MORPHIC p-GROUPS

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ABSTRACT. A group G is called morphic if every endomorphism $\alpha : G \to G$ for which $G\alpha$ is normal in G satisfies $G/G\alpha \cong ker(\alpha)$. This concept for modules was first investigated by G. Ehrlich in 1976. Since then the concept has been extensively studied in module and ring theory. A recent paper of Li, Nicholson and Zan investigated the idea in the category of groups. A characterization for a finite nilpotent group to be morphic was obtained, and some results about when a small p-group is morphic were given. In this paper, we continue the investigation of the general finite morphic p-groups. Necessary and sufficient conditions for a morphic p-group of order $p^n (n > 3)$ to be abelian are given. Our main results show that if G is a morphic p-group of order p^n with $n > 3$ such that either $d(G) = 2$ or $|G^{'}| < p^3$, then G is abelian, where $d(G)$ is the minimal number of generators of G. As consequences of our main results we show that any morphic p-groups of order p^4, p^5 and p^6 are abelian.

1. Introduction and Preliminaries

Call a group G *morphic* if every endomomorphism α of G for which $G\alpha$ is normal in G satisfies $G/G\alpha \cong ker(\alpha)$. This condition for modules was introduced in 1976 by Gertrude Ehrlich [4] to characterize when the endomorphism ring of a module is unit regular. A group-theoretic version of Ehrlich's theorem was given in [6]. The condition $M/M\alpha \cong ker(\alpha)$ was studied in the context of rings in [8], for group rings in [3] and for modules in [9]. In the recent paper [7] this condition was studied in the category of groups. One of the interesting results proved in that paper is regarding morphic nilpotent groups. It was shown that a finite nilpotent group is morphic if and only if its Sylow subgroups are morphic. This motivated the study of morphic p-groups and the characterizations of morphic groups of order p^3 and p^4 were given. In the present paper, we continue the investigation of finite morphic p-groups.

If H is a subgroup of a group G, we write $H \lhd G$ to indicate that H is a normal subgroup of G, we write $Z(G)$ for the centre of G, and we write $G^{'}$ for the commutator (or derived) subgroup of G. We recall some basic properties of morphic groups which will be used in the sequel.

Lemma 1.1. [7, Lemma 5] *The following are equivalent for a group G :*
(1) G is morphic.
(2) If $K \lhd G$ is such that $G/K \cong N \lhd G$, then $G/N \cong K$.

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Lemma 1.2. [7, Lemma 17] *The following are equivalent for a morphic group G :*
(1) Every normal subgroup of G is isomorphic to an image of G.
(2) Every image of G is isomorphic to a normal subgroup of G.

A group G is called *strongly morphic* if it is morphic and the conditions in the above lemma are satisfied. Note that not every morphic group is strongly morphic. For example, the alternating group A_4 is morphic, but not strongly morphic.

Lemma 1.3. [7, Theorem 37 (1)] *Let G be a finite morphic group of order p^n. Then all subgroups and images of G of order p^{n-1} are isomorphic.*

The morphic groups of order p^3 are well understood.

Lemma 1.4. [7, Propsition 39] *Let G be a group of order p^3. Then G is morphic if and only if either G is cyclic, or elementary abelian, or nonabelain with $exp(G) = p$. Moreover, in this case, we have:*
If $p = 2$, then G is cyclic or elementary abelian.
If $p > 2$, then G is either cyclic, or elementary abelian, or
$G = \langle a, b, c \mid a^p = b^p = c^p = 1, [a, b] = c, ac = ca, bc = cb\rangle$.

The Frattini subgroup of a finite group G, denoted by $\Phi(G)$, is defined to be the intersection of all the maximal subgroups of G. Denote by $d(G)$ the minimal number of generators of G. For a finite p-group G, $\Phi(G) = G^{'}G^p$ where $G^p = \langle g^p | g \in G\rangle$, and $\Phi(G)$ is the smallest normal subgroup of G such that the factor group $G/\Phi(G)$ is an elementary abelian p-group, so $d(G)$ is equal to $dim(G/\Phi(G)) = rank(G/\Phi(G))$.

In Section 2, we study general finite morphic p-groups. It is first shown that any finite morphic p-group is strongly morphic. Necessary and sufficient conditions for a morphic p-group of order $p^n (n > 3)$ to be abelian are also given. Our main results are Theorems 2.10 and 2.13, which show that if G is a morphic p-group of order p^n with $n > 3$ such that either $d(G) = 2$ or $|G^{'}| < p^3$, then G is abelian. In section 3, the main results in the above section are applied to show that any morphic p-groups of order p^4, p^5 and p^6 are abelian.

2. The Classification of morphic p-Groups

In this section, we investigate finite morphic p-groups of order p^n where p is a prime. We begin with several preliminary results.

Proposition 2.1. *If G is a morphic p-group of order p^n, then G is strongly morphic. Moreover, each subgroup of G is isomorphic to a normal subgroup of G.*

Proof. We first show that each subgroup of G is an image of G. Assume that for each subgroup N of order p^{k+1}, N is an image. Let H be a subgroup of order p^k. Then there exists a maximal normal subgroup $\mathfrak{m}$ containing H. Note that $G/\mathfrak{m} \cong \langle z\rangle$ where z is any central element of order p. Thus $\mathfrak{m} \cong G/\langle z\rangle$ by Lemma 1.1 as G is morphic. So there exists a subgroup N of G with $|N| = p^{k+1}$ such that $H \cong N/\langle z\rangle$. Since N is an image of G, the above implies that H is an image, so G is strongly morphic. Note that in such a group G, every image is isomorphic to a normal subgroup by Lemma 1.2, so each subgroup of G is isomorphic to a normal subgroup of G as desired. □

Lemma 2.2. *Let G be a finite morphic p-group with $d(G) = k$. Then for every subgroup N of G, $d(N) \leq k$. Moreover, if $\Phi(G) \neq 1$, then $d(\mathfrak{m}) = k$ for every maximal subgroup $\mathfrak{m}$ of G.*

Proof. It was shown in the proof of Proposition 2.1 that every subgroup N of G is an image of G, so $d(N) \leq d(G)$. If $\Phi(G) \neq 1$, then there exists a central element z of order p of G such that $z \in \Phi(G)$. Thus $G/\Phi(G) \cong (G/\langle z\rangle)/(\Phi(G)/\langle z\rangle)$. By Lemma 1.3, all the maximal subgroups $\mathfrak{m}$ of G are isomorphic and $\mathfrak{m} \cong G/\langle z\rangle$. Therefore, $d(G) = d(G/\Phi(G)) \leq d(\mathfrak{m})$, implying $d(G) = d(\mathfrak{m})$ as desired. □

Lemma 2.3. [12, Lemma 2.2] *Let G be a finite nonabelian p-group. Then the following statements are equivalent.*
(1) All the maximal subgroups of G are abelian;
(2) $d(G) = 2$ and $|G^{'}| = p$;
(3) $d(G) = 2$ and $Z(G) = \Phi(G)$.

Proposition 2.4. *Let G be a morphic p-group of order p^n with $n > 3$. Then the following statements are equivalent:*
(1) $|G^{'}| < p^2$;
(2) All the maximal subgroups of G are abelian;
(3) G has an abelian maximal subgroup;
(4) G is abelian.

Proof. (1) $\Rightarrow$ (2). If $G^{'} = 1$, then G is abelian, so the result holds. If $|G^{'}| = p$ and M is any maximal subgroup of G, then $G/M \cong G^{'}$, so $M \cong G/G^{'}$ is abelian.
(2) $\Rightarrow$ (3) is obvious.
(3) $\Rightarrow$ (4). Since G has an abelian maximal subgroup and all maximal subgroups of G are isomorphic, we conclude that all maximal subgroups of G are abelian. If $Z(G)$ contains two distinct subgroups $\langle z\rangle$ and $\langle t\rangle$ of order p, then since $G/\langle z\rangle \cong M \cong G/\langle t\rangle$ and M is abelian, we conclude that $G^{'} \subseteq \langle z\rangle \bigcap \langle t\rangle = 1$. Thus G is abelian. If $Z(G)$ contains exactly one subgroup of order p, then $Z(G)$ is cyclic. If G is non-abelian, by Lemma 2.3, $d(G) = 2$ and $\Phi(G) = Z(G)$, so $Z(G) = \langle z\rangle$ has order p^{n-2}. Let $G = \langle a, b\rangle$. Then since $G/Z(G) \cong C_p \times C_p$, we have $a^p, b^p \in \Phi(G) = Z(G)$. We claim that

$$a^p, b^p \in \langle z^p\rangle \ \ (*).$$

For otherwise, we may assume that $a^p \notin \langle z^p\rangle$. Then $|a^p| = |z| = p^{n-2}$ and thus $|a| = p^{n-1}$. Therefore, G contains a cyclic maximal subgroup, so by [7, Proposition 38] G is cyclic, a contradiction. Since $|G^{'}| \leq p$, $G^{'} \subseteq \langle z^p\rangle$, so $G/\langle z^p\rangle$ is abelian of order p^3 with $exp(G/\langle z^p\rangle) = p$ and thus $G/\langle z^p\rangle \cong C_p \times C_p \times C_p$, contradicting $d(G/\langle z^p\rangle) \leq 2$.
(4) $\Rightarrow$ (1) is obvious. □

Lemma 2.5. *Let G be a finite morphic p-group. Then G has exactly one subgroup of order p if and only if G is cyclic.*

Proof. Suppose that G has exactly one subgroup of order p. By [10, 5.3.6], G is cyclic or G is a generalized quaternion group. It follows from [7, Example 16] that no generalized quaternion group is morphic, so G is cyclic. The converse is clear. □

Lemma 2.6. [1, Theorem 4] *Let G be a p-group. If both G and $G^{'}$ can be generated by two elements, then $G^{'}$ is abelian.*

Lemma 2.7. *Let G be a morphic p-group of order p^n with $d(G) = 2$ and $|G^{'}| = p^k$. Then every subgroup of order p^{n-k} is abelian.*

Proof. Let K be a subgroup of order p^{n-k}. Then there exist subgroups $K_0, K_1, ..., K_k = G$ such that $K = K_0 \subset K_1 \subset \cdots \subset K_k = G$. If K is not abelian, then K_j is not abelian for $0 \leq j \leq k$. By Lemma 2.2 $d(K_j) = 2$, for all $0 \leq j \leq k$. It follows from [13, Lemma 2.8] that $|K_0'| < |K_1'| < \cdots < |K_k'| = p^k$, so $K' = 1$, a contradiction. Thus K is abelian and we are done. □

A group G is metacyclic if it has a cyclic normal subgroup K such that G/K is cyclic.

Lemma 2.8. *Let G be a metacyclic morphic p-group of order p^n with $n > 3$ and $d(G) = 2$. Then G is abelian.*

Proof. We first assume that $|G^{'}| \geq p^2$. Since G is metacyclic, $G = \langle a, b\rangle$ where $\langle a\rangle$ is a cyclic normal subgroup of G and $b \in G$. Since $G/\langle a\rangle = \langle b\langle a\rangle\rangle = \langle \bar{b}\rangle$ is abelian, $G^{'} \subseteq \langle a\rangle$, and so $G^{'}$ is cyclic. Let $|G^{'}| = p^k$. We claim that $2k \leq n$. Since $|\bar{b}| \leq |b|$, G has a cyclic subgroup of order $|\bar{b}|$. By Proposition 2.1, G has a normal subgroup $\langle c\rangle$ of order $|\bar{b}|$. Since $G/\langle a\rangle = \langle \bar{b}\rangle \cong \langle c\rangle$, $G/\langle c\rangle \cong \langle a\rangle$ as G is morphic. Thus $G^{'} \subseteq \langle c\rangle$ and so $|G^{'}| \leq |\langle c\rangle| = |\bar{b}|$. Therefore, $p^k = |G^{'}| \leq min\{|a|, |c|\}$, so $2k \leq n$ as $|a||c| = p^n$. If G has exactly one subgroup of order p, then by Lemma 2.5 G is cyclic, a contradiction. Thus G must have a subgroup N which is isomorphic to $C_p \times C_p$. By Proposition 2.1 we may assume that N is normal. Since G/N is also a p-group, G has a normal subgroup K of order p^k such that $N \subseteq K$. By Proposition 2.1 G is strongly morphic, so $K \cong G/M$ where M is a normal subgroup of G. Therefore, $G/K \cong M$. Since $|K| = p^k$, $|M| = p^{n-k}$. It follows from Lemma 2.7 that every subgroup of order p^{n-k} is abelian and so is M. Therefore, $G^{'} = K$, a contradiction as $G^{'}$ is cyclic. Thus we must have $|G^{'}| < p^2$. It follows from Proposition 2.4 that G is abelian and we are done. □

Proposition 2.9. *Let G be a finite morphic p-group. Then the following hold:*
(1) $exp(G/G^{'}) = exp(G)$.
(2) $d(G/G^{'}) = d(G)$.
(3) If G is nonabelian, then $\mathfrak{m}/\mathfrak{m}^{'} \cong G/G^{'}$ for every maximal subgroup $\mathfrak{m}$ of G.

Proof. (1) Suppose that $exp(G) = p^k$. Then there exists a cyclic subgroup of order p^k, so by Proposition 2.1, we conclude that there exists a cyclic normal subgroup $\langle b\rangle$ of order p^k. Since every subgroup is an image (by Proposition 2.1), there exists a normal subgroup N such that $G/N \cong \langle b\rangle$. Thus $G^{'} \subseteq N$ and $exp(G) = exp(G/N)$. Since $(G/G^{'})/(N/G^{'}) \cong G/N$, $exp(G/G^{'}) \geq exp(G/N) = exp(G)$. Therefore $exp(G) = exp(G/G^{'})$.

(2) Clearly $d(G/G^{'}) \leq d(G)$. Since $G/\Phi(G) \cong (G/G^{'})/(\Phi(G)/G^{'})$, we conclude that $d(G/G^{'}) \geq d(G/\Phi(G)) = d(G)$, so $d(G/G^{'}) = d(G)$.

(3) Let $z \in G^{'} \cap Z(G)$ be an element of order p. For any maximal subgroup $\mathfrak{m}$ of G, by Lemma 1.3, $\mathfrak{m} \cong G/\langle z\rangle$. Clearly, $(G/\langle z\rangle)^{'} = G^{'}/\langle z\rangle$. So we conclude that $\mathfrak{m}/\mathfrak{m}^{'} \cong (G/\langle z\rangle)/(G^{'}/\langle z\rangle) \cong G/G^{'}$. □

Now we are ready to prove our first main result.

Theorem 2.10. *Let G be a morphic p-group of order p^n $(n > 3)$ and $d(G) = 2$. Then G is abelian.*

Proof. Suppose on the contrary that G is nonabelian. Then by Proposition 2.4 we have $|G'| \geq p^2$. We next divide the proof into the following two cases.

Case 1: $|G'| = p^2$. By Lemma 2.5 G has at least two distinct subgroups of order p. As before, we conclude that G has a normal subgroup which is isomorphic to $C_p \times C_p$. Since G has an abelian subgroup of order p^3 (every group of order p^4 has an abelian subgroup of order p^3 by [11, Proposition 6.5.1]) and by Lemma 2.2 every subgroup of G can be generated by two elements, we conclude that G has a subgroup which is isomorphic to C_{p^2}. By Proposition 2.1, there exist normal subgroups H and K such that $H \cong C_p \times C_p$ and $K \cong C_{p^2}$. Since G is strongly morphic, G/H and G/K are isomorphic to normal subgroups of order p^{n-2} that are abelian by Lemma 2.7. Therefore, $G' \subseteq H \bigcap K$, a contradiction.

Case 2: $|G'| \geq p^3$. Then $n \geq 5$. We first show that $G' = \Phi(G)$. If $n = 5$, then since $d(G) = 2$, we have $|G'| = p^3 = |\Phi(G)|$ and thus $G' = \Phi(G)$. So we may assume that $n \geq 6$. By Lemma 2.2 every subgroup of G can be generated by two elements, so it follows from [2, Theorem 4.2, and Theorem 5.1] that G is either metacyclic or $G' = \Phi(G)$. If G is metacyclic, then G is abelian by Lemma 2.8, a contradiction. Hence, $G' = \Phi(G)$. Note that $|G'| = |\Phi(G)| = p^{n-2}$. Since G' is an image, $G' \cong G/N$ for some normal subgroup N. Since G and G' can be generated two elements, by Lemma 2.6 G' is abelian. So we have $G' \subseteq N$. But $|G'| = p^{n-2}$ and $|N| = p^2$, a contradiction.

In both cases, we have found contradictions. Thus G must be abelian. □

A group G is called the semidirect product of two subgroups A and K, denoted by $G = A \rtimes K$, if A is a normal subgroup such that $G = AK$ and $A \bigcap K = 1$.

Proposition 2.11. *Let G be a morphic p-group of order p^n. Then the following hold:*

(1) $G = A \rtimes K$ where A is a normal subgroup, $K \cong C_{p^k}$ is a subgroup and $p^k = exp(G)$.

(2) If every cyclic subgroup of order p^k is normal, then G is abelian.

Proof. (1) Let $\langle a \rangle$ be a cyclic subgroup of G of order p^k such that $p^k = exp(G)$. By Proposition 2.1 every subgroup is an image, so we obtain that $G/A \cong \langle a \rangle$ for some normal subgroup A of G. Thus there exists an element $b \in G$ such that $G/A = \langle bA \rangle$ and $|\langle bA \rangle| = p^k$. We have $|b| = p^k$ because $exp(G) = p^k$, so $\langle b \rangle \cap A = 1$. Thus $G = A \rtimes \langle b \rangle$ and so (1) holds with $K = \langle b \rangle$.

(2) If every cyclic subgroup of order p^k is normal, then by (1) we have $G = A \times K$ with $K = \langle b \rangle$ and $|b| = p^k$. To show that G is abelian, it suffices to show A is abelian. Let a be any element of A. Then ab has order p^k as b is a central element. Let $H = \langle ab \rangle$. Then $G = A \rtimes H$. Since H is normal, $G = A \times H$. We conclude that ab is central, hence a is central, implying that A is a central subgroup. Therefore, G is abelian and we are done. □

Proposition 2.12. *Let G be a morphic p-group of order p^n $(n > 3)$ and $exp(G) = p^k < p^n$. Then the following hold:*

(1) If $k > 1$, then every subgroup of G of order p^{k+1} is abelian if and only if G is abelian.

(2) If $k = 1$, then every subgroup of G of order p^3 is abelian if and only if G is elementary abelian.

Proof. Clearly in both cases we need only show the forward directions.

(1) Assume that every subgroup of G of order p^{k+1} is abelian. We may also assume that $n \geq k + 2$. Let N be a subgroup of G such that $|N| = p^r \geq p^{k+2}$. If every subgroup of G of order p^{r-1} is abelian, then we show that N is abelian. Assume to the contrary that N is non-abelian. By Lemma 2.3, we have $d(N) = 2$, $|N'| = p$, and $Z(N) = \Phi(N)$. If $Z(N)$ contains

two distinct subgroups $\langle z\rangle$ and $\langle t\rangle$ of order p, then $N/\langle z\rangle$, as an image of G, is isomorphic to a subgroup of G of order p^{r-1} because G is strongly morphic (Proposition 2.1). Since every subgroup of G of order p^{r-1} is abelian, we have $N^{'} \subseteq \langle z\rangle$. Similarly, we have $N^{'} \subseteq \langle t\rangle$. Thus $N^{'} = 1$, a contradiction as $|N^{'}| = p$. Hence, $Z(N) = \langle z\rangle$ is cyclic. Let $N = \langle a, b\rangle$. Since $[a, b] = g \in N^{'}$, $|\langle g\rangle| = p$. Thus $a^p, b^p \in \langle z^p\rangle$. For otherwise, as in the proof of Proposition 2.4, N must contain an element of order $p^{r-1} > p^k$, a contradiction to $exp(G) = p^k$. Since $exp(N/\langle z^p\rangle) = p$, $|N| \leq p^{r-1}$, a contradiction. Therefore, N must be abelian. Now, since every subgroup of G of order p^{k+1} is abelian, by what we just proved every subgroup of G of order p^{k+2} is abelian. By repeating this argument, we conclude that G is abelian as desired.

(2) Assume that N is a subgroup of G of order p^r $(r > 3)$ and every subgroup of G of order p^{r-1} is abelian. We claim that N must be abelian. For otherwise, if N is non-abelain, as above, we have $d(N) = 2$ and $Z(N) = \phi(N)$ and $|N^{'}| = p$. Since $exp(N) = p$, $N^{'} = \Phi(N)$, so $|N| = |\Phi(N)|p^2 = p^3$, a contradiction. Since every subgroup of G of order p^3 is abelian, by what we just proved, we have that every subgroup of G of order p^4 is abelian. By repeating this argument, we conclude that G is abelian. Since $exp(G) = p$, G is elementary abelian. □

We now prove our second main result.

Theorem 2.13. *Let G be a morphic p-group of order p^n with $n > 3$. If $|G^{'}| < p^3$, then G is abelian.*

Proof. Assume to the contrary that G is non-abelian. By Proposition 2.4, $|G^{'}| = p^2$. By Proposition 2.9, we have $G/G^{'} \cong \mathfrak{m}/\mathfrak{m}^{'}$ for each maximal subgroup $\mathfrak{m}$ of G, so $|\mathfrak{m}^{'}| = p$. By [13, Theorem 3.1], we have either $d(G) = 2$ or $d(G) = 3$ and $G^{'} \subseteq Z(G)$ with $G^{'} \cong C_p \times C_p$. If the former is true, i.e. $d(G) = 2$, then G is abelian by Theorem 2.10, a contradiction. Thus we have the latter case.

Next, we show that $G^p \subseteq Z(G)$. For any $x \in G^p$, $y \in G$, $x = a^p$ for some $a \in G$. Since $y^{-1}ay = az$ where $z \in G^{'} \subseteq Z(G)$ and $z^p = 1$, we have $y^{-1}xy = y^{-1}a^p y = a^p = x$, so x is a central element and thus $G^p \subseteq Z(G)$. Now $\Phi(G) = G^{'}G^p \subseteq Z(G)$. By [5, Lemma 1] $Z(G) \subseteq \Phi(G)$, so $Z(G) = \Phi(G)$. If $d(Z(G)) = 3$, then there exists a subgroup

$$H = \langle z_1\rangle \times \langle z_2\rangle \times \langle z_3\rangle \text{ of } Z(G)$$

such that $|\langle z_i\rangle| = p$, $1 \leq i \leq 3$. Since $|\mathfrak{m}^{'}| = p$ and $\mathfrak{m} \cong G/\langle z_i\rangle$ for each i, we conclude that $|G^{'}\langle z_i\rangle/\langle z_i\rangle| = p$. This together with $|G^{'}| = p^2$ implies that $\langle z_i\rangle \subseteq G^{'}$, so $\langle z_1\rangle \times \langle z_2\rangle \times \langle z_3\rangle \subseteq G^{'}$, a contradiction to $d(G^{'}) = 2$. Thus we must have $d(Z(G)) = 2$.

Let $Z(G) = \langle z\rangle \times \langle x\rangle$ such that $|\langle z\rangle| = p^k \geq |\langle x\rangle| = p^s$. We now show that $exp(G) = p^k$. Since $Z(G)$ is an image, there exists a normal subgroup N of G such that $G/N \cong Z(G)$. Since G is morphic, we have $N \cong G/Z(G) = G/\Phi(G) \cong C_p \times C_p \times C_p$. For any subgroup H of G containing $G^{'}$, denote $H/G^{'}$ by $\overline{H}$. Since $G/N \cong (G/G^{'})/(N/G^{'}) = \overline{G}/\overline{N} \cong \langle z\rangle \times \langle x\rangle$, there exist $\overline{a}, \overline{b} \in G/G^{'}$ such that $\overline{G}/\overline{N} = \langle \overline{a}\overline{N}\rangle \times \langle \overline{b}\overline{N}\rangle$ where $|\overline{a}\overline{N}| = p^k$ and $|\overline{b}\overline{N}| = p^s$. Thus $\overline{G} = \langle \overline{a}\rangle\langle \overline{b}\rangle\overline{N}$. If $\langle \overline{a}\rangle\langle \overline{b}\rangle \cap \overline{N} \neq \overline{1}$, then since $|\overline{N}| = p$, $\overline{N} \subseteq \langle \overline{a}\rangle\langle \overline{b}\rangle$. Thus $\overline{G} = \langle \overline{a}\rangle\langle \overline{b}\rangle$ and so $d(G/G^{'}) = d(\overline{G}) = 2$. However, by Proposition 2.9, $d(G/G^{'}) = d(G) = 3$, so we have a contradiction. Thus $\langle \overline{a}\rangle\langle \overline{b}\rangle \cap \overline{N} = \overline{1}$, so $\overline{G} = \langle \overline{a}\rangle\langle \overline{b}\rangle \times \overline{N}$ and $|\langle \overline{a}\rangle\langle \overline{b}\rangle| = p^{k+s}$. Since $\langle \overline{a}\rangle\langle \overline{b}\rangle \cap \overline{N} = \overline{1}$, $\langle \overline{a}\rangle \cap \overline{N} = \overline{1}$ and $\langle \overline{b}\rangle \cap \overline{N} = \overline{1}$. Thus $|\langle \overline{a}\rangle| = |\langle \overline{a}\overline{N}\rangle| = p^k$ and $|\langle \overline{b}\rangle| = |\langle \overline{b}\overline{N}\rangle| = p^s$. Since $|\langle \overline{a}\rangle\langle \overline{b}\rangle| = p^{k+s}$, we have $\langle \overline{a}\rangle \cap \langle \overline{b}\rangle = \overline{1}$. Thus $\overline{G} = \langle \overline{a}\rangle \times \langle \overline{b}\rangle \times \overline{N} \cong C_{p^k} \times C_{p^s} \times C_p$. Since $k \geq s$ and $exp(G/G^{'}) = exp(G)$ (by Proposition 2.9), we have $exp(G) = p^k = exp(\Phi(G))$.

Finally, we divide our proof into two cases according to whether $k = 1$ or not.

Case 1: $k = 1$. Then $|G| = p^5$ because $|\Phi(G)| = |G^{'}| = p^2$ and $d(G) = 3$. Let H be a subgroup of G of order p^3. By Proposition 2.1 $H \cong K$ where K is a normal subgroup of G. Since K is an image, there exists a normal subgroup N of order p^2 such that $G/N \cong K$. Thus $N \cong G/K$ as G is morphic. Since $exp(G) = p$, $N \cong C_p \times C_p \cong G^{'}$. Note that $G/K \cong N \cong G^{'}$. Thus $K \cong G/G^{'}$ is abelian and so is H. Therefore, every subgroup of G of order p^3 is abelian, so it follows from Proposition 2.12 that G is abelian, a contradiction.

Case 2: $k > 1$. Since $exp(G) = p^k$ and $G^p \subseteq \Phi(G) = Z(G)$, we have $exp(G^p) = p^{k-1} \geq p$. This together with $exp(G^{'}) = p$ shows that $exp(\Phi(G)) < p^k$, a contradiction to $exp(\Phi(G)) = p^k$.

In all cases, we have found contradictions. Thus G must be abelian and we are done. □

3. Morphic p-Groups of order p^4, p^5, and p^6

We now apply Theorems 2.10 and 2.13 to prove that a morphic p-group of order p^n with $4 \leq n \leq 6$ is abelian. The following result was first proved in [7] by using [7, Theorem 37 (2)]. However, the proof for [7, Theorem 37 (2)] is incomplete. Here we give a short proof.

Proposition 3.1. [7, Proposition 40] *Let G be a morphic p-group of order p^4. Then G is abelian.*

Proof. Since $|G^{'}| < p^3$, it follows from Theorem 2.13 that G is abelian. □

Theorem 3.2. *Let G be a morphic p-group of order p^5. Then G is abelian.*

Proof. If $d(G) < 3$, by Theorem 2.10, G is abelian. If $d(G) \geq 3$, then $|\Phi(G)| < 3$ and thus $|G^{'}| < p^3$. By Theorem 2.13, G is abelian. □

Theorem 3.3. *Let G be a morphic p-group of order p^6. Then G is abelian.*

Proof. Assume to the contrary that G is non-abelian. In term of Theorems 2.10 and 2.13, we may assume that $d(G) \geq 3$ and $|G^{'}| \geq p^3$. Since $|G| = p^6$, we have $d(G) = 3$ and $|G^{'}| = p^3$, so $G^{'} = \Phi(G)$. By Proposition 2.9, $exp(G) = exp(G/G^{'}) = p$. We first show that all subgroups of G of order p^4 are isomorphic. Let H_1 and H_2 be any two normal subgroups of G of order p^4. By Proposition 2.1, H_1 and H_2 are images of G. So there exist normal subgroups K_1 and K_2 of G of order p^2 such that $G/K_1 \cong H_1$ and $G/K_2 \cong H_2$. Thus $G/H_1 \cong K_1$ and $G/H_2 \cong K_2$. Since $exp(G) = p$, we have $K_1 \cong K_2 \cong C_p \times C_p$. By [7, Theorem 19] $H_1 \cong H_2$ (as G is strongly morphic). Note that it follows from Proposition 2.1 that every subgroup of G is isomorphic to a normal subgroup of G. Thus all subgroups of G of order p^4 are isomorphic. We remark that no subgroup of G of order p^4 is abelian. For otherwise, each such subgroup is abelian, so is every subgroup of G of order p^3. It follows from Proposition 2.12(2) that G is abelian, a contradiction.

We next show that if H is any subgroup of G of order p^4, then $d(H) = 3$ and $|H^{'}| = p$. Since $G/\Phi(G) \cong C_p \times C_p \times C_p$ and G is strongly morphic, G has a normal subgroup N such that $N \cong C_p \times C_p \times C_p$. Note that $N \subseteq \mathfrak{m}$ and $G/\langle z \rangle \cong \mathfrak{m}$ where $\mathfrak{m}$ is a maximal subgroup of G and z is a central element of G of order p. So $N \cong H_0/\langle z \rangle$ for some subgroup H_0 of G of order p^4. Thus $3 = d(N) \leq d(H_0) \leq d(G) = 3$. Since all subgroups H of G of order p^4 are isomorphic, we have $d(H) = 3$ and $|H^{'}| = p$.

Now, by Proposition 2.9, $\mathfrak{m}/\mathfrak{m}^{'} \cong G/G^{'}$ for every maximal subgroup $\mathfrak{m}$ of G. So we have $|\mathfrak{m}^{'}| = p^2$. For every maximal subgroup H of $\mathfrak{m}$, $H^{'} \lhd \mathfrak{m}$. Since $|H^{'}| = p$ (as $|H| = p^4$), $H^{'} \subseteq$

$Z(\mathfrak{m})$. Given elements $x, y \in \mathfrak{m}$, the fact that $d(\mathfrak{m}) = 3$ shows that $x, y \in H$ for some maximal subgroup H of $\mathfrak{m}$. Hence their commutator $[x, y] \in H' \subseteq Z(\mathfrak{m})$ and thus $\mathfrak{m}' \leq Z(\mathfrak{m})$. Since all the maximal subgroups of $\mathfrak{m}$ are isomorphic, [5, Lemma 1] shows that $Z(\mathfrak{m}) \subseteq \Phi(\mathfrak{m}) = \mathfrak{m}'$ (as $exp(\mathfrak{m}) = p$). Thus $\mathfrak{m}' = Z(\mathfrak{m})$ and $|Z(\mathfrak{m})| = p^2$. Since $Z(\mathfrak{m}) = \Phi(\mathfrak{m}) = \mathfrak{m}' \subseteq \Phi(G) \subseteq \mathfrak{m}$, we have $Z(\mathfrak{m}) \subseteq Z(\Phi(G))$. Since $p^2 = |Z(\mathfrak{m})| \leq |Z(\Phi(G))|$ and $|\Phi(G)| = p^3$, we have $\Phi(G)/Z(\Phi(G))$ is cyclic, so $\Phi(G)(= G')$ is abelian. Let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be any two distinct maximal subgroups of G. If $Z(\mathfrak{m}_1) \neq Z(\mathfrak{m}_2)$, then since $Z(\mathfrak{m}_1) \subseteq \Phi(G)$ and $Z(\mathfrak{m}_2) \subseteq \Phi(G)$, $Z(\mathfrak{m}_1)Z(\mathfrak{m}_2) = \Phi(G)$, so $|Z(\mathfrak{m}_1)Z(\mathfrak{m}_2)| = p^3$. Since $Z(\mathfrak{m}_1)Z(\mathfrak{m}_2) = \Phi(G) \leq \mathfrak{m}_1 \cap \mathfrak{m}_2$, we conclude that $Z(\mathfrak{m}_1)Z(\mathfrak{m}_2) \subseteq Z(\mathfrak{m}_1 \cap \mathfrak{m}_2)$, so

$$|Z(\mathfrak{m}_1 \cap \mathfrak{m}_2)| \geq p^3 \quad (*).$$

Note that $|\mathfrak{m}_1 \cap \mathfrak{m}_2| = |\mathfrak{m}_1||\mathfrak{m}_2|/|\mathfrak{m}_1\mathfrak{m}_2| = p^4$. By $(*)$, $\mathfrak{m}_1 \cap \mathfrak{m}_2/Z(\mathfrak{m}_1 \cap \mathfrak{m}_2)$ is cyclic, so $\mathfrak{m}_1 \cap \mathfrak{m}_2$ is an abelian subgroup of order p^4, contradicting our early remark. Next, we assume that $Z(\mathfrak{m}_1) = Z(\mathfrak{m}_2)$ (i.e. ${\mathfrak{m}_1}' = {\mathfrak{m}_2}'$) for any two distinct maximal subgroups $\mathfrak{m}_1$ and $\mathfrak{m}_2$ of G. Then every maximal subgroup of $G/{\mathfrak{m}_1}'$ is of the form $\mathfrak{m}/{\mathfrak{m}_1}' = \mathfrak{m}/\mathfrak{m}'$ which is abelian. Since $|(G/\mathfrak{m}_1')'| = |G'/\mathfrak{m}_1'| = p$, $G/\mathfrak{m}_1'$ is a minimal non-abelian group. By Lemma 2.3, $d(G/\mathfrak{m}_1') = 2$. Since G is strongly morphic and $|G/\mathfrak{m}_1'| = p^4$, $G/\mathfrak{m}_1' \cong H$ where H is a subgroup of order p^4. Thus $d(H) = d(G/\mathfrak{m}_1') = 2$. However, we have shown that $d(H) = 3$, a contradiction. Thus G must be abelian and we are done. □

The question of whether $G \times G$ is morphic given that G is morphic was raised in [7]. The following example provides a negative answer to this question.

Example 3.4. Let $p > 2$ be a prime, and let G be a nonabelian group of order p^3 with presentation $G = \langle a, b, c \mid a^p = b^p = c^p = 1, [a, b] = c, ac = ca, bc = cb\rangle$. Then G is morphic, but $G \times G$ is not morphic.

Proof. First, G is morphic by [7, Proposition 39]. Note that $G \times G$ is a non-abelian p-group of order p^6. By Theorem 3.3 it is not morphic. □

Much evidence suggests that any morphic p-group of order p^n $(n > 3)$ is abelian. We close this paper by making this a conjecture.

Conjecture 3.5. *If G is a morphic group of order p^n with $n > 3$, then G is abelian.*

With Lemma 1.4, this completely describes the finite morphic p-groups, as the abelian case is well known [7, Theorem 14].

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