_r$.

(4) $J \subseteq A_r$ and $Z_l \subseteq A_r$.

Proof. (1). Clearly A_r is a right ideal; it is a left ideal by (2) below and Corollary 5.

(2). Write $X = \{k_1+k_2+\cdots+k_n \mid k_i \in K_r$ for each i; $n \geq 1\}$. If $x \in A_r$ then $x \in K_1+\cdots+K_n$ where $K_i \subseteq^{as} R_R$ for each i. If $x = k_1 + \cdots + k_n$, $k_i \in K_i$, then $k_i R \subseteq K_i$ so $k_i R \subseteq^{as} R_R$ by Lemma 1. Hence $k_i \in K_r$ for each i, proving that $A_r \subseteq X$. Conversely, if $k \in K_r$ then $kR \subseteq^{as} R_R$, so $k \in A_r$. It follows that $X \subseteq A_r$.

(3). This follows from (1) and (2) and the fact that $K_r \subseteq A_r$.

(4). This follows from Proposition 2. □

Example 15 below shows that:

Example 9. *$A_r \nsubseteq A_l$ and $A_l \nsubseteq A_r$ can happen.*

Note that $A_r = R$ is possible, indeed $A_r(\mathbb{Z}) = \mathbb{Z}$, so the ideal A_r need not be a-small as a right ideal. Also, the sum of a-small right ideals need not be a-small (consider $2\mathbb{Z} + 3\mathbb{Z} \subseteq \mathbb{Z}$). These properties are equivalent.

Theorem 10. *The following are equivalent for a ring R :*

(1) If $K \subseteq^{as} R_R$ and $L \subseteq^{as} R_R$ then $K + L \subseteq^{as} R_R$.

(2) K_r is closed under addition, that is a sum of right a-small elements is right a-small.

(3) $A_r = K_r$.

(4) $A_r \subseteq^{as} R_R$.

If this is the case, we have:

(a) A_r is the unique largest a-small right ideal of R.

(b) $A_r = \{k \mid \mathtt{l}(1 - kr) = 0$ for all $r \in R\} = \{k \mid \mathtt{l}(1 - rk) = 0$ for all $r \in R\}$.

(c) $A_r = \cap\{M \subseteq^{max} R_R \mid A_r \subseteq M\}$.

Proof. (1)⇒(2). This follows from Lemma 1 because $(k + l)R \subseteq kR + lR$.

(2)⇒(3). Always $K_r \subseteq A_r$; and $A_r \subseteq K_r$ by (2) and Theorem 8(2).

(3)⇒(4). Let $A_r + X = R$, so $K_r + X = R$ by (3). If $1 = k + x$ with $k \in K_r$ and $x \in X$, then $R = kR + X$ and $kR \subseteq^{as} R_R$. Hence $\mathtt{l}(X) = 0$, proving (4).

(4)⇒(1). Let $K \subseteq^{as} R_R$ and $L \subseteq^{as} R_R$. Then $K \subseteq A_r$ and $L \subseteq A_r$, so $K + L \subseteq A_r$. Hence $K + L \subseteq^{as} R_R$ by (4) and Lemma 1.

Finally, (a) is clear by (4), and (b) follows from (3) and Lemma 4. As to (c): If $a \notin A_r$ then aR is not right a-small by (3) so $aR + X = R$ for some right ideal X with $\mathtt{l}(X) \neq 0$. Since $A_r \subseteq^{as} R_R$ by (4), it follows that $A_r + X \neq R$. If $A_r + X \subseteq M \subseteq^{max} R_R$, then $a \notin M$, proving (c). □

We say that $A_r = A_r(R)$ is **closed** if the conditions in Theorem 10 are satisfied.

Corollary 11. *A_r is closed if $A_r \subseteq J + Z_l$. The converse is false.*

Proof. The first assertion is because $J+Z_l$ is a-small as a right ideal by Proposition 2. Example 15 below provides a counterexample to the converse. $\square$

It follows from Lemma 1 and the definition of A_r that $A_r = J$ if and only if every a-small right ideal is small. Hence Corollary 11 gives

Corollary 12. *If every a-small right ideal is small then A_r is closed (and $A_r = J$).*

4. Examples

We have referred to Example 15 several times above. This example comes from the following result about the *split-null extension* $\begin{bmatrix} R & V \\ 0 & S \end{bmatrix}$ of two rings R and S by a bimodule $V = {}_RV_S$ using matrix operations.

Proposition 13. *Consider the split-null extension $T = \begin{bmatrix} R & V \\ 0 & S \end{bmatrix}$. Then*

(1) $\begin{bmatrix} A_r(R) & V \\ 0 & 0 \end{bmatrix} \subseteq A_r(T) \subseteq \begin{bmatrix} A_r(R) & V \\ 0 & A_r(S) \end{bmatrix}$.

(2) $\begin{bmatrix} A_l(R) \cap \mathtt{l}_R(V) & V \\ 0 & A_l(S) \end{bmatrix} \subseteq A_l(T) \subseteq \begin{bmatrix} A_l(R) & V \\ 0 & A_l(S) \end{bmatrix}$.

Proof. (1). As $A_r(R)$ is a sum of a-small right ideals of R, it suffices for the first inclusion to show that $\begin{bmatrix} K & V \\ 0 & 0 \end{bmatrix} \subseteq^{as} T_T$ whenever $K \subseteq^{as} R_R$. By Proposition 7, if $\begin{bmatrix} k & v \\ 0 & 0 \end{bmatrix} \in \begin{bmatrix} K & V \\ 0 & 0 \end{bmatrix}$ we must show that $\mathtt{l}_T \begin{bmatrix} 1-k & -v \\ 0 & 1 \end{bmatrix} = 0$. But if $\begin{bmatrix} r & w \\ 0 & s \end{bmatrix} \begin{bmatrix} 1-k & -v \\ 0 & 1 \end{bmatrix} = 0$ we have $r(1-k) = 0$, $w = rv$ and $s = 0$, so $\begin{bmatrix} r & w \\ 0 & s \end{bmatrix} = 0$ because $\mathtt{l}_R(1-k) = 0$.

Turning to the second inclusion, let $\begin{bmatrix} a & v \\ 0 & b \end{bmatrix} \in A_r(T)$; we must show that $a \in A_r(R)$ and $b \in A_r(S)$. Since $\begin{bmatrix} a & v \\ 0 & b \end{bmatrix}$ is a sum of right a-small matrices, we may assume that $\begin{bmatrix} a & v \\ 0 & b \end{bmatrix}$ is right a-small in T, and hence that $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is right a-small in T (because $\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \in J(T) \subseteq A_r(T)$). Given $r \in R$ and $s \in S$, let $c \in \mathtt{l}_R(1-ar)$ and $d \in \mathtt{l}_S(1-bs)$. Then $\begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} \in \mathtt{l}_M \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix} \right\}$, so $c = 0 = d$. Hence $\mathtt{l}_R(1-ar) = 0$ and $\mathtt{l}_S(1-bs) = 0$, so $a \in A_r(R)$ and $b \in A_r(S)$ as required.

(2). For the first inclusion, let $\begin{bmatrix} a & v \\ 0 & b \end{bmatrix} \in \begin{bmatrix} A_l(R) \cap \mathtt{l}_R(V) & V \\ 0 & A_l(S) \end{bmatrix}$; we must show that $\begin{bmatrix} a & v \\ 0 & b \end{bmatrix} \in A_l(T)$; equivalently that $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in A_l(T)$. Given $\begin{bmatrix} r & w \\ 0 & s \end{bmatrix}$ in T, we have $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} r & w \\ 0 & s \end{bmatrix} = \begin{bmatrix} 1-ar & 0 \\ 0 & 1-bs \end{bmatrix}$. Hence $\begin{bmatrix} p & u \\ 0 & q \end{bmatrix} \in \mathtt{r}_M \begin{bmatrix} 1-ar & 0 \\ 0 & 1-bs \end{bmatrix}$ gives $p \in \mathtt{r}_R(1-ar) = 0$, $q \in \mathtt{r}_R(1-bs) = 0$ and $u = a(ru) = 0$, as required.

For the other inclusion, let $\begin{bmatrix} a & v \\ 0 & b \end{bmatrix} \in A_l(T)$; we must show that $a \in A_l(R)$ and $b \in A_l(S)$. We may assume that $\begin{bmatrix} a & v \\ 0 & b \end{bmatrix}$ is right a-small in T, whence $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is right a-small in T. Suppose that $p \in \mathtt{r}_R(1-ar) = 0$ and $q \in \mathtt{r}_R(1-bs)$. Then we see that $\begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$ lies in $\mathtt{r}_T \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix} \right\} = 0$ so $p = 0$ and $q = 0$, as required. $\square$

Example 14. *Let* $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$. *Then:*

(1) $A_r(R) = \begin{bmatrix} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$ *and* $A_l(R) = \begin{bmatrix} 2\mathbb{Z} & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$.

(2) $A_r(R)$ *is not closed but* $A_l(R)$ *is closed.*

Proof. We have $A_r(\mathbb{Z}) = \mathbb{Z} = A_l(\mathbb{Z})$, $A_r(\mathbb{Z}_2) = 0 = A_l(\mathbb{Z}_2)$ and $\mathtt{l}_{\mathbb{Z}}(\mathbb{Z}_2) = 2\mathbb{Z}$. Hence, by Proposition 13, $\begin{bmatrix} 2\mathbb{Z} & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix} \subseteq A_k(R) \subseteq \begin{bmatrix} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$. But if $\begin{bmatrix} n & a \\ 0 & 0 \end{bmatrix} \in A_l(R)$, it is easy to show that n is even. Next, the idempotent $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is in $A_r(R)$, so $A_r(R)$ is not closed by Corollary 6. Finally, to see that $A_l(R)$ is closed, let $\begin{bmatrix} 2n & x \\ 0 & 0 \end{bmatrix} \in A_l(R)$. Then we see that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2n & x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k & p \\ 0 & q \end{bmatrix} = \begin{bmatrix} 1-2nk & xq \\ 0 & 1 \end{bmatrix}$ and this has zero right annihilator in R. Hence $\begin{bmatrix} 2n & x \\ 0 & 0 \end{bmatrix}$ is left a-small in R by Proposition 7 so $A_l(R) \subseteq^{sm} {}_R R$. □

Example 15. *Let* R *be a ring. Then:*

(1) $A_r \not\subseteq A_l$ *and* $A_l \not\subseteq A_r$ *can happen.*

(2) An element can be a-small on one side but not on the other.

(3) If A_r *is closed it need not happen that* $A_r \subseteq J + Z_l$.

Proof. We work in the ring R in Example 14

(1). We have $J = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$ so $J \subset A_l \subset A_r$. Similarly we can have $J \subset A_r \subset A_l$.

(2).The matrix $k = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \in R$ is left a-small by Proposition 7 because $k \in A_l(R)$ which is closed by Example 14. On the other hand, k is not right a-small because $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$ has nonzero left annihilator.

(3). $A_l = \begin{bmatrix} 2\mathbb{Z} & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$ is closed but $A_l \neq J + Z_r$ because $J = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$ and $Z_r = 0$—in fact $T \subseteq^{ess} R_R$ if and only if $T \supseteq T_m$ for some $m \neq 0$ where $T_m = \begin{bmatrix} 2m\mathbb{Z} & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$. □

If R is a ring and ${}_R X_R$ is a bimodule, the *trivial extension* of R by X is defined to be the additive abelian group $T = T(R, X) = R \oplus X$ with multiplication $(r, x)(r', x') = (rr', rx' + xr')$. We identify $R \subseteq T$ and $X \subseteq T$ so that $T = R \oplus X$, X is an ideal of T, $X^2 = 0$, and $T/X \cong R$.

Proposition 16. *Let* $T = R \oplus X$ *be the trivial extension of the ring* R *by the bimodule* ${}_R X_R$.

(1) $A_r(T) \subseteq A_r(R) \oplus X$.

(2) If $A_r(R) = J(R)$ *then* $A_r(T) = J(T)$.

(3) If $\mathtt{r}_R(x) = 0$, $x \in X$, *implies that* $x = 0$, *then* $A_r(T) = A_r(R) \oplus X$.

Proof. Observe first that if $a \in R$ then $(r + x)a = 0$ if and only if $ra = 0$ and $xa = 0$, that is

$$\mathtt{l}_T(a) = \mathtt{l}_R(a) \oplus \mathtt{l}_X(a). \qquad (*)$$

(1). It suffices to show that if $a + x$ is right a-small in T then a is right a-small in R. Since $x \in J(T)$ then a is right a-small in T. Hence, if $r \in R$, we have $\mathtt{l}_T(1 - ar) = 0$ so $\mathtt{l}_R(1 - ar) = 0$ by (*). It follows that $a \in A_r(R)$.

(2). If $A_r(R) = J(R)$ then $A_r(T) \subseteq J(R) \oplus X = J(T)$ using (1), proving (2).

(3). We have $X \subseteq A_r(T)$ because $X \subseteq J(T)$. If $a \in A_r(R)$ we must show that $a \in A_r(T)$, and we may assume that a is right a-small in R. If $t = r + x$ in T then (*) gives

$$\mathtt{l}_T(1 - at) = \mathtt{l}_T[(1 - ar) - ax] = \mathtt{l}_R(1 - ar) + \mathtt{l}_X(1 - ar) = 0 + \mathtt{l}_X(1 - ar).$$

But $\mathtt{l}_X(1-ar)=0$ by hypothesis because $1-ar\neq 0$ (in fact, $\mathtt{l}_R(1-ar)=0$). □

5. Relation to the Total

If a is an element in a ring R, it is easy to see that Ra contains a nonzero idempotent if and only if aR contains a nonzero idempotent. Kasch and Mader [3] say that a has a *partial inverse* in this case, and they define the *total* of the ring R as follows:

$$\mathrm{tot}(R)=\{a\in R\mid a \textit{ has no partial inverse}\}$$

The total is a semi-ideal but may not be closed under addition. In fact, if 0 and 1 are the only idempotents in R, then $\mathrm{tot}(R)$ is the set of nonunits. In particular, $\mathrm{tot}(R)=K_r(R)$ if R is a domain.

Call a subset I of a ring R *idempotent-free* if it contains no nonzero idempotent. If I is idempotent free, and either $aR\subseteq I$ or $Ra\subseteq I$ for each $a\in I$, we claim that $I\subseteq\mathrm{tot}(R)$. Indeed, if $a\in I$ but $a\notin\mathrm{tot}(R)$, then a has a partial inverse, so there exists $0\neq e^2=e\in Ra$ and $0\neq f^2=f\in aR$, contrary to assumption. In particular, $\mathrm{tot}(R)$ contains $J(R)$, $Z_r(R)$ and $Z_l(R)$ for any ring R.

Since (by Corollary 6) K_r and K_l are idempotent free subsets, Corollary 5 gives

Proposition 17. *If R is any ring then $K_r\subseteq tot(R)$ and $K_l\subseteq tot(R)$.*

If I is a subset of a ring R, we say that R is I-*semipotent* (see [6]) if every right (equivalently left) ideal not contained in I contains a nonzero idempotent, equivalently if every element $a\notin I$ has a partial inverse.

Lemma 18. *Let I be a subset of a ring R. The following are equivalent:*
(1) R is I-semipotent.
(2) $tot(R)\subseteq I$.

Proof. If $a\in\mathrm{tot}(R)$ then a has no partial inverse by definition, so $a\in I$ by (1). This proves that (1)⇒(2). Conversely, if $a\notin I$ then $a\notin\mathrm{tot}(R)$ by (2), so a has a partial inverse, proving that (2)⇒(1). □

A $J(R)$-semipotent ring R is called *semipotent.* The following answers a question of Kasch and Schneider [4, Page 173].

Corollary 19. *If R is any ring then R is semipotent if and only if $J(R)=tot(R)$.*

Proof. If R is semipotent, we have $\mathrm{tot}(R)\subseteq J$ by Lemma 18; the other inclusion always holds. Conversely, if $J=\mathrm{tot}(R)$ then R is J-semipotent by the definition of $\mathrm{tot}(R)$. □

Theorem 20. *The following are equivalent for a ring R :*
(1) R is K_r-semipotent.
(2) If $\mathtt{l}(b)\neq 0$ where $b\in R$, then $\mathtt{l}(b)$ contains a nonzero idempotent.
(3) If $\mathtt{l}(b)\neq 0$ where $b\in R$, then $(1-b)R$ contains a nonzero idempotent.
(4) $K_r=tot(R)$.

Proof. (4)⇒(1) is clear from the definitions, and (1)⇒(4) by Proposition 17 and Lemma 18.

(1)⇒(2). Let $ab = 0$ where $a \neq 0$. Then $a \notin K_r$, since otherwise $R = \mathtt{l}(1) = \mathtt{l}(1 - ab) = 0$ by Lemma 4. So there exists $0 \neq e^2 = e \in Ra$ by (1), whence $eb \in (Ra)b = 0$, that is $e \in \mathtt{l}(b)$.

(2)⇒(3). If $0 \neq e^2 = e \in \mathtt{l}(b)$, then $e = e(1 - b) \in R(1 - b)$. Now (3) follows.

(3)⇒(1). Let $a \notin A_r$. By (4) of Lemma 4 we have $\mathtt{l}(1 - ra) = 0$ for some $r \in R$. By (3) let $0 \neq e^2 = e \in \mathtt{l}(1 - ra)$. Hence $e(1 - ra) = 0$, and so $e = era \in Ra$. □

Note that $K_r(\mathbb{Z}) = \mathbb{Z} - \{1, -1\}$, and so $K_r(\mathbb{Z}) = \text{tot}(\mathbb{Z})$ because $\text{tot}(\mathbb{Z})$ contains no units. However, $A_r(\mathbb{Z}) = \mathbb{Z} \neq K_r(\mathbb{Z})$ so $A_r(\mathbb{Z})$ is not closed.

Recall that A_r is closed if it is contained in $J + Z_l$ because this ideal is always right a-small. Many of our examples arise in this way. If A_r is closed it contains no nonzero idempotent (Corollary 6), and this points to a large class of rings in which $A_r = A_l$ is closed. The ring R is called *semipotent* if it is J-semipotent. The class of semipotent rings is large, including all *exchange* rings (for every $a \in R$, there exists $e^2 = e \in aR$ such that $1 - e \in (1 - a)R$), and all *semiregular* rings (R/J is (von Neumann) regular and idempotents lift modulo J).

Proposition 21. *If R is semipotent then $A_r = J = A_l = tot(R)$. In particular, both A_r and A_l are closed.*

Proof. We always have $J \subseteq K_r$, with equality by Corollary 6 because R is J-semipotent. Thus $K_r = J$ is closed under addition, so A_r is closed and $A_r = K_r = J$ by Theorem 10. Moreover, this shows that R is K_r-semipotent, so $J = \text{tot}(R)$ by Theorem 20. Finally, a similar argument shows that $A_l = J$. □

More generally, the proof of Proposition 21 goes through to show that if R is I-semipotent where $I = Z_r$ or $I = Z_r + J$, then $A_r = I = \text{tot}(R)$ is closed.

6. Other Results

A ring R is called *reversible* if $ab = 0$ in R implies $ba = 0$; equivalently if, for any $a \in R$, we have $\mathtt{l}(a) = 0$ if and only if $\mathtt{r}(a) = 0$. Every commutative is clearly reversible, as is every *reduced* ring (no nonzero nilpotent elements). If R is reversible then $A_r = A_l$ by Lemma 4. There is a natural class of rings that contains the reversible ones: A ring R is called *semicommutative* if $ab = 0$ in R implies that $aRb = 0$; equivalently if $\mathtt{l}(a)$ (respectively $\mathtt{r}(a)$) is an ideal for each $a \in R$. This raises the question:

Question. *If R is semicommutative is $A_r = A_l$?*

Note that the converse is false. Call a ring R *directly finite* if $ab = 1$ in R implies $ba = 1$. If R is regular but not directly finite then $A_r = A_l$ by Proposition 21 but R is not even reversible. Indeed, if $ab = 1$ and $ba \neq 1$, then $\mathtt{l}(a) = 0$ but $\mathtt{r}(a) = (1 - ba)R \neq 0$.

Recall that the right socle of a ring is denoted S_r, and that $J \subseteq \mathtt{r}(S_r)$ always holds. The ideal A_r lies between J and $\mathtt{r}(S_r)$.

Proposition 22. *For any ring R we have $S_rA_r = 0$, so that $J \subseteq A_r \subseteq \mathtt{r}(S_r)$.*

Proof. We have already observed that $J \subseteq A_r$, so it remains to show that $S_r A_r = 0$. By Theorem 8, it suffices to show that $bk = 0$ whenever $bR \subseteq R$ is simple and k is right a-small. But if $bk \neq 0$ then $bkR = bR$ because bR is simple, say $b = bkr$. Hence $b \in \mathtt{l}(1 - kr) = 0$ by Lemma 4, a contradiction. □

Observe that, while $S_r A_r = 0$, we need not have $\mathtt{r}(S_r) = A_r$ or $\mathtt{l}(A_r) = S_r$. Indeed:

(1) If $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ then $\mathtt{r}(S_r) = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix} = A_r$.

(2) If $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ then $\mathtt{r}(A_l) = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{Z}_2 \end{bmatrix} \neq \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix} = S_l$.

Note that (1) also shows that $S_r \cap A_r \neq 0$ can happen (because $S_r = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$ in that case).

It is a routine matter to verify that

$$\begin{aligned} \mathtt{r}(S_r) &= \{a \in R \mid Ba = 0 \text{ for every simple module } B_R \text{ that embeds in } R_R\} \\ &= \cap\{M \subseteq^{max} R_R \mid \mathtt{l}(M) \neq 0\}. \end{aligned}$$

In view of condition (c) in Theorem 10, it is tempting to ask whether $A_r = \mathtt{r}(S_r)$ implies that A_r is closed, or conversely. However both implications are false:

If $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ where F is a field, then $M = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$ and $N = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$ are the only maximal right ideals of R, $\mathtt{l}(M) \neq 0$, and $\mathtt{l}(N) = 0$. Hence, in this case, $\mathtt{r}(S_r) = M$ but $A_r = J = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix} \neq \mathtt{r}(S_r)$. On the other hand, if $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ then the maximal right ideals are $M = \begin{bmatrix} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$ and $M_p = \begin{bmatrix} p\mathbb{Z} & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ where p is a prime. Since $\mathtt{l}(M_p) = 0$ and $\mathtt{l}(M) \neq 0$ we have $\mathtt{r}(S_r) = M = A_r$ in this case (using Example 14 below), but A_r is not closed by Corollary 6.

If $a \in R$ is right a-small and $X \subseteq^{max} R_R$ satisfies $\mathtt{l}(X) \neq 0$, then $a \in X$ (since otherwise $aR + X = R$ and so $\mathtt{l}(X) = 0$). It follows that $A_r \subseteq \cap\{X \subseteq^{max} R_R \mid \mathtt{l}(X) \neq 0\}$, but the next example shows that this need not be equality, even in a right artinian ring for which $A_r = J$ is closed.

Example 23. *Let* $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ *where* F *is a field. Then*

$$A_r = J = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix} \quad \textit{while} \quad \cap\{M \subseteq^{max} R_R \mid \mathtt{l}(M) \neq 0\} = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}.$$

Proof. Note first that $A_r = J = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$ by Proposition 21. The only maximal right ideals of R are $X_1 = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix} = e_{11}R$, and $X_2 = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix} = S_r$. Since $\mathtt{l}(X_1) = R(1 - e_{11}) \neq 0$ and $\mathtt{l}(X_2) = 0$, the result follows. □

Remark. If $K \subseteq^{as} R_R$, Zorn's lemma provides maximal members of $\{T \mid K \subseteq T \subseteq^{as} R_R\}$; call them **max-as** right ideals. Then $A_r(R) = \Sigma\{M \mid M \text{ is max-as}\}$, and $A_r(R)$ is closed if and only if $A_r(R)$ is the unique max-as right ideal. Moreover, $Z_l + J \subseteq \cap\{M \mid M \text{ is max-as}\}$ because $M + Z_l + J \subseteq^{as} R_R$ by Proposition 2. But $Z_l + J \neq \cap\{M \mid M \text{ is max-as}\}$ is possible because $J \nsubseteq Z_l$ can happen: For example if $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ where F is a field then $Z_l = 0 = Z_r$ but $J = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$.

Lemma 24. *The following are equivalent for a maximal left ideal* $M \subseteq^{max} {}_RR$:
(1) $\mathtt{r}(M) \subseteq^{as} R_R$.
(2) $M \subseteq^{ess} {}_RR$.

Proof. (1)⇒(2). If (2) fails, let $M \cap K = 0$ where $K \neq 0$. Since M is maximal, it follows that $M = Re$ where $e^2 = e$. Hence $\mathtt{r}(M) = (1-e)R$, so (1) shows that $1 - e \in A_r$. Hence $e = 1$ by Corollary 6, a contradiction.

(2)⇒(1). If $\mathtt{r}(M) + X = R_R$ then $0 = \mathtt{l}(R) = \mathtt{lr}(M) \cap \mathtt{l}(X)$. Since $M \subseteq \mathtt{lr}(M)$, (2) gives $\mathtt{l}(X) = 0$, as required. □

It follows from Theorem 8 that $A_r = J$ if and only if every a-small right ideal of a ring R is small. This holds if R is semipotent (Proposition 21) and we are going to give several other cases where it is true. The unifying result is:

Proposition 25. *If* R *is a ring in which* $\mathtt{l}(a) = 0$, $a \in R$, *implies* $aR = R$, *then* $A_r = J$.

Proof. Always $J \subseteq K_r$; we show that this is equality (then K_r is closed under addition so A_r is closed and $A_r = K_r = J$). If $k \in K_r$ then k is right a-small so $\mathtt{l}(1 - kr) = 0$ for all $r \in R$ by Lemma 4. Hence $(1 - kr)R = R$ by hypothesis, and it follows that $k \in J$. □

The converse to Proposition 25 is false. If $R = \mathbb{Z}_2[[x]]$ is the power series ring then R is local (hence semipotent) and so $A_r = J$. However $\mathtt{l}(x) = 0$ but x is not a unit in R.

A ring R is called *right Kasch* if each simple right module embeds in R; equivalently if $\mathtt{l}(M) \neq 0$ for every maximal right ideal M of R. Call R left *principally injective* if every R-linear map $Ra \to R$, $a \in R$, extends to $R \to R$; equivalently if aR is a right annihilator in R for each $a \in R$. Finally, call R a left *C2 ring* if every left ideal that is isomorphic to a direct summand of R is itself a direct summand.

Example 26. *In each of the following cases we have* $J = A_r$ *(so* A_r *is closed).*
(1) R *is semipotent.*
(2) R *is right Kasch.*
(3) R *is left principally injective.*
(4) R *is a left C2 ring.*

Proof. Proposition 21 gives (1); for the rest we show that $\mathtt{l}(a) = 0$, $a \in R$, implies $aR = R$, and invoke Proposition 25.

(2). If $\mathtt{l}(a) = 0$, $a \in R$, and $aR \neq R$ then $aR \subseteq M \subseteq^{max} R_R$ so $\mathtt{l}(M) = 0$ contrary to the Kasch hypothesis. Hence $aR = R$ and Proposition 25 applies.

(3). If $\mathtt{l}(a) = 0$, $a \in R$, write $aR = \mathtt{r}(X)$ by (3). Then $X \subseteq \mathtt{l}(a) = 0$ so $aR = \mathtt{r}(X) = R$, as required.

(4). If $\mathtt{l}(a) = 0$, $a \in R$, then $aR \cong R$ so a is regular by (4), say $aba = a$. But then $0 = \mathtt{l}(a) = \mathtt{l}(ab) = R(1 - ab)$, so $aR = R$ and again Proposition 25 applies. □

Note that Example 15 shows that the fact that A_r is closed does not imply that $A_r = J$. Note further that in case (3) of Example 26 we have $J = Z_l = A_r$ by [5, Theorem 2.1].

A ring R with $A_r(R) = J(R)$ need not be in any of the classes in Example 26. We have $A_r(\mathbb{Z}) = J(\mathbb{Z}) = \mathbb{Z} - \{1, -1\}$ but $\mathbb{Z}$ is not semipotent; the localization $\mathbb{Z}_{(2)}$ of the integers at 2 is a domain that satisfies $A_r = J$ (it is local and so semipotent) but, as is readily verified, it is not right Kasch, not left principally injective, and not left C2.

We conclude this section by identifying some modules for which the endomorphism ring has $A_r = J$. Recall that a module ${}_RM$ is said to *generate* a module ${}_RK$ if $K = \Sigma\{M\theta \mid \theta : M \to K\}$, and that M *cogenerates* K if K given $0 \neq k \in K$ then $k\sigma \neq 0$ for some $\sigma : K \to M$.

Proposition 27. *Consider a module ${}_RM$ and write $E = \text{end}({}_RM)$.*
(1) If M generates $\ker(\alpha)$ for each $\alpha \in E$, and monomorphisms in E are epic, then $A_r(E) = J(E)$.
(2) If M cogenerates $M/M\alpha$ for each $\alpha \in E$, and epimorphisms in E are monic, then $A_l(E) = J(E)$.

Proof. (1). If $\alpha \in E$ is right a-small, then $\mathtt{l}_E(1 - \alpha\rho) = 0$ for all $\rho \in E$, so it suffices by hypothesis to show that $\mathtt{l}_E(\sigma) = 0$, $\sigma \in E$, implies that σ is monic. Write $K = \ker(\sigma)$ and, by hypothesis, let $K = \Sigma\{M\theta \mid \theta : M \to K\}$. But given $\theta : M \to K$ we have $(M\theta)\sigma \subseteq K\sigma = 0$, so $\theta\sigma = 0$. Hence $\theta \in \mathtt{l}_E(\sigma) = 0$, and it follows that $K = 0$, as required.

(2). If $\alpha \in E$ is left a-small, then $\mathtt{r}_E(1 - \rho\alpha) = 0$ for all $\rho \in E$, so it suffices by hypothesis to show that $\mathtt{r}_E(\lambda) = 0$, $\lambda \in E$ implies that λ is epic. But if $m_0 \notin M\lambda$ choose $\sigma : M/M\lambda \to M$ such that $(m_0 + M\lambda)\sigma \neq 0$. If $\beta \in E$ is defined by $m\beta = (m + M\lambda)\sigma$ for all $m \in M$, then $(M\lambda)\beta = 0$ so $\beta \in \mathtt{r}_E(\lambda) = 0$, a contradiction because $m_0\beta = 0$. □

7. Related Rings

The ideal $A_r(R)$ is relatively well behaved when the ring R changes. We begin with direct products.

Proposition 28. *If $R = \Pi_{i\in I}R_i$ is a direct product of rings, then $A_r(R) = \Pi_i A_r(R_i)$.*

Proof. Observe first that if $a = \langle a_i\rangle \in R$ then a is right a-small in R if and only if each a_i is right a-small in R_i. Indeed, given $r = \langle r_i\rangle \in R$ we have $\mathtt{l}_R(1-ar) = \Pi_i \mathtt{l}_{R_i}(1-a_ir_i)$. The result follows because $a \in A_r(R)$ if and only if it is a finite sum of right a-small elements. □

Proposition 29. *Let R denote any ring and let $e^2 = e \in R$.*
(1) $eA_r(R)e \subseteq A_r(eRe)$.
(2) If $ReR = R$ then $eA_r(R)e = A_r(eRe)$.

Proof. For convenience denote $S = eRe$.

(1). Let $a \in eA_r(R)e \subseteq A_r(R)$, say $a = \Sigma_i a_i$ where each a_i is right a-small in R. Then $a = \Sigma_i e a_i e$ so it suffices to prove that, if a is right a-small in R, then eae is right a-small in S. So let $s \in S$ be arbitrary and consider $x \in \mathtt{l}_S[e - (eae)s]$. Then $x = xe \in \mathtt{l}_S[1 - (ae)s] = 0$ because $ae \in A_r(R)$.

(2). Now assume that $ReR = R$, and let $a \in A_r(eRe)$; it suffices to show that $a \in A_r(R)$, We may assume that a is a-small in S, and we show that a is right a-small in R, that is $\mathtt{l}_R(1 - ra) = 0$ for all $r \in R$. If $b(1 - ra) = 0$ then $b = be$ and we have

$$0 = exb(1 - ra) = exbe(1 - ra) = exb(e - era) = exb(e - (ere)a).$$

Hence $exb = 0$ (because $a \in A_r(S)$), and it follows that $ReRb = 0$. Thus $b = 0$ by hypothesis. □

Example 30. *If* $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ *and* $e = \begin{bmatrix} 1 & \bar{1} \\ 0 & 0 \end{bmatrix}$ *then* $eA_l(R)e \subset A_l(eRe)$.

Proof. By Example 14, $eA_l(R)e = e\begin{bmatrix} 2\mathbb{Z} & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}e = \begin{bmatrix} 2\mathbb{Z} & 0 \\ 0 & 0 \end{bmatrix}$, so it suffices to show that $A_l(eRe) = eRe$. We have $eRe = \left\{\begin{bmatrix} n & \bar{n} \\ 0 & 0 \end{bmatrix} \mid n \in \mathbb{Z}\right\}$. Since $e = \begin{bmatrix} 3 & \bar{3} \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ it suffices to show that both $a = \begin{bmatrix} 3 & \bar{3} \\ 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ are left a-small in eRe. But a routine calculation shows that $\mathtt{r}_{eRe}(e - ax) = 0 = \mathtt{r}_{eRe}(e - ax)$ for all $x \in eRe$. □

Let $M_n(R)$ denote the ring of $n \times n$ matrices over the ring R.

Proposition 31. *If* $n \geq 1$ *then* $A_r[M_n(R)] = M_n[A_r(R)]$ *for any ring* R.

Proof. Because of Proposition 29, it suffices to prove it for $n = 2$. Write $S = M_n(R)$.

If $a = [a_{ij}] \in A_r(S)$ we must show that $\alpha \in M_n[A_r(R)]$; we may assume that α is right a-small in S. Let ε_{ij} denote the matrix units in S. Hence $a_{ij}\varepsilon_{11} = \varepsilon_{1i}\alpha\varepsilon_{j1}$ is right a-small in S by Corollary 5, so it suffices to show that $\alpha\varepsilon_{11}$ right a-small in S implies that α is right a-small in R. But if $r \in R$ is arbitrary and $q \in \mathtt{l}(1 - ar)$, then $q\varepsilon_{11} \in \mathtt{l}(1_S - \alpha\varepsilon_{11}r\varepsilon_{11}) = 0$ so $q = 0$.

Conversely, let $\alpha = [a_{ij}] \in M_n[A_r(R)]$; we must show that $\alpha \in A_r(S)$. Since each $a_{ij} \in A_r(R)$, we may assume that a_{ij} is right a-small in R for all i and j. Moreover, since $\alpha = \Sigma_{i,j}a_{ij}\varepsilon_{ij}$ and $a_{ij}\varepsilon_{ij} = \varepsilon_{i1}(a_{ij}\varepsilon_{11})\varepsilon_{1j}$, it suffices to show that a right a-small in R implies that $a\varepsilon_{11} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ is right a-small in S. Hence we must show that $\mathtt{l}_S\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} r & s \\ t & q \end{bmatrix}\right\} = 0$ for all $r, s, t, q \in R$. But if

$$\begin{bmatrix} u & v \\ w & z \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} r & s \\ t & q \end{bmatrix}\right) = \begin{bmatrix} u & v \\ w & z \end{bmatrix}\begin{bmatrix} 1-ar & -as \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} u(1-ar) & v-as \\ w(1-ar) & z-ws \end{bmatrix}$$

vanishes we obtain $u = v = w = z = 0$ because $\mathtt{l}_R(1 - ar) = 0$. □

If we combine Propositions 29 and 31, we obtain

Theorem 32. *The following conditions on a ring* R *are Morita invariants:*

(1) $A_r(R) = 0$.
(2) $A_r(R) = R$.
(3) $A_r(R) = J$.
(4) $A_r(R) = Z_l$.
(5) $A_r(R) \subseteq^{ess} R_R$.

Proof. If we write $M = M_n(R)$ and $e^2 = e \in M$ satisfies $MeM = M$, we must show that $S = eMe$ inherits each of these conditions. Then Propositions 29 and 31 give (1) and (2), while (3) and (4) are because $J[M_n(R)] = M_n(J)$, $J(eRe) = eJe$, $Z_l[M_n(R)] = M_n[Z_l(R)]$ and (if $ReR = R$) $Z_l(eRe) = eZ_l(R)e$. Finally, (5) follows because if $T \subseteq^{ess} R_R$, T a right ideal of the ring R, then $M_n(T) \subseteq^{ess} M_n(R)$ and (if $ReR = R$) $eTe \subseteq^{ess} eRe$. □

If R is a ring let $R[[x]]$ denote formal power series ring over R in an indeterminant x.

Proposition 33. *If* x *is an indeterminant in* R *then* $A_r(R[[x]]) = A_r(R) + xR[[x]]$.

Proof. Write $S = R[[x]]$. We begin with:

Claim. x is right a-small in S.

Proof. We show that $\mathtt{l}(1 - xf) = 0$ for every $f \in S$. If $g = b_0 + b_1x + \cdots \in \mathtt{l}(1 - xf)$ where each $b_i \in R$, we have $g = gxf$ so $b_0 = 0$ and hence $g = xg_1$ where $g_1 = b_1 + b_2x + \cdots$. But then $xg_1(1 - xf) = 0$, so $g_1 \in \mathtt{l}(1 - xf)$ and we obtain $b_1 = 0$. Continuing in this way, we see that $g = 0$, proving the Claim.

Write $I = A_r(R) + xS$. To show that $I \subseteq A_r(S)$, let $f = a + xg \in I$ where $a \in A_r(R)$, $g \in S$; we must show that $f \in A_r(S)$. We may assume that a is a-small in R, and we show that $f = a + xg \in A_r(S)$. By the claim, it suffices to show that a is a-small in S. To this end, let $aS + \mathcal{T} = S$ where $\mathcal{T}$ is a right ideal of S. Looking at constant coefficients, we see that $aR + T = R$ where $T = \{t \in R \mid t + xg \in \mathcal{T}$ for some $g \in S\}$. But then $\mathtt{l}_R(T) = 0$ by hypothesis, and we claim that $\mathtt{l}_{R[x]}(\mathcal{T}) = 0$. If not, let $f = a_0x^m + a_1x^{m+1} + \cdots \in \mathtt{l}_{R[x]}(\mathcal{T})$ where $a_0 \neq 0$. Then for each $t \in T$, $f(t + xg) = 0$ for some $g \in S$, which implies that $a_0t = 0$. Hence $a_0 \in \mathtt{l}_R(T)$, a contradiction. This proves that $I \subseteq A_r(S)$.

Conversely, let $f = a + xg \in A_r(S)$ where $a \in R$ and $g \in S$; we must show that $a \in A_r(R)$. Since f is a sum of a-small elements in $R[x]$, comparing constant coefficients shows that we may assume that $f = a + xg$ is a-small in S. We show that a is a-small in R. So let $aR + T = R$, T a right ideal of R. Then $aS + T[x] = S$, so $(a + gx)S + T[[x]] + xS = S$. Hence $(a + gx)S + T[[x]] = S$ because $xS \subseteq J(S)$. But $(a + gx)$ is a-small in S, so $\mathtt{l}_S(T) = 0$. This gives $\mathtt{l}_R(T) = 0$, and so proves that $a \in A_r(R)$. □

The situation for the polynomial ring $R[x]$ is quite different from $R[[x]]$. A ring R is called an *Armendariz ring* if whenever $(\Sigma a_ix^i)(\Sigma b_jx^j) = 0$ in $R[x]$ we have $a_ib_j = 0$ for all i and j. Examples of Armendariz rings include reduced rings, $R[x]/(x^n)$ where R is reduced, $\mathbb{Z}_n$ for $n \geq 2$, and polynomial rings over Armendariz rings.

Proposition 34. *If R is an Armendariz ring then $A_r(R[x]) = R[x]$.*

Proof. Write $S = R[x]$. The proof of the Claim in Proposition 33 goes through to show that x is a-small in S. Hence $-x$ is a-small in S so it suffices to show that $a = 1 + x$ is a-small in S (since then $1 = (1 + x) + (-x) \in A_r(S)$). Hence we prove the

Claim. $\mathtt{l}_S(1 + ab) = 0$ for all $b \in S$.

Write $b = b_0 + b_1x + \cdots + b_n$ where $n \geq 0$. Then

$$1 + ab = (1 + b_0) + (b_0 + b_1)x + (b_1 + b_2)x^2 + \cdots + (b_{n-1} + b_n)x^n + b_nx^{n+1}.$$

Suppose that $\mathtt{l}_S(1 + ab) \neq 0$. Then there exists $0 \neq c_0 + c_1x + \cdots + c_mx^m$ in S such that

$$\begin{aligned} 0 &= (c_0 + c_1x + \cdots + c_mx^m)(1 + ab) \\ &= (c_0 + c_1x + \cdots + c_mx^m)((1 + b_0) + (b_0 + b_1)x + \cdots + (b_{n-1} + b_n)x^n + b_nx^{n+1}). \end{aligned}$$

We may assume that $c_0 \neq 0$ with no loss in generality. Since R is Armendariz, it follows that

$$\begin{aligned} c_0(1 + b_0) &= 0 \\ c_0(b_0 + b_1) &= 0 \\ &\vdots \\ c_0(b_{n-1} + b_n) &= 0 \\ c_0(b_n) &= 0 \end{aligned}$$

This gives $c_0 b_i = 0$ for $i = n,\ n-1,\ \cdots, 2,\ 1,\ 0$. So $c_0 = 0$, a contradiction. This proves the Claim, and hence the proposition. □

A result of Anderson and Camillo [1] shows that if R is Armendariz then $R[x_1, x_2, \cdots, x_n]$ is Armendariz for each n. Hence:

Corollary 35. *If R is Armendariz then $A_r(R[x_1, x_2, \cdots, x_n]) = R[x_1, x_2, \cdots, x_n]$.*

If R is a ring, let $T_n(R) = \begin{bmatrix} R & R & \cdots & R \\ 0 & R & \cdots & R \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R \end{bmatrix}$ denote the ring of $n \times n$ upper triangular matrices over R.

Proposition 36. *If R is any ring then*

$$A_r[T_n(R)] = \begin{bmatrix} A_r(R) & R & \cdots & R \\ 0 & A_r(R) & \cdots & R \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_r(R) \end{bmatrix} \quad \text{and} \quad A_l[T_n(R)] = \begin{bmatrix} A_l(R) & R & \cdots & R \\ 0 & A_l(R) & \cdots & R \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_l(R) \end{bmatrix}.$$

Proof. Induct on n, the result being clear if $n = 1$. If $n \geq 2$ write $T_n(R) = \begin{bmatrix} T_{n-1}(R) & V \\ 0 & R \end{bmatrix}$ where $V = R^{n-1}$, written as column matrices, is a $T_{n-1}(R)$-R bimodule in a natural way. By Proposition 13 we have $\begin{bmatrix} A_r[T_{n-1}(R)] & V \\ 0 & 0 \end{bmatrix} \subseteq A_r[T_n(R)] \subseteq \begin{bmatrix} A_r[T_{n-1}(R)] & V \\ 0 & A_r(R) \end{bmatrix}$. By induction, this gives

$$\begin{bmatrix} A_r(R) & R & \cdots & R & R \\ 0 & A_r(R) & \cdots & R & R \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & & A_r(R) & R \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \subseteq A_r[T_n(R)] \subseteq \begin{bmatrix} A_r(R) & R & \cdots & R & R \\ 0 & A_r(R) & \cdots & R & R \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & & A_r(R) & R \\ 0 & 0 & \cdots & 0 & A_r(R) \end{bmatrix}$$

If $k \in K_r$ and let $A = [a_{ij}] \in T_n(R)$, write kE_{nn} for the matrix in $T_n(R)$ with k in the (n, n)-position and 0s elsewhere. Then, for each $B = [b_{ij}] \in T_n(R)$ we have

$$B \in \mathtt{l}[1 - A(kE_{nn})] \quad \text{if and only if} \quad \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1\,n-1} & b_{1n}(1 - a_{nn}k) \\ 0 & b_{22} & \cdots & b_{2\,n-1} & b_{2n}(1 - a_{nn}k) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{n-1\,n-1} & b_{n\,n-1}(1 - a_{nn}k) \\ 0 & 0 & \cdots & 0 & b_{nn}(1 - a_{nn}k) \end{bmatrix} = 0.$$

Since $\mathtt{l}_R(1 - a_{nn}k) = 0$, this holds if and only if $B = 0$. Hence $kE_{nn} \in A_r[T_n(R)]$, that is $A_r(R)\, E_{nn} \subseteq A_r(R)$. This proves the first of the assertions in the Proposition; the proof of the other assertion is similar. □

Corollary 37. *Let R be an Armendariz ring, and write $T = T_n(R)$. Then $A_r(T[x]) = T[x]$.*

Proof. We haved $T[x] \cong T_n(R[x])$ via a natural isomorphism. Since R is Armendariz, Proposition 36 gives

$$A_r(T_n(R[x])) = \begin{bmatrix} A_r(R[x]) & R[x] & \cdots & R[x] \\ 0 & A_r(R[x]) & \cdots & R[x] \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_r(R[x]) \end{bmatrix} = \begin{bmatrix} R[x] & R[x] & \cdots & R[x] \\ 0 & R[x] & \cdots & R[x] \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R[x] \end{bmatrix} = T_n(R[x])$$

as required. □

Remark. Note that it is known that $T_n(R)$ is not Armendariz unless $n = 1$. Hence being Armendariz is sufficient, but not necessary, in showing that $A_r(R[x]) = R[x]$.

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References

[1] D.D. Anderson and V.P. Camillo, *Armendariz rings and Gaussian rings*, Comm. in Algebra **26** (1998), 2265-2272.

[2] V.P. Camillo, W.K. Nicholson and M.F. Yousif, *Ikeda-Nakayama rings*, J. Algebra **226** (2000), 1001-1010.

[3] F. Kasch and A. Mader, "Rings, Modules, and the Total", Frontiers in Mathematics. Birkhauser, Basel, 2004.

[4] F. Kasch and W. Schneider, *Exchange rings and the total*, Advances in Ring Theory. Birkhauser, Boston, 1997.

[5] W.K. Nicholson and M.F. Yousif, *Principally injective rings*, J. Algebra **174** (1995), 77-93.

[6] W.K. Nicholson and Y. Zhou, *Strong lifting*, J. Algebra **285** (2005), 795-818.