Strong Law of Large Numbers and Central Limit Theorems for functionals of inhomogeneous Semi-Markov processes

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Abstract: Limit theorems for functionals of classical (homogeneous) Markov renewal and semi-Markov processes have been known for a long time, since the pioneering work of R. Pyke and R. Schaufele (1964). Since then, these processes, as well as their time-inhomogeneous generalizations, have found many applications, for example in finance and insurance. Unfortunately, no limit theorems have been obtained for functionals of inhomogeneous Markov renewal and semi-Markov processes as of today, to the best of the authors' knowledge. In this paper, we provide strong law of large numbers and central limit theorem results for such processes. In particular, we make an important connexion of our results with the theory of ergodicity of inhomogeneous Markov chains. Finally, we provide an application to risk processes used in insurance by considering a inhomogeneous semi-Markov version of the well-known continuous-time Markov chain model, widely used in the literature.

Keywords: Markov renewal process, semi-Markov process, inhomogeneous, non-homogeneous, limit theorems, strong law of large numbers, central limit theorem.

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1. Introduction

Consider a classical homogeneous positive J-X process $(J_n, X_n)_{n\in\mathbb{N}}$ taking value in $\mathbb{J}\times\mathbb{R}_+$ (see e.g. [13] section 3.1), where $\mathbb{J}$ is a finite set, $\mathbb{N}$ is the set of non negative integers and $\mathbb{R}_+ := [0,\infty)$. Denoting $T_n := \sum_{k=0}^n X_k$, the process $(J_n, T_n)_{n\in\mathbb{N}}$ is referred to as the associated homogeneous Markov renewal process, or simply "Markov renewal process". Such a process is characterized by the existence of a kernel Q (satisfying certain regularity conditions) such that for $j \in \mathbb{J}$ and $t \in \mathbb{R}_+$:

$$\begin{aligned}&\mathbb{P}[J_{n+1} = j, X_{n+1} \le t | J_k, T_k : k \in [|0,n|]]\\ &= \mathbb{P}[J_{n+1} = j, X_{n+1} \le t | J_n] = Q(J_n, j, t) \text{ a.e.}\end{aligned} \tag{1.1}$$

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For such processes, limit theorems (strong law of large numbers, central limit theorem) for the functionals W_f are well established (see e.g. [9], [13], [14], [18]), where:

$$W_f(n) := \sum_{k=1}^{n} f(J_{k-1}, X_k) \quad \text{or} \quad W_f(n) := \sum_{k=1}^{n} f(J_{k-1}, J_k, X_k). \tag{1.2}$$

Indeed, under some assumptions and denoting $N(t) := \sup\{n \in \mathbb{N} : T_n \leq t\}$ the counting function of the associated semi-Markov process, we know that:

$$\lim_{n\to\infty} \frac{W_f(n)}{n} \text{ and } \lim_{t\to\infty} \frac{W_f(N(t))}{t} \tag{1.3}$$

exist almost surely, and that:

$$\frac{W_f(n) - \alpha n}{\sqrt{n}} \text{ and } \frac{W_f(N(t)) - \widetilde{\alpha} t}{\sqrt{t}} \tag{1.4}$$

converge weakly to a normal distribution for some constants α, $\widetilde{\alpha}$. The question we want to answer in this paper is whether or not (and under which conditions) such results can be obtained in the case of inhomogeneous Markov renewal processes, i.e. when the joint distribution of (J_{n+1}, X_{n+1}) given information at time n depends not only on J_n but on both J_n and $T_n := \sum_{k=0}^{n} X_k$. Inhomogeneous Markov renewal processes and semi-Markov processes have been introduced in various works and have many applications in finance and insurance (see e.g. [4],[5], [13] chapter 3, section 21), and that is why limit theorems related to those are of interest. To the best of the authors' knowledge, no such theorems have been obtained as of today. In [13], it is said (Remark 21.1, chapter 3): "*In the non-homogeneous case, it is much more difficult to obtain asymptotic results (see for example [1], [17], [20]) for interesting theoretical results)*". Unfortunately, these latter papers deal mainly with limits of expected quantities, sometimes under very strong assumptions, and do not provide any strong law of large numbers or central limit theorem results.

Limit theorems in the homogeneous case are based on two pillars:

1. an ergodicity condition on the Markov chain $\{J_n\}$,
2. the independence of the sojourn times $\{X_n\}$, conditionally on the Markov chain $\{J_n\}$.

In the inhomogeneous case, $\{J_n\}$ is not a Markov chain anymore, but it can be viewed - as we will see - as an inhomogeneous Markov chain, conditionally on the sequence $\{T_n\}$. Therefore, for the first condition above, we will be able to use the theory of ergodicity of inhomogeneous Markov chains (see e.g. [3] section 6.8, [6], [19]). For the second condition, the X_n's will not be independent anymore in the inhomogeneous case (conditionally on the J_n's). Therefore, we will typically require that they are asymptotically uncorrelated in some sense to be made clear.

Our law of large numbers results will use [16], which investigates conditions under which the law of large numbers holds for a (weakly) correlated sequence of random variables. The price to pay for the non independence of the X_n's (conditionally on the

J_n's) will be the L^2 boundedness of the functionals f. For our central limit theorem results, we will use a martingale approach similar to the one presented in [9], [14], but of course different because of the time-inhomogeneity. In particular, we will have to estimate rates of convergence of various time-dependent quantities.

Finally, we apply our results to risk processes used in insurance: indeed, in several cases, risk exhibits seasonality, and the use of a time-dependent model is particularly relevant. In particular, one might desire to have a different transition matrix between states depending on the time of the year to have, say, more "bad" states in the winter than in the summer. We consider a inhomogeneous semi-Markov version of the well-known continuous-time Markov chain model (which is a specific case of homogeneous semi-Markov process), which has had various applications in the literature, see for example [2], [10], [21] for interesting applications in finance.

Throughout the paper, we let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and we consider on it an inhomogeneous J-X process $(J_n, X_n)_{n\in\mathbb{N}}$ taking value in $\mathbb{J}\times\mathbb{R}_+$ (where $\mathbb{J}$ is a finite set) with kernel $Q = \{Q_s(i,j,t) : i,j,s,t \in \mathbb{J}^2\times\mathbb{R}_+^2\}$ (see [13] chapter 3, section 21). We let $X_0 := 0$, $T_n := \sum_{k=0}^n X_k$ so that $(J_n, T_n)_{n\in\mathbb{N}}$ is an inhomogeneous Markov renewal process (in the sense of [13] chapter 3, section 21), namely for $j \in \mathbb{J}$ and $t \in \mathbb{R}_+$:

$$\begin{aligned} &\mathbb{P}[J_{n+1} = j, X_{n+1} \le t | J_k, T_k : k \in [|0,n|]] \\ &= \mathbb{P}[J_{n+1} = j, X_{n+1} \le t | J_n, T_n] = Q_{T_n}(J_n, j, t) \text{ a.e.} \end{aligned} \tag{1.5}$$

We also introduce the following notations which will be useful, for $i, j \in \mathbb{J}$ and $s, t \in \mathbb{R}_+$:

$$P_s(i,j) := \mathbb{P}[J_{n+1} = j | J_n = i, T_n = s] = \lim_{t\to\infty} Q_s(i,j,t), \tag{1.6}$$

$$H_s(i,j,t) := \mathbb{P}[X_{n+1} \le t | J_n = i, J_{n+1} = j, T_n = s], \tag{1.7}$$

$$F_s(i,t) := \mathbb{P}[X_{n+1} \le t | J_n = i, T_n = s] = \sum_{j\in\mathbb{J}} Q_s(i,j,t), \tag{1.8}$$

so that $Q_s(i,j,t) = P_s(i,j)H_s(i,j,t)$.

In the following, we will focus on the general case $W_f(n) := \sum_{k=1}^n f(J_{k-1}, J_k, X_k)$, and present the results related to the case $W'_f(n) := \sum_{k=1}^n f(J_{k-1}, X_k)$ as corollaries, highlighting when needed the differences in terms of assumptions required.

2. Strong law of large numbers for functionals of inhomogeneous Markov renewal and semi-Markov processes

Let us first recall corollary 11 of [16], which will be used in this section:

Theorem 2.1. ***([16], corollary 11)*** *Assume that the Z_n's are random variables satisfying $\sup_n var(Z_n) < \infty$ and that there exists a non negative function h such that* *for every non negative integers n, m:*

$$cov(Z_n, Z_m) \leq h(|m-n|), \tag{2.1}$$

$$with \sum_{n\geq 1} \frac{h(n)}{n} < \infty. \tag{2.2}$$

Then the strong law of large numbers holds, namely:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (Z_k - \mathbb{E}(Z_k)) = 0 \text{ } \boldsymbol{a.e.} \tag{2.3}$$

Other results similar to theorem 2.1 exist in the literature (see e.g. [12]). They allow $var(Z_n)$ to be unbounded as $n \to \infty$, and the price to pay for this is that the denominator in the series of condition (2.2) is replaced by $n^{1-\delta}$ for some $\delta > 0$. However, in our case, we will typically have $\sup_n var(Z_n) < \infty$, and therefore the most appropriate result to use will be theorem 2.1.

Similarly to the concept of ergodicity for inhomogeneous Markov chains, we will assume that $(J_n, T_n)_{n\in\mathbb{N}}$ is uniformly ergodic in the following sense:

Assumption 2.2. *There exists a probability measure $\pi := \{\pi(i)\}_{i\in\mathbb{J}}$ on $\mathbb{J}$ and a (non negative) function ψ such that for all non negative integers $m \geq n$:*

$$\sup_{s\in\mathbb{R}_+} ||P_s^{n,m} - \Pi|| \leq \psi(m-n), \tag{2.4}$$

$$with \lim_{n\to\infty} \psi(n) = 0 \text{ } and \sum_{n\geq 1} \frac{\psi(n)}{n} < \infty, \tag{2.5}$$

where Π is the matrix with all rows equal to π, $P_s^{n,m}$ the matrix with entries:

$$P_s^{n,m}(i,j) := \mathbb{P}[J_m = j | J_n = i, T_n = s], \tag{2.6}$$

and $||\cdot||$ is the matrix norm:

$$||A|| := \max_i \sum_j |A_{i,j}|. \tag{2.7}$$

Remark 2.3. *When $P_s = P$ doesn't depend on s, we have $P_s^{n,m} = P^{m-n}$ and assumption 2.2 coincides with the classical notion of uniform ergodicity for a homogeneous Markov chain, with $\psi(n) = C\delta^n$ for some $\delta \in (0,1)$ and $C \geq 0$.*

It is natural to wonder under which conditions assumption 2.2 is satisfied. The distribution of J_{n+1} conditionally on information available at time n, $\mathcal{F}_n := \sigma\{J_k, T_k : 0 \leq k \leq n\}$ is determined by the kernel P_{T_n}, where we recall that $P_{T_n}(J_n, j) = \mathbb{P}[J_{n+1} = j | J_n, T_n]$. Assume for a moment only that the T_n's are deterministic. In this case, $\{J_n\}$ would be a classical inhomogeneous Markov chain with the associated family of kernels $\{P_{T_n}\}$. Precisely, starting at (J_n, T_n), the distribution of J_m given $\mathcal{F}_n$ would be determined by the product of kernels $\Pi_{k=n}^{m-1} P_{T_k}$. Ergodicity of such inhomogeneous Markov chains has been studied in various works, such as [3] section 6.8, [19], or [6] which provides quantitative bounds for the convergence of such inhomogeneous Markov chains. Now, in our case, the T_n's are not deterministic, but we would still want to relate assumption 2.2 to the ergodicity of inhomogeneous Markov chains. As mentioned earlier, the distribution of J_m given $\mathcal{F}_n$ is roughly speaking determined by $\Pi_{k=n}^{m-1} P_{T_k}$. Therefore, intuitively, one could think that (2.4) would be satisfied if we required the *a priori* stronger condition:

$$\sup_{s \in \mathbb{R}_+} \sup_{t \geq s} \sup_{\rho \in \rho([s,t], m-n)} \left|\left| \prod_{k=0}^{m-n-1} P_{t_k^\rho} - \Pi \right|\right| \leq \psi(m-n), \tag{2.8}$$

where the product operator $\prod$ applies the product of the *right* (that is, $\prod_{k=0}^{m-n-1} P_{t_k^\rho} = P_{t_0^\rho} ... P_{t_{n-m-1}^\rho}$), $\rho([s,t], m-n)$ is the set of partitions ρ of the interval $[s,t]$ consisting of $m-n$ points and $\{t_k^\rho\}_{k=0..m-n-1}$ the points of the partition ρ, where $t_0^\rho = s$ and $t_{m-n-1}^\rho = t$. In the previous formula, we recall that P_t represents the matrix with entries $P_t(i,j)$. We will prove in the next proposition that indeed, (2.4) is satisfied if (2.8) is. Although it seems relatively obvious, this remark is crucial because it allows us to make a direct link between assumption 2.2 and the ergodicity of inhomogeneous Markov chains.

Proposition 2.4. *For any $n, m \in \mathbb{N}$ and $s \in \mathbb{R}_+$ we have:*

$$||P_s^{n,n+m} - \Pi|| \leq \sup_{t \geq s} \sup_{\rho \in \rho([s,t], m)} \left|\left| \prod_{k=0}^{m-1} P_{t_k^\rho} - \Pi \right|\right|, \tag{2.9}$$

where we have used the notations of (2.4) and (2.8).

Proof. The proof is a little heavy, and cannot be done by induction (to the best of the authors' knowledge). The case $m = 1$ is trivial, because $P_s^{n,n+1} = P_s$. We will first prove the case $m = 2$ with care. The method will be identical for general m (except that notations will be heavier). For $m = 2$ we want to prove that:

$$||P_s^{n,n+2} - \Pi|| \leq \sup_{t \geq s} ||P_s P_t - \Pi|| \,. \tag{2.10}$$

We have:

$$P_s^{n,n+2}(i,j) = \mathbb{P}[J_{n+2} = j | J_n = i, T_n = s] \tag{2.11}$$

$$= \int_0^\infty \sum_{r \in \mathbb{J}} \mathbb{P}[J_{n+2} = j | J_{n+1} = r, T_{n+1} = t + s] \mathbb{P}[J_{n+1} = r, X_{n+1} \in dt | J_n = i, T_n = s] \tag{2.12}$$

$$= \int_0^\infty \sum_{r \in \mathbb{J}} P_{s+t}(r,j) Q_s(i,r,dt) = \int_0^\infty \sum_{r \in \mathbb{J}} P_{s+t}(r,j) P_s(i,r) H_s(i,r,dt), \tag{2.13}$$

and therefore, denoting $\overline{H}_s(t) := \max_{i,j} H_s(i,j,t)$:

$$\mathbb{P}[J_{n+2} = j | J_n = i, T_n = s] - \pi_j = \int_0^\infty \sum_{r \in \mathbb{J}} P_{s+t}(r,j) P_s(i,r) H_s(i,r,dt) - \pi_j \tag{2.14}$$

$$\leq \int_0^\infty \underbrace{\sum_{r \in \mathbb{J}} P_{s+t}(r,j) P_s(i,r)}_{P_s P_{s+t}(i,j)} \overline{H}_s(dt) - \pi_j \tag{2.15}$$

$$= \int_0^\infty [P_s P_{s+t}(i,j) - \pi_j] \overline{H}_s(dt) \leq \int_0^\infty |P_s P_{s+t}(i,j) - \pi_j| \overline{H}_s(dt). \tag{2.16}$$

Now, denote $\underline{H}_s(t) := \min_{i,j} H_s(i,j,t)$. We have on the other hand:

$$\mathbb{P}[J_{n+2} = j | J_n = i, T_n = s] - \pi_j \tag{2.17}$$

$$\geq \int_0^\infty \sum_{r \in \mathbb{J}} P_{s+t}(r,j) P_s(i,r) \underline{H}_s(dt) - \pi_j = \int_0^\infty [P_s P_{s+t}(i,j) - \pi_j] \underline{H}_s(dt) \tag{2.18}$$

$$\geq -\int_0^\infty |P_s P_{s+t}(i,j) - \pi_j| \underline{H}_s(dt) \geq -\int_0^\infty |P_s P_{s+t}(i,j) - \pi_j| \overline{H}_s(dt) \tag{2.19}$$

This proves that:

$$|\mathbb{P}[J_{n+2} = j | J_n = i, T_n = s] - \pi_j| \leq \int_0^\infty |P_s P_{s+t}(i,j) - \pi_j| \overline{H}_s(dt) \tag{2.20}$$

$$\Rightarrow \sum_{j \in \mathbb{J}} |\mathbb{P}[J_{n+2} = j | J_n = i, T_n = s] - \pi_j| \leq \int_0^\infty \sum_{j \in \mathbb{J}} |P_s P_{s+t}(i,j) - \pi_j| \overline{H}_s(dt) \tag{2.21}$$

$$\Rightarrow ||P_s^{n,n+2} - \Pi|| \leq \int_0^\infty ||P_s P_{s+t} - \Pi|| \, \overline{H}_s(dt) \tag{2.22}$$

$$\Rightarrow ||P_s^{n,n+2} - \Pi|| \leq \sup_{t \geq s} ||P_s P_t - \Pi|| \, . \tag{2.23}$$

The general case m is proved exactly the same way but with heavier notations. We have, denoting $t_{n+p} := s + \sum_{k=1}^p x_{n+k}$:

$$P_s^{n,n+m}(i,j) = \mathbb{P}[J_{n+m} = j | J_n = i, T_n = s] \tag{2.24}$$

$$= \int_0^\infty \sum_{r_{n+1} \in \mathbb{J}} \mathbb{P}[J_{n+m} = j | J_{n+1} = r_{n+1}, T_{n+1} = t_{n+1}] \tag{2.25}$$

$$\times \mathbb{P}[J_{n+1} = r_{n+1}, X_{n+1} \in dx_{n+1} | J_n = i, T_n = s] \tag{2.26}$$

$$= \int_0^\infty \sum_{r_{n+1} \in \mathbb{J}} P_{t_{n+1}}^{n+1,n+m}(r_{n+1}, j) Q_s(i, r_{n+1}, dx_{n+1}) \tag{2.27}$$

$$= \int_0^\infty \sum_{r_{n+1} \in \mathbb{J}} P_{t_{n+1}}^{n+1,n+m}(r_{n+1}, j) P_s(i, r_{n+1}) H_s(i, r_{n+1}, dx_{n+1}) \tag{2.28}$$

Now, we iterate this method by replacing in the previous formula the term $P_{t_{n+1}}^{n+1,n+m}(r_{n+1}, j)$ as we did in (2.24)-(2.25). We get overall:

$$P_s^{n,n+m}(i,j) = \int_{\mathbb{R}_+^{m-1}} \sum_{r_{n+1},\ldots,r_{n+m-1} \in \mathbb{J}} P_s(i, r_{n+1}) P_{t_{n+1}}(r_{n+1}, r_{n+2}) \ldots P_{t_{n+m-1}}(r_{n+m-1}, j)$$
$$\times H_{t_{n+m-2}}(r_{n+m-2}, r_{n+m-1}, dx_{n+m-1}) \ldots H_{t_{n+1}}(r_{n+1}, r_{n+2}, dx_{n+2}) H_s(i, r_{n+1}, dx_{n+1}) \tag{2.29}$$

We get the exact same way we did for $m = 2$:

$$P_s^{n,n+m}(i,j) - \pi_j \leq \int_{\mathbb{R}_+^{m-1}} |P_s P_{t_{n+1}} \ldots P_{t_{n+m-1}}(i,j) - \pi_j| \times \overline{H}_{t_{n+m-2}}(dx_{n+m-1}) \ldots \overline{H}_s(dx_{n+1}), \tag{2.30}$$

and also:

$$P_s^{n,n+m}(i,j) - \pi_j \geq -\int_{\mathbb{R}_+^{m-1}} |P_s P_{t_{n+1}} \ldots P_{t_{n+m-1}}(i,j) - \pi_j| \times \underline{H}_{t_{n+m-2}}(dx_{n+m-1}) \ldots \underline{H}_s(dx_{n+1}) \tag{2.31}$$

$$\geq -\int_{\mathbb{R}_+^{m-1}} |P_s P_{t_{n+1}} \ldots P_{t_{n+m-1}}(i,j) - \pi_j| \times \overline{H}_{t_{n+m-2}}(dx_{n+m-1}) \ldots \overline{H}_s(dx_{n+1}) \tag{2.32}$$

which yields:

$$||P_s^{n,n+m} - \Pi|| \leq \int_{\mathbb{R}_+^{m-1}} ||P_s P_{t_{n+1}} \ldots P_{t_{n+m-1}} - \Pi|| \times \overline{H}_{t_{n+m-2}}(dx_{n+m-1}) \ldots \overline{H}_s(dx_{n+1}) \tag{2.33}$$

Taking the supremum over $t \geq s$ and over the partitions ρ of $[s,t]$ with m points yields the desired result. □

Remark 2.5. *Because of the latter proposition 2.4, assume, in the spirit of [19], that the Markov kernels $\{P_s : s \in \mathbb{R}_+\}$ take value in a finite set of Markov kernels, say $\widetilde{P} := \{\widetilde{P}_n : n \in [|1, M|]\}$. Then assumption 2.2 is satisfied if $(\widetilde{P}, \pi)$ is ergodic in the sense of [19], definition 2.2, the latter being characterized explicitly in terms of eigenvalues in their theorem 3.4 (the proof of the latter theorem gives us that the convergence of $P_s^{n,m}$ to Π is actually geometric).*

The question we want to answer is whether or not we can get - as in the homogeneous case - a strong law of large numbers for functionals of the type:

$$W_f(n) := \sum_{k=1}^{n} f(J_{k-1}, J_k, X_k) = \sum_{i,j \in \mathbb{J}} \sum_{k=1}^{N_{i,j}(n)} f(i, j, X_{p(k,i,j)}), \tag{2.34}$$

where $N_{i,j}(n)$ represents the number of times that $(J_{k-1}, J_k) = (i, j)$ for $1 \leq k \leq n$; and $\{p(k, i, j) : k \geq 1\}$ the successive indexes for which $(J_{p(k,i,j)-1}, J_{p(k,i,j)}) = (i, j)$. We will give as corollaries the results related to the functionals $W'_f(n) := \sum_{k=1}^{n} f(J_{k-1}, X_k)$, highlighting the main differences between the two cases. As pointed out in [18], in the homogeneous case, limit theorems for W_f directly follow from those for W'_f by considering the Markov renewal process $(\widetilde{J_n}, T_n)$, where $\widetilde{J_n} := (J_n, J_{n+1})$. In the inhomogeneous case, we cannot use this trick because of the dependence of the kernel Q on $\{T_n\}$. Regarding the Strong Law of Large numbers, the main difference between the cases $W'_f(n)$ and $W_f(n)$ will be that the latter requires the convergence of the kernels P_t to some kernel P (cf. assumption 2.6 below), whereas the former doesn't. In particular, the renewal theorem for inhomogeneous semi-Markov processes - that is, the a.e. convergence of $t^{-1}N(t)$ - will not require the convergence of the kernels P_t to some kernel P and will be established under assumptions 2.2 and 2.20 only. We now introduce the following assumption:

Assumption 2.6. *There exists a Markov kernel $P = \{P(i, j) : i, j \in \mathbb{J}\}$ on $\mathbb{J}$ such that:*

$$\sum_{n \geq 1} \frac{F^{(n)}\psi'}{n} < \infty, \tag{2.35}$$

where:

$$\psi'(t) := \sup_{s \geq t} ||P_s - P||, \tag{2.36}$$

$F^{(n)}$ denotes the cumulative distribution function of T_n and $F^{(n)}\psi' := \mathbb{E}[\psi'(T_n)]$.

Lemma 2.7. *Under assumptions 2.2, 2.6, we have for every $i, j \in \mathbb{J}$:*

$$\lim_{n \to \infty} \frac{N_{i,j}(n)}{n} = \pi(i)P(i, j) \text{ a.e.} \tag{2.37}$$

Proof.

$$\frac{N_{i,j}(n)}{n} = \frac{1}{n}\sum_{k=1}^{n} 1_{\{J_{k-1}=i, J_k=j\}} \tag{2.38}$$

Letting $Y_k := 1_{\{J_{k-1}=i,J_k=j\}}$, we have that $var(Y_k) \leq 1$. If we prove that $cov(Y_m, Y_n) \leq \alpha(|m-n|)$ for some function α satisfying $\sum_{n\geq 1} \frac{\alpha(n)}{n} < \infty$, then we have by theorem 2.1 that:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}(Y_k - \mathbb{E}(Y_k)) = 0 \text{ a.e.} \tag{2.39}$$

But:

$$\begin{aligned}
&\mathbb{E}(Y_k) = \mathbb{E}(1_{\{J_{k-1}=i\}}\mathbb{E}(1_{\{J_k=j\}}|J_{k-1}, T_{k-1})) \\
&= \mathbb{E}(1_{\{J_{k-1}=i\}}P_{T_{k-1}}(i,j)) \\
&= \mathbb{E}(1_{\{J_{k-1}=i\}}(P_{T_{k-1}}(i,j) - P(i,j))) + P(i,j)\mathbb{P}(J_{k-1}=i) \\
&\Rightarrow |\mathbb{E}(Y_k) - P(i,j)\pi(i)| \leq F^{(k-1)}\psi' + |\mathbb{P}(J_{k-1}=i) - \pi(i)|
\end{aligned} \tag{2.40}$$

By assumption 2.2, $\mathbb{P}(J_{k-1}=i) \to \pi(i)$ as $k\to\infty$. By assumption 2.6, $F^{(k)}\psi' \to 0$ as $k\to\infty$, since ψ' is non increasing. This shows that $\mathbb{E}(Y_k) \to P(i,j)\pi(i)$ as $k\to\infty$. Now, we have for $m > n$:

$$\begin{aligned}
&\mathbb{E}(Y_mY_n) = \mathbb{E}(1_{\{J_{m-1}=i,J_m=j\}}1_{\{J_{n-1}=i,J_n=j\}}) \\
&= \mathbb{E}(1_{\{J_{n-1}=i,J_n=j\}}1_{\{J_{m-1}=i\}}\mathbb{E}(1_{\{J_m=j\}}|J_k, T_k : k \leq m-1)) \\
&= \mathbb{E}(1_{\{J_{n-1}=i,J_n=j\}}1_{\{J_{m-1}=i\}}P_{T_{m-1}}(i,j)) \\
&\leq P(i,j)\mathbb{E}(1_{\{J_{n-1}=i,J_n=j\}}1_{\{J_{m-1}=i\}}) + F^{(m-1)}\psi' \\
&= P(i,j)\mathbb{E}(1_{\{J_{n-1}=i,J_n=j\}}\mathbb{E}(1_{\{J_{m-1}=i\}}|J_k, T_k : k \leq n)) + F^{(m-1)}\psi' \\
&= P(i,j)\mathbb{E}(1_{\{J_{n-1}=i,J_n=j\}}P^{n,m-1}_{T_n}(j,i)) + F^{(m-1)}\psi' \\
&\leq \psi(m-1-n) + P(i,j)\Pi(j,i)\mathbb{E}(Y_n) + F^{(m-1)}\psi' \\
&= \psi(m-1-n) + P(i,j)\pi(i)\mathbb{E}(Y_n) + F^{(m-1)}\psi'.
\end{aligned} \tag{2.41}$$

Therefore:

$$cov(Y_m, Y_n) \leq \psi(m-1-n) + |P(i,j)\pi(i) - \mathbb{E}(Y_m)| + F^{(m-1)}\psi' \tag{2.42}$$

We have shown in (2.40) that:

$$|P(i,j)\pi(i) - \mathbb{E}(Y_m)| \leq F^{(m-1)}\psi' + |\mathbb{P}(J_{m-1}=i) - \pi(i)|. \tag{2.43}$$

We have, using assumption 2.2:

$$\pi(i) - \mathbb{P}(J_m = i) = \int_0^\infty \sum_{j\in\mathbb{J}}(\pi(i) - \mathbb{P}(J_m = i|J_n = j, T_n = u))\mathbb{P}(J_n = j, T_n \in du) \tag{2.44}$$

$$\Rightarrow |\pi(i) - \mathbb{P}(J_m = i)| \leq \int_0^\infty \sum_{j\in\mathbb{J}} \psi(m-n)\mathbb{P}(J_n = j, T_n \in du) = \psi(m-n), \tag{2.45}$$

which yields $|\mathbb{P}(J_{m-1}=i)-\pi(i)| \leq \psi(m-1-n)$. Also, since ψ' is non increasing we get $F^{(m-1)}\psi' \leq F^{(m-n)}\psi'$. Overall we obtain:

$$cov(Y_m, Y_n) \leq 2\psi(m-1-n) + 2F^{(m-n)}\psi' =: \widetilde{\phi}(m-n). \tag{2.46}$$

Since $\sum_{n\geq 1} \frac{\widetilde{\phi}(n)}{n} < \infty$ by assumption, we are done. □

Now, if we denote $F^{(n)}(t) := \mathbb{P}(T_n \leq t)$, we can obtain the latter by defining the quantities $Q^{(n)}$ recursively the following way:

$$Q^{(1)}(j,t) := \mathbb{P}(T_1 \leq t, J_1 = j) = \sum_{i\in\mathbb{J}} Q_0(i,j,t)\mathbb{P}(J_0 = i), \tag{2.47}$$

$$Q^{(n)}(j,t) := \mathbb{P}(T_n \leq t, J_n = j) = \int_0^\infty \sum_{i\in\mathbb{J}} Q_s(i,j,t-s)Q^{(n-1)}(i,ds), \tag{2.48}$$

$$F^{(n)}(t) = \sum_{j\in\mathbb{J}} Q^{(n)}(j,t). \tag{2.49}$$

For a measure μ on some probability space, denote $||\cdot||_{p,\mu}$ the $L^p(\mu)-$norm. If μ is a measure on $\mathbb{R}_+$ and $g \in L^1(\mathbb{R}_+,\mu)$, we denote:

$$\mu g := \int_0^\infty g(t)\mu(dt). \tag{2.50}$$

For the next theorem, we would like that the sojourn times X_n are asymptotically uncorrelated (conditionally on $\{J_n\}$) in some sense, to be able to use theorem 2.1. The sojourn times are strictly uncorrelated (conditionally on $\{J_n\}$) when the cumulative distribution functions $\{H_t(i,j,\cdot) : t \in \mathbb{R}_+, i,j \in \mathbb{J}\}$ do not depend on t. We will therefore require that these functions tend - in some sense - to some cumulative distribution function $H(i,j,\cdot)$ as $t \to \infty$.

Let $\{H(i,j,\cdot)\}_{i,j\in\mathbb{J}}$ a collection of cumulative distribution functions on $\mathbb{R}_+$ (fixed throughout the rest of the paper). For a function $g : \mathbb{R}_+ \to \mathbb{R}$ such that $g \in \bigcap_{i,j\in\mathbb{J}} L^1(H(i,j,\cdot))$ and $\sup_{\substack{t\in\mathbb{R}_+\\ i,j\in\mathbb{J}}} H_t(i,j,\cdot)|g| = \sup_{\substack{t\in\mathbb{R}_+\\ i,j\in\mathbb{J}}} \int_0^\infty |g(u)|H_t(i,j,du) < \infty$, we let:

$$\phi_g(t) := \sup_{\substack{s\geq t\\ i,j\in\mathbb{J}}} |H_s(i,j,\cdot)g - H(i,j,\cdot)g| = \sup_{\substack{s\geq t\\ i,j\in\mathbb{J}}} \left| \int_0^\infty g(u)H_s(i,j,du) - \int_0^\infty g(u)H(i,j,du) \right|. \tag{2.51}$$

If the domain of g is $\mathbb{J}^n \times \mathbb{R}_+$ for some n, then we will simply let $\phi_g(t) := \max_{i\in\mathbb{J}^n} \phi_{g(i,\cdot)}(t)$.

Remark 2.8. *We didn't choose $\phi_g(t)$ to be equal to $\max_{i,j\in\mathbb{J}} |H_t(i,j,\cdot)g - H(i,j,\cdot)g|$ because in the proof of the next theorem (theorem 2.14), we will need that ϕ_g is non* *increasing (in the variable t). Theoretically, one could choose ϕ_g to be any non increasing function (in the variable t) of the quantity $\max_{i,j\in\mathbb{J}}|H_t(i,j,\cdot)g - H(i,j,\cdot)g|$.*

Assumption 2.9. *We say that a function $g : \mathbb{R}_+ \to \mathbb{R}$ satisfies this assumption if there exists $p, q \in [1,\infty]$ with $\frac{1}{p}+\frac{1}{q} = 1$ such that:*

$$g \in \bigcap_{i,j\in\mathbb{J}} L^1(H(i,j,\cdot)) \text{ and } \sup_{\substack{t\in\mathbb{R}_+ \\ i,j\in\mathbb{J}}} ||g||_{2\vee q, H_t(i,j,\cdot)} < \infty, \tag{2.52}$$

$$\sum_{n\geq 1} \frac{||\phi_g||_{p,F^{(n)}}}{n} < \infty, \tag{2.53}$$

where we recall - since $F^{(n)}(t) = \mathbb{P}(T_n \leq t)$ by definition - that $||\phi_g||_{p,F^{(n)}} = ||\phi_g(T_n)||_{p,\mathbb{P}} = \mathbb{E}[|\phi_g(T_n)|^p]^{\frac{1}{p}}$, and:

$$||g||_{2\vee q, H_t(i,j,\cdot)} = \left(\int_0^\infty |g(u)|^{2\vee q} H_t(i,j,du)\right)^{\frac{1}{2\vee q}}. \tag{2.54}$$

Remark 2.10. *In the previous assumption, note that $||\phi_g||_{p,F^{(n)}}$ is well defined since:*

$$||\phi_g||_{p,F^{(n)}} \leq \sup_{\substack{t\in\mathbb{R}_+ \\ i,j\in\mathbb{J}}} |H_t(i,j,\cdot)g| + |H(i,j,\cdot)g| < \infty. \tag{2.55}$$

Remark 2.11. *When $H_t(i,j,\cdot) = H(i,j,\cdot)$ $\forall i,j \in \mathbb{J}$ doesn't depend on t (homogeneous case), we have $\phi_g = 0$ and therefore assumption 2.9 is automatically fulfilled with $(p,q) = (\infty, 1)$ for every function $g \in \bigcap_{i,j\in\mathbb{J}} L^2(H(i,j,\cdot))$. The price to pay for the obtention of a law of large numbers for inhomogeneous Markov renewal processes is a L^2 integrability condition on g, which is not required in the homogeneous case where the X_n's are independent (conditionally on the Markov chain $\{J_n\}$).*

Remark 2.12. *Note that assumption 2.9 does not require the convergence of the kernels Q_{T_n} as $n \to \infty$ to some kernel Q, but only the convergence of H_{T_n} to H in some L^p sense, namely $||\phi_g||_{p,F^{(n)}} \to 0$ (indeed, since $||\phi_g||_{p,F^{(n)}}$ is non increasing in n, the convergence of the series in assumption 2.9 implies $||\phi_g||_{p,F^{(n)}} \to 0$). In particular, this assumption doesn't require the convergence of the probabilities $P_{T_n}(i,j) = \mathbb{P}[J_{n+1} = j|J_n = i, T_n]$.*

Remark 2.13. *Take the case where $\lim_{n\to\infty} T_n = +\infty$ a.e.. In this case, the Markov renewal process is said to be regular, see definition 2.15 below. Assumption 2.9 requires in particular $||\phi_g||_{p,F^{(n)}} \to 0$, which is true if $\lim_{t\to\infty} \phi_g(t) = 0$ since ϕ_g is non increasing and $\lim_{n\to\infty} T_n = +\infty$ a.e.. But $\lim_{t\to\infty} \phi_g(t) = 0$ means $\mathbb{E}[g(X(i,j,t))] \to \mathbb{E}[g(X(i,j))]$ as $t \to \infty$ for every $i,j \in \mathbb{J}$, where the random variables $X(i,j,t)$,*

$X(i,j)$ have respective laws $H_t(i,j,\cdot), H(i,j,\cdot)$. By the dominated convergence theorem for uniformly integrable random variables, sufficient conditions to have the latter are $g(X(i,j,t)) \Rightarrow g(X(i,j))$ as $t \to \infty$ for every $i,j \in \mathbb{J}$ - where $\Rightarrow$ denotes weak convergence - and the collection of random variables $\{g(X(i,j,t)) : i,j \in \mathbb{J}, t \in \mathbb{R}_+\}$ are uniformly integrable (see [7], appendix, proposition 2.3).

Example. Assumption 2.9 requires a knowledge of the behavior of $F^{(n)}$. A simple case is the following: if we have $\mathbb{P}(X_n \geq \underline{\tau}) = 1$ for some $\underline{\tau} > 0$ and $\phi_g(t) = O(t^{-\delta})$ for some $\delta > 0$, then we have $T_n \geq n\underline{\tau}$ a.e. and because ϕ_g is non negative and non increasing, then $||\phi_g||_{\infty,F^{(n)}} = O(n^{-\delta})$. Therefore assumption 2.9 is fulfilled with $(p,q) = (\infty, 1)$ for every function g satisfying:

$$\sup_{\substack{t\in\mathbb{R}_+ \\ i,j\in\mathbb{J}}} ||g||_{2,H_t(i,j,\cdot)} < \infty. \tag{2.56}$$

The following theorem is a strong law of large numbers for functionals of inhomogeneous Markov renewal processes and is one of the main results of the paper:

Theorem 2.14. *Assume that assumptions 2.2, 2.6 hold true, that f is a function $\mathbb{J} \times \mathbb{J} \times \mathbb{R}_+ \to \mathbb{R}$ such that for every $i,j \in \mathbb{J}$, $f(i,j,\cdot)$ satisfies assumption 2.9. Then we have:*

$$\lim_{n\to\infty} \frac{W_f(n)}{n} = \sum_{i,j\in\mathbb{J}} \pi(i)Q(i,j,\cdot)f(i,j,\cdot) \ \textit{a.e.}, \tag{2.57}$$

where $Q(i,j,\cdot) := P(i,j)H(i,j,\cdot)$.

Proof. We have:

$$\frac{W_f(n)}{n} = \sum_{i,j\in\mathbb{J}} \frac{N_{i,j}(n)}{n} \frac{1}{N_{i,j}(n)} \sum_{k=1}^{N_{i,j}(n)} f(i,j,X_{p(k,i,j)}) \tag{2.58}$$

Because of Lemma 2.7, we only need to show that:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(i,j,X_{p(k,i,j)}) = H(i,j,\cdot)f(i,j,\cdot) \text{ a.e.} \tag{2.59}$$

Again, we want to apply theorem 2.1. First we will show that $\lim_{k\to\infty} \mathbb{E}[f(i,j,X_{p(k,i,j)})] = H(i,j,\cdot)f(i,j,\cdot)$. Because $f(i,j,\cdot)$ satisfies assumption 2.9, the integrals below exist and we have (recall that by definition of the index $p(k,i,j)$ we have $J_{p(k,i,j)-1} = i$ and

$J_{p(k,i,j)} = j$):

$$\begin{aligned}
&|\mathbb{E}[f(i,j,X_{p(k,i,j)})] - H(i,j,\cdot)f(i,j,\cdot)| \\
=&|\mathbb{E}[\mathbb{E}[f(i,j,X_{p(k,i,j)})|J_{p(k,i,j)-1} = i, J_{p(k,i,j)} = j, T_{p(k,i,j)-1}]] - H(i,j,\cdot)f(i,j,\cdot)| \\
=&|\mathbb{E}[H_{T_{p(k,i,j)-1}}(i,j,\cdot)f(i,j,\cdot) - H(i,j,\cdot)f(i,j,\cdot)]| \\
\leq&\mathbb{E}[\phi_f(T_{p(k,i,j)-1})] \leq \mathbb{E}[\phi_f(T_{k-1})] = F^{(k-1)}\phi_f.
\end{aligned} \tag{2.60}$$

The last inequality holds true because ϕ_f is non increasing and we have $k \leq p(k,i,j)$ a.e. for every i,j, which implies $T_{k-1} \leq T_{p(k,i,j)-1}$ a.e..

Because $||\phi_f||_{p,F^{(n)}}$ is non negative and non increasing in n, assumption 2.9 implies that:

$$\lim_{n\to\infty} ||\phi_f||_{p,F^{(n)}} = 0 \tag{2.61}$$

Because by Hölder's inequality we have $0 \leq F^{(n)}\phi_f \leq ||\phi_f||_{p,F^{(n)}}$, we get that:

$$\lim_{n\to\infty} F^{(n)}\phi_f = 0 \tag{2.62}$$

and therefore we obtain $\lim_{k\to\infty} \mathbb{E}[f(i,j,X_{p(k,i,j)})] = H(i,j,\cdot)f(i,j,\cdot)$.

Now let $\eta_r := \sup_{k\geq 1, i,j\in\mathbb{J}} ||f(i,j,X_{p(k,i,j)})||_{r,\mathbb{P}}$. Since we have:

$$||f(i,j,X_{p(k,i,j)})||_{r,\mathbb{P}} = \left(\mathbb{E}[|f(i,j,X_{p(k,i,j)})|^r]\right)^{\frac{1}{r}} \tag{2.63}$$

$$= \left(\mathbb{E}[\mathbb{E}[|f(i,j,X_{p(k,i,j)})|^r | J_{p(k,i,j)-1} = i, J_{p(k,i,j)} = j, T_{p(k,i,j)-1}]]\right)^{\frac{1}{r}} \tag{2.64}$$

$$= \left(\mathbb{E}[H_{T_{p(k,i,j)-1}}(i,j,\cdot)|f(i,j,\cdot)|^r]\right)^{\frac{1}{r}}, \tag{2.65}$$

we get:

$$\eta_r = \sup_{k\geq 1, i,j\in\mathbb{J}} \left(\mathbb{E}[H_{T_{p(k,i,j)-1}}(i,j,\cdot)|f(i,j,\cdot)|^r]\right)^{\frac{1}{r}} \leq \sup_{s\in\mathbb{R}_+, i,j\in\mathbb{J}} ||f(i,j,\cdot)||_{r,H_s(i,j,\cdot)} \tag{2.66}$$

Therefore $\eta_2 < \infty$ by assumption, which fulfills the first condition of theorem 2.1. Now it remains to prove that $cov(f(i,j,X_{p(m,i,j)}), f(i,j,X_{p(n,i,j)})) \leq \alpha(|m-n|)$ for some function α satisfying $\sum_{n\geq 1} \frac{\alpha(n)}{n} < \infty$.

We have for $m > n$:

$$\begin{aligned}
&\mathbb{E}[f(i,j,X_{p(m,i,j)})f(i,j,X_{p(n,i,j)})] = \mathbb{E}[f(i,j,X_{p(n,i,j)})\mathbb{E}[f(i,j,X_{p(m,i,j)})|J_k, T_k : k \leq p(m,i,j)-1]] \\
=&\mathbb{E}[f(i,j,X_{p(n,i,j)})\mathbb{E}[f(i,j,X_{p(m,i,j)})|J_{p(m,i,j)-1} = i, J_{p(m,i,j)} = j, T_{p(m,i,j)-1})]] \\
=&\mathbb{E}[f(i,j,X_{p(n,i,j)})H_{T_{p(m,i,j)-1}}(i,j,\cdot)f(i,j,\cdot)] \\
=&\mathbb{E}[f(i,j,X_{p(n,i,j)})(H_{T_{p(m,i,j)-1}}(i,j,\cdot)f(i,j,\cdot) - H(i,j,\cdot)f(i,j,\cdot))] + F(i,j,\cdot)f(i,j,\cdot)\mathbb{E}[f(i,j,X_{p(n,i,j)})]
\end{aligned} \tag{2.67}$$

Because $\eta_2 < \infty$, we have $\eta_1 < \infty$ by Hölder's inequality. We get, again using the fact that ϕ_f is non negative, non increasing and $m \leq p(m,i,j) \Rightarrow T_{m-1} \leq T_{p(m,i,j)-1}$ a.e.:

$$\begin{aligned}&|cov(f(i,j,X_{p(m,i,j)}), f(i,j,X_{p(n,i,j)}))|\\ \leq&\mathbb{E}[|f(i,j,X_{p(n,i,j)})|\phi_f(T_{m-1})] + \eta_1|H(i,j,\cdot)f(i,j,\cdot) - \mathbb{E}[f(i,j,X_{p(m,i,j)})]|\end{aligned} \tag{2.68}$$

We have shown in (2.60) that:

$$|H(i,j,\cdot)f(i,j,\cdot) - \mathbb{E}[f(i,j,X_{p(m,i,j)})]| \leq F^{(m-1)}\phi_f. \tag{2.69}$$

Therefore we have ($\eta_q < \infty$ by assumption):

$$\begin{aligned}&|cov(f(i,j,X_{p(m,i,j)}), f(i,j,X_{p(n,i,j)}))|\\ \leq&\eta_q||\phi_f||_{p,F^{(m-1)}} + \eta_1 F^{(m-1)}\phi_f \text{ (by Hölder's inequality)}\\ \leq&2\eta_q||\phi_f||_{p,F^{(m-1)}} \leq 2\eta_q||\phi_f||_{p,F^{(m-n)}},\end{aligned} \tag{2.70}$$

and the proof is complete as $f(i,j,\cdot)$ satisfies assumption 2.9.

□

To get our next result, we will need the notion of regular inhomogeneous Markov renewal process, which will be defined similarly to the homogeneous case (see e.g. [18]).

Definition 2.15. *Similarly to the homogeneous case, the inhomogeneous Markov renewal process $(J_n, T_n)_{n\in\mathbb{N}}$ is said to be regular if $\lim_{n\to\infty} T_n = +\infty$ a.e.. or equivalently, $N(t) < \infty$ a.e. for every $t \in \mathbb{R}_+$, where we denote $N(t) := \sup\{n \in \mathbb{N} : T_n \leq t\}$ (observe that $\{T_n \leq t\} = \{N(t) \geq n\}$).*

Proposition 2.16. *Assume that there exists $\tau > 0$ and $\beta > 0$ such that:*

$$\sup_{\substack{t\in\mathbb{R}_+\\ i\in\mathbb{J}}} F_t(i,\tau) < 1-\beta. \tag{2.71}$$

Then the inhomogeneous Markov renewal process $(J_n, T_n)_{n\in\mathbb{N}}$ is regular.

Proof. The proof is classical and consists in finding a suitable standard renewal process that dominates our inhomogeneous Markov renewal process (see e.g. [15], chapter 4, theorem 4.2). Indeed, consider the standard renewal process $T'_n := \sum_{k=0}^n X'_k$ with $\mathbb{P}(X'_k = 0) = 1-\beta$ and $\mathbb{P}(X'_k = \tau) = \beta$ (the X'_n's being i.i.d.). Because T'_n is a renewal process, we have $T'_n \to \infty$ a.e. and by assumption we have $\mathbb{P}(T_n \leq t) \leq \mathbb{P}(T'_n \leq t)$ for every $t \in \mathbb{R}_+$. Taking the limit as $n \to \infty$ for fixed t and using the dominated convergence theorem, we get $\lim_{n\to\infty} \mathbb{P}(T'_n \leq t) = 0$ and therefore $\lim_{n\to\infty} \mathbb{P}(T_n \leq t) = 0$. By the dominated convergence theorem the latter limit is equal to $\mathbb{P}(\lim_{n\to\infty} T_n \leq t)$, which shows that $T_n \to \infty$ a.e.. □

Corollary 2.17. *Assume that the cumulative distribution functions* $\{F_t(i,\cdot) : t \in \mathbb{R}_+, i \in \mathbb{J}\}$ *take value in a finite set of cumulative distribution functions, say* $\{\widetilde{F}_n : n \in [|1, M|]\}$*. Then* $(J_n, T_n)_{n\in\mathbb{N}}$ *is regular.*

If $(J_n, T_n)_{n\in\mathbb{N}}$ is a regular inhomogeneous Markov renewal process, we can define the associated regular inhomogeneous semi-Markov process $Z(t) := J_{N(t)}$ (similarly to the homogeneous case), where $N(t)$ is the usual counting function up to time t. Since $\{T_n \leq t\} = \{N(t) \geq n\}$, we get that $\forall t$, $N(t) < \infty$ a.e..

Corollary 2.18. *Assume that* $(J_n, T_n)_{n\in\mathbb{N}}$ *is a regular inhomogeneous Markov renewal process and that theorem 2.14 holds true for the function* $t \to t$*. Then we have:*

$$\lim_{t\to\infty} \frac{t}{N(t)} = \Pi m \text{ } a.e., \tag{2.72}$$

where $\Pi m := \sum_{i\in\mathbb{J}} m(i)\pi(i)$ *and* $m(i) := \int_0^\infty tF(i, dt)$*. If in addition theorem 2.14 holds true for the function* $f : \mathbb{J} \times \mathbb{J} \times \mathbb{R}_+ \to \mathbb{R}$ *and* $\Pi m > 0$*, then we have:*

$$\lim_{t\to\infty} \frac{W_f(N(t))}{t} = \frac{1}{\Pi m} \sum_{i,j\in\mathbb{J}} \pi(i)Q(i,j,\cdot)f(i,j,\cdot) \text{ } a.e., \tag{2.73}$$

where $Q(i, j, \cdot) := P(i, j)H(i, j, \cdot)$*.*

Proof. Because the Markov renewal process is regular, we have $N(t) < \infty$ a.e. for each t and $\tau := \inf\{t : N(t) > 0\} = \inf\{T_n : T_n > 0\} > 0$ a.e.. Denote $\Omega^* := \bigcap_{t\in\mathbb{Q}_+} \{N(t) < \infty\} \cap \{\tau > 0\}$. For each $\omega \in \Omega^*$, we have for $t \geq \tau(\omega)$:

$$T_{N(t)} \leq t \leq T_{N(t)+1} \tag{2.74}$$

$$\Rightarrow \frac{1}{N(t)} \sum_{k=1}^{N(t)} X_k \leq \frac{t}{N(t)} \leq \frac{N(t)+1}{N(t)} \frac{1}{N(t)+1} \sum_{k=1}^{N(t)+1} X_k. \tag{2.75}$$

Since $t \leq T_{N(t)+1}$ a.e., taking the limit as $t \to \infty$ yields $\lim_{t\to\infty} T_{N(t)+1} = \infty$ a.e. and therefore $\lim_{t\to\infty} N(t) = \infty$ a.e. by regularity of the Markov renewal process. Therefore $\lim_{t\to\infty} \frac{N(t)+1}{N(t)} = 1$ a.e.. Now, because theorem 2.14 holds true for the function $t \to t$ by assumption, we get:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} X_k = \Pi m \text{ a.e.} \tag{2.76}$$

Since $\lim_{t\to\infty} N(t) = \infty$ a.e. we get:

$$\lim_{t\to\infty} \frac{t}{N(t)} = \Pi m \text{ a.e.} \tag{2.77}$$

Now to get the second result, we must just observe that:

$$\frac{W_f(N(t))}{t} = \frac{W_f(N(t))}{N(t)} \frac{N(t)}{t}. \tag{2.78}$$

and apply theorem 2.14. □

To comment briefly on the case $W'_f(n) := \sum_{k=1}^{n} f(J_{k-1}, X_k)$, it turns out that assumption 2.6 - i.e. the convergence of the kernels P_t to some P - will not be required. Indeed, denoting $N_i(n) := \sum_{j\in\mathbb{J}} N_{i,j}(n)$, we have the following lemma which is similar to lemma 2.7 and that we quote without proof:

Lemma 2.19. *Under assumption 2.2, we have for every $i \in \mathbb{J}$:*

$$\lim_{n\to\infty} \frac{N_i(n)}{n} = \pi(i) \ a.e. \tag{2.79}$$

Further, because we have:

$$W'_f(n) = \sum_{k=1}^{n} f(J_{k-1}, X_k) = \sum_{i\in\mathbb{J}} \frac{N_i(n)}{n} \frac{1}{N_i(n)} \sum_{k=1}^{N_i(n)} f(i, X_{p(k,i)}), \tag{2.80}$$

where $\{p(k,i) : k \geq 1\}$ are the successive indexes for which $J_{p(k,i)-1} = i$, it turns out that the equivalents of theorem 2.14 and corollary 2.18 can be proved under an assumption similar to 2.9, involving the convergence of the CDF's $\{F_t(i,\cdot)\}$ to some $\{F(i,\cdot)\}$, instead of the convergence of $\{H_t(i,j,\cdot)\}$ to $\{H(i,j,\cdot)\}$. Therefore we let $\{F(i,\cdot)\}_{i\in\mathbb{J}}$ a collection of cumulative distribution functions on $\mathbb{R}_+$ and we introduce the following assumption:

Assumption 2.20. *We say that a function $g : \mathbb{R}_+ \to \mathbb{R}$ satisfies this assumption if there exists $p, q \in [1,\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$ such that:*

$$g \in \bigcap_{i\in\mathbb{J}} L^1(F(i,\cdot)) \ \text{and} \ \sup_{\substack{t\in\mathbb{R}_+\\ i\in\mathbb{J}}} ||g||_{2\vee q, F_t(i,\cdot)} < \infty, \tag{2.81}$$

$$\sum_{n\geq 1} \frac{||\phi'_g||_{p,F^{(n)}}}{n} < \infty, \tag{2.82}$$

where:

$$\phi'_g(t) := \sup_{\substack{s\geq t\\ i\in\mathbb{J}}} |F_s(i,\cdot)g - F(i,\cdot)g| = \sup_{\substack{s\geq t\\ i\in\mathbb{J}}} \left| \int_0^\infty g(u)F_s(i,du) - \int_0^\infty g(u)F(i,du) \right|. \tag{2.83}$$

We can now quote the following equivalents of theorem 2.14 and corollary 2.18 without proof:

Theorem 2.21. *Assume that assumption 2.2 holds true, that f is a function $\mathbb{J}\times\mathbb{R}_+ \to \mathbb{R}$ such that for every $i \in \mathbb{J}$, $f(i,\cdot)$ satisfies assumption 2.20. Then we have:*

$$\lim_{n\to\infty} \frac{W_f'(n)}{n} = \sum_{i\in\mathbb{J}} \pi(i)F(i,\cdot)f(i,\cdot) \ a.e.. \tag{2.84}$$

Corollary 2.22. *Assume that $(J_n, T_n)_{n\in\mathbb{N}}$ is a regular inhomogeneous Markov renewal process and that theorem 2.21 holds true for the function $t \to t$. Then we have:*

$$\lim_{t\to\infty} \frac{t}{N(t)} = \Pi m \ a.e., \tag{2.85}$$

where $\Pi m := \sum_{i\in\mathbb{J}} m(i)\pi(i)$ and $m(i) := \int_0^\infty tF(i,dt)$. If in addition theorem 2.21 holds true for the function $f : \mathbb{J}\times\mathbb{R}_+ \to \mathbb{R}$ and $\Pi m > 0$, then we have:

$$\lim_{t\to\infty} \frac{W_f'(N(t))}{t} = \frac{1}{\Pi m} \sum_{i\in\mathbb{J}} \pi(i)F(i,\cdot)f(i,\cdot) \ a.e., \tag{2.86}$$

In particular, the latter establishes the renewal theorem - i.e. the a.e. convergence of $t^{-1}N(t)$ - in the inhomogeneous case under assumptions 2.2 and 2.20 (for the function $t \to t$) only.

3. Central limit theorem for functionals of inhomogeneous Markov renewal and semi-Markov processes

In this section, we will use a martingale method similar to the one of [9], [14], but of course different because of the time-inhomogeneity. In particular, we will need a series of estimations to compute rates of convergence of various time-dependent quantities that we provide in the next lemmas. Also, because we will need lemma 2.19 of the previous section, we keep throughout this section assumption 2.2 of the previous section. But first we recall the definition of an irreducible Markov kernel:

Definition 3.1. *A Markov kernel K on $\mathbb{J}$ is irreducible if for any $i,j \in \mathbb{J}$ there exists $n = n(i,j)$ and a finite $\mathbb{J}-$valued sequence $(x_k)_{k=0..n}$ with $x_0 = i$, $x_n = j$ and $K(x_k, x_{k+1}) > 0$, $k = 0,...,n-1$. It is well known (see e.g. [19]) that an irreducible Markov kernel on a finite set $\mathbb{J}$ admits a unique invariant probability measure and this measure is positive (i.e., gives positive mass to every element of $\mathbb{J}$).*

We introduce the following assumption, similar to assumption 2.6:

Assumption 3.2. *The Markov kernels $(P_t)_{t\in\mathbb{R}_+}$ are irreducible. Further, there exists an irreducible Markov kernel $P = \{P(i,j) : i,j \in \mathbb{J}\}$ on $\mathbb{J}$ such that:*

$$\lim_{n\to\infty} F^{(n)}\psi' = \mathbb{E}[\psi'(T_n)] = 0, \tag{3.1}$$

where $\psi'(t) := \sup_{s\geq t} ||P_s - P||$.

Remark 3.3. *If the Markov renewal process is regular, i.e. $T_n \to \infty$ a.e., then assumption 3.2 is equivalent to $\lim_{t\to\infty} \psi'(t) = 0$, since ψ' is non increasing.*

Lemma 3.4. *Assume that assumptions 2.2, 3.2 hold true. Then π is the unique positive invariant probability measure of P. Further, let $(\pi_t)_{t\in\mathbb{R}_+}$ the unique positive invariant probability measures of the Markov kernels $(P_t)_{t\in\mathbb{R}_+}$. Denoting by $\lesssim$ an inequality up to a positive multiplicative constant, there exists a constant $\delta_P \in (0,1)$ depending only on the kernel P such that we have for any integer n and $t \in \mathbb{R}_+$:*

$$||\Pi - \Pi_t|| \lesssim n\psi'(t) + \delta_P^n \tag{3.2}$$

Proof. By assumption 3.2, let $\widetilde{\pi}$ the unique positive invariant probability measure of P. We have (using notations of assumption 2.2):

$$||\Pi - \widetilde{\Pi}|| \leq \mathbb{E}||\Pi - P_{T_n}^{n,n+m}|| + \mathbb{E}||P_{T_n}^{n,n+m} - P^m|| + ||P^m - \widetilde{\Pi}||. \tag{3.3}$$

Assume we have shown that $\lim_{n\to\infty} \mathbb{E}||P_{T_n}^{n,n+m} - P^m|| = 0$ for every m. By definition of $\widetilde{\Pi}$ we have $\lim_{m\to\infty} ||P^m - \widetilde{\Pi}|| = 0$ and by assumption 2.2 we have $\lim_{m\to\infty} \sup_n \mathbb{E}||\Pi - P_{T_n}^{n,n+m}|| = 0$. This proves that $\Pi = \widetilde{\Pi}$, i.e. that π is the unique positive invariant probability measure of P. Now let's show that $\lim_{n\to\infty} \mathbb{E}||P_{T_n}^{n,n+m} - P^m|| = 0$ for every m. We will do so by induction on m. For $m = 1$ we have $\mathbb{E}||P_{T_n}^{n,n+m} - P^m|| = \mathbb{E}||P_{T_n} - P|| \leq \mathbb{E}\psi'(T_n) \to 0$ by assumption 3.2. Now assume that the induction hypothesis holds true for some m. We have:

$$
\begin{aligned}
&|P_{T_n}^{n,n+m+1}(i,j) - P^{m+1}(i,j)| && (3.4)\\
=&|\mathbb{P}\left[J_{n+m+1} = j | T_n, J_n = i\right] - P^{m+1}(i,j)| && (3.5)\\
=&|\mathbb{E}\left[\mathbb{P}[J_{n+m+1} = j | T_k, J_k : k \leq n+m] | T_n, J_n = i\right] - P^{m+1}(i,j)| && (3.6)\\
=&|\mathbb{E}\left[P_{T_{n+m}}(J_{n+m}, j) | T_n, J_n = i\right] - P^{m+1}(i,j)| && (3.7)\\
\leq&|\mathbb{E}\left[P_{T_{n+m}}(J_{n+m}, j) - P(J_{n+m}, j) | T_n, J_n = i\right]| \\
&+ |\mathbb{E}\left[P(J_{n+m}, j) | T_n, J_n = i\right] - P^{m+1}(i,j)| && (3.8)
\end{aligned}
$$

The first term on the left-hand side of the above is $\leq \mathbb{E}(\psi'(T_{n+m})) \leq \mathbb{E}(\psi'(T_n))$, and for the second term we have:

$$|\mathbb{E}\left[P(J_{n+m}, j)|T_n, J_n = i\right] - P^{m+1}(i,j)| \tag{3.9}$$

$$= \left|\sum_{k\in\mathbb{J}} P(k,j)P^{n,n+m}_{T_n}(i,k) - P^m(i,k)P(k,j)\right| \tag{3.10}$$

$$\leq ||P^{n,n+m}_{T_n} - P^m||. \tag{3.11}$$

And therefore we get overall:

$$\mathbb{E}||P^{n,n+m+1}_{T_n} - P^{m+1}|| \leq \mathbb{E}\psi'(T_n) + \mathbb{E}||P^{n,n+m}_{T_n} - P^m||. \tag{3.12}$$

The first term goes to 0 by assumption 3.2, and the second term goes to 0 by induction hypothesis. This completes the proof that $\Pi = \widetilde{\Pi}$. Now, we have the following, noticing that $\Pi_t P_t = \Pi_t$ by definition of Π_t and therefore $\Pi_t P_t^n = \Pi_t$:

$$||\Pi_t - \Pi|| = ||\Pi_t P_t^n - \Pi|| \tag{3.13}$$

$$\leq ||\Pi_t P_t^n - \Pi_t P^n|| + ||\Pi_t P^n - \Pi_t \Pi|| + ||\Pi_t \Pi - \Pi|| \tag{3.14}$$

$$\leq ||P_t^n - P^n|| + ||P^n - \Pi|| + ||\Pi_t \Pi - \Pi|| \tag{3.15}$$

Since $\Pi_t \Pi = \Pi$, the last term above is 0. By the standard theory of ergodic homogeneous Markov chains, the term $||P^n - \Pi||$ is bounded by $C_P \delta_P^n$ for some constants $C_P \geq 0$ and $\delta_P \in (0,1)$ depending only on the kernel P. It remains to study the term $||P_t^n - P^n||$. We are going to show by induction that $||P_t^n - P^n|| \leq n\psi'(t)$. It is obviously true for $n = 0$. Now assume it is true at stage n. We have:

$$||P_t^{n+1} - P^{n+1}|| \leq ||P_t^n P_t - P_t^n P|| + ||P_t^n P - P^{n+1}|| \tag{3.16}$$

$$\leq ||P_t - P|| + ||P_t^n - P^n|| \leq \psi'(t) + n\psi'(t) = (n+1)\psi'(t). \tag{3.17}$$

□

We let in the following $Q(i,j,\cdot) := P(i,j)H(i,j,\cdot)$ and we will use the following notations - provided they are well-defined - if $f : \mathbb{J} \times \mathbb{J} \times \mathbb{R}_+ \to \mathbb{R}$:

$$\alpha_f := \frac{1}{\Pi m} \sum_{i,j \in \mathbb{J}} \pi(i) Q(i,j,\cdot) f(i,j,\cdot), \tag{3.18}$$

$$m(i) := \int_0^\infty u F(i,du), \tag{3.19}$$

$$b(i) := \sum_{j \in \mathbb{J}} Q(i,j,\cdot) f(i,j,\cdot) - \alpha_f m(i), \tag{3.20}$$

$$\alpha_f(t) := \frac{1}{\Pi_t m_t} \sum_{i,j \in \mathbb{J}} \pi_t(i) Q_t(i,j,\cdot) f(i,j,\cdot), \tag{3.21}$$

$$m_t(i) := \int_0^\infty u F_t(i,du), \tag{3.22}$$

$$b(i,t) := \sum_{j \in \mathbb{J}} Q_t(i,j,\cdot) f(i,j,\cdot) - \alpha_f(t) m_t(i), \tag{3.23}$$

so that $\Pi_t b(\cdot,t) := \sum_{i \in \mathbb{J}} \pi_t(i) b(i,t) = 0$ and $\Pi b := \sum_{i \in \mathbb{J}} \pi(i) b(i) = 0$ by construction. Now, for $t \in \mathbb{R}_+$ and assuming that assumptions 2.2, 3.2 hold true, we will denote by g and $g(\cdot,t)$ the unique solutions of the following Poisson equations associated to the kernels P and $(P_t)_{t \in \mathbb{R}_+}$, respectively (see e.g. [9], equation (7)):

$$(P - I) g = b, \tag{3.24}$$

$$(P_t - I) g(\cdot,t) = b(\cdot,t). \tag{3.25}$$

Notation 3.5. *In the previous Poisson equations, P and P_t stand for the bounded linear operators acting on the Banach space of bounded Borel measurable functions $\mathbb{J} \to \mathbb{R}$ by (we abusively use the same notations for matrices and their associated bounded linear operators):*

$$Pg(i) := \sum_{j \in \mathbb{J}} g(j) P(i,j) \text{ and } P_t g(\cdot,t)(i) := \sum_{j \in \mathbb{J}} g(j,t) P_t(i,j). \tag{3.26}$$

The matrix $P_t + \Pi_t - I$ (resp. $P + \Pi - I$) is often called the fundamental matrix associated to the Poisson equation (3.25) (resp. (3.24)), and it is well-known that it is invertible (for each t). We will introduce the following assumption, which will prove to be useful in the remaining of the paper.

Assumption 3.6. *The following conditions are satisfied:*

$$\sup_{t \in \mathbb{R}_+} ||(P_t + \Pi_t - I)^{-1}|| < \infty, \tag{3.27}$$

$$\inf_{t \in \mathbb{R}_+} \Pi_t m_t > 0 \text{ and } \Pi m > 0. \tag{3.28}$$

Lemma 3.7. *Assume that* $f : \mathbb{J} \times \mathbb{J} \times \mathbb{R}_+ \to \mathbb{R}$*, that assumptions 2.2, 3.2, 3.6 hold true and that:*

$$\sup_{\substack{t\in\mathbb{R}_+\\ i,j\in\mathbb{J}}} H_t(i,j,\cdot)|f(i,j,\cdot)| < \infty \text{ and } \sup_{\substack{t\in\mathbb{R}_+\\ i\in\mathbb{J}}} m_t(i) < \infty. \tag{3.29}$$

Then we have for all $t \in \mathbb{R}_+$*, denoting* $\lesssim$ *an inequality up to a positive multiplicative constant and* $||g(\cdot,t) - g|| := \max_{i\in\mathbb{J}} |g(i,t) - g(i)|$*:*

$$||g(\cdot,t) - g|| \lesssim \phi_f(t) + ||m_t - m|| + \psi'(t) + ||\Pi_t - \Pi||, \tag{3.30}$$

where $\phi_f(t) := \max_i \phi_{f(i,\cdot)}(t)$ *as in (2.51).*

Proof. By standard results on the Poisson equations (3.24), (3.25), $P_t + \Pi_t - I$ and $P + \Pi - I$ are invertible with bounded inverse and we have further $\Pi g = \Pi_t g(\cdot,t) = 0$, which yields:

$$(P + \Pi - I)g = b \text{ and } (P_t + \Pi_t - I)g(\cdot,t) = b(\cdot,t), \tag{3.31}$$

which implies:

$$(P_t + \Pi_t)g(\cdot,t) - (P + \Pi)g - (g(\cdot,t) - g) = b(\cdot,t) - b \tag{3.32}$$

$$\Rightarrow (P_t + \Pi_t - I)(g(\cdot,t) - g) = b(\cdot,t) - b - (P_t + \Pi_t - P - \Pi)g \tag{3.33}$$

$$\Rightarrow ||g(\cdot,t) - g|| \le ||(P_t + \Pi_t - I)^{-1}|| \left[||b(\cdot,t) - b|| + ||P_t - P|| + ||\Pi_t - \Pi||\right]. \tag{3.34}$$

By assumption we have $\sup_{t\in\mathbb{R}_+} ||(P_t + \Pi_t - I)^{-1}|| < \infty$ and therefore we get the following inequality up to a multiplicative constant:

$$||g(\cdot,t) - g|| \lesssim ||b(\cdot,t) - b|| + \psi'(t) + ||\Pi_t - \Pi||. \tag{3.35}$$

Our assumptions imply that $|H_t(i,j,\cdot)f(i,j,\cdot)|$, $|\alpha_f(t)|$ are uniformly bounded and therefore we get:

$$|b(i,t) - b(i)| = |\sum_{j\in\mathbb{J}} Q_t(i,j,\cdot)f(i,j,\cdot) - \alpha_f(t)m_t(i) - \sum_{j\in\mathbb{J}} Q(i,j,\cdot)f(i,j,\cdot) + \alpha_f m(i)|. \tag{3.36}$$

$$\lesssim \phi_f(t) + \psi'(t) + |\alpha_f(t) - \alpha_f| + ||m_t - m||. \tag{3.37}$$

We also have $\inf_{t\in\mathbb{R}_+} \Pi_t m_t > 0$ and therefore:

$$|\alpha_f(t) - \alpha_f| \lesssim \phi_f(t) + ||\Pi_t - \Pi|| + ||m_t - m|| + \psi'(t). \tag{3.38}$$

Overall we get:

$$||g(\cdot,t) - g|| \lesssim \phi_f(t) + ||m_t - m|| + \psi'(t) + ||\Pi_t - \Pi||. \tag{3.39}$$

□

Now we are ready to express $R_n := W_f(n) - \alpha_f T_n = \sum_{k=1}^n [f(J_{k-1}, J_k, X_k) - \alpha_f X_k]$ as the sum of a martingale and of another term which will prove to vanish when proving the central limit theorem for W_f.

Lemma 3.8. *Assume that $f : \mathbb{J} \times \mathbb{J} \times \mathbb{R}_+ \to \mathbb{R}$, that assumptions 2.2, 3.2, 3.6 hold true, that the function $t \to t$ is integrable with respect to $\{F_t(i,\cdot), F(i,\cdot) : i \in \mathbb{J}, t \in \mathbb{R}_+\}$ and that the functions $\{t \to f(i,j,t) : i,j \in \mathbb{J}\}$ are integrable with respect to $\{H_t(i,j,\cdot), H(i,j,\cdot) : i,j \in \mathbb{J}, t \in \mathbb{R}_+\}$. Let $\mathcal{F}_n := \sigma(J_k, X_k : k \in [0,n])$. Then there exists a $\mathcal{F}_n$-martingale $\{M_n\}$ such that for $n \geq 0$:*

$$R_n := W_f(n) - \alpha_f T_n \tag{3.40}$$

$$= M_n + \sum_{k=1}^n g(J_k, T_{k-1}) - g(J_{k-1}, T_{k-1}) + (\alpha_f(T_{k-1}) - \alpha_f) m_{T_{k-1}}(J_{k-1}). \tag{3.41}$$

Further, we have for $n \geq 0$ (as usual, $\sum_{k=1}^0 := 0$ so that $M_0 = 0$):

$$M_n = \sum_{k=1}^n f(J_{k-1}, J_k, X_k) - \alpha_f X_k - (\alpha_f(T_{k-1}) - \alpha_f) m_{T_{k-1}}(J_{k-1}) - g(J_k, T_{k-1}) + g(J_{k-1}, T_{k-1}). \tag{3.42}$$

Proof. We have:

$$W_f(n) - \alpha_f T_n = \underbrace{\sum_{k=1}^n [f(J_{k-1}, J_k, X_k) - \alpha_f X_k] - \mathbb{E}[f(J_{k-1}, J_k, X_k) - \alpha_f X_k | \mathcal{F}_{k-1}]}_{M_1(n)} + \sum_{k=1}^n \mathbb{E}[f(J_{k-1}, J_k, X_k) - \alpha_f X_k | \mathcal{F}_{k-1}]. \tag{3.43}$$

We have $\mathbb{E}[f(J_{k-1}, J_k, X_k) - \alpha_f X_k | \mathcal{F}_{k-1}] = \sum_{j \in \mathbb{J}} Q_{T_{k-1}}(J_{k-1}, j, \cdot) f(J_{k-1}, j, \cdot) - \alpha_f m_{T_{k-1}}(J_{k-1})$ and $M_1(n)$ is by construction a martingale. Therefore we have, by definition of b:

$$\mathbb{E}[f(J_{k-1}, J_k, X_k) - X_k | \mathcal{F}_{k-1}] = b(J_{k-1}, T_{k-1}) + (\alpha_f(T_{k-1}) - \alpha_f) m_{T_{k-1}}(J_{k-1}). \tag{3.44}$$

Overall we have:

$$W_f(n) - \alpha_f T_n = M_1(n) + \sum_{k=1}^{n} b(J_{k-1}, T_{k-1}) + (\alpha_f(T_{k-1}) - \alpha_f) m_{T_{k-1}}(J_{k-1}). \quad (3.45)$$

We can prove that the following process M_2 is a martingale:

$$M_2(n) := \sum_{k=1}^{n} b(J_{k-1}, T_{k-1}) - g(J_k, T_{k-1}) + g(J_{k-1}, T_{k-1}). \quad (3.46)$$

Indeed, we have (recall Notation 3.5 for the operator P_{T_n} below):

$$M_2(n+1) - M_2(n) = b(J_n, T_n) - g(J_{n+1}, T_n) + g(J_n, T_n) \quad (3.47)$$

$$\Rightarrow \mathbb{E}[M_2(n+1) - M_2(n) | \mathcal{F}_n] = b(J_n, T_n) - \mathbb{E}[g(J_{n+1}, T_n) | \mathcal{F}_n] + g(J_n, T_n) \quad (3.48)$$

$$= b(J_n, T_n) - P_{T_n} g(\cdot, T_n)(J_n) + g(J_n, T_n). \quad (3.49)$$

Because $g(\cdot, T_n)$ is the unique solution of the Poisson equation (3.25) and that $\Pi_t b(\cdot, t) = 0$, we have:

$$P_{T_n} g(\cdot, T_n)(J_n) = g(J_n, T_n) + b(J_n, T_n), \quad (3.50)$$

which completes the proof that $\mathbb{E}[M_2(n+1) - M_2(n) | \mathcal{F}_n] = 0$, and therefore that M_2 is a martingale. Overall we obtain, denoting $M_n := M_1(n) + M_2(n)$:

$$\begin{aligned} & W_f(n) - \alpha_f T_n \\ & = M_n + \sum_{k=1}^{n} g(J_k, T_{k-1}) - g(J_{k-1}, T_{k-1}) + (\alpha_f(T_{k-1}) - \alpha_f) m_{T_{k-1}}(J_{k-1}). \end{aligned} \quad (3.51)$$

□

Recalling that $R_n := W_f(n) - \alpha_f T_n$ and as in [14], define the following process for $\eta, t \in \mathbb{R}_+$:

$$U_\eta(t) := \eta^{-1/2} \left[(1 - \lambda_{\eta,t}) R_{\lfloor \eta t \rfloor} + \lambda_{\eta,t} R_{\lfloor \eta t \rfloor + 1} \right], \quad (3.52)$$

$$U_\eta^M(t) := \eta^{-1/2} \left[(1 - \lambda_{\eta,t}) M_{\lfloor \eta t \rfloor} + \lambda_{\eta,t} M_{\lfloor \eta t \rfloor + 1} \right], \quad (3.53)$$

$$\text{where } \lambda_{\eta,t} := \eta t - \lfloor \eta t \rfloor. \quad (3.54)$$

Our goal will be to show that $U_\eta \Rightarrow \sigma W$ in $C(\mathbb{R}_+)$ as $\eta \to \infty$ (the space of continuous functions $\mathbb{R}_+ \to \mathbb{R}$), where σ is a constant, W a Brownian motion. To this end, we will show that $U_\eta^M \Rightarrow \sigma W$, and that $\sup_{t\in[0,T]} |U_\eta(t) - U_\eta^M(t)| \Rightarrow 0$ for every $T \in \mathbb{R}_+$. To show that $U_\eta^M \Rightarrow \sigma W$, we will use the following lemma ([11], theorem 4.1):

Lemma 3.9. ***([11], theorem 4.1).*** *Let $(M_n, \mathcal{F}_n)$ a square-integrable martingale. If the following conditions hold for some constant σ, denoting $D_n := M_n - M_{n-1}$:*

1. $\lim_{n\to\infty} n^{-1} \sum_{k=1}^n \mathbb{E}[D_k^2|\mathcal{F}_{k-1}] = \sigma^2$ *in probability.*
2. $\lim_{n\to\infty} n^{-1} \sum_{k=1}^n \mathbb{E}[D_k^2 I(D_k^2 \geq \epsilon n)] = 0$, $\forall \epsilon > 0$,

where I is the indicator function, then $U_\eta^M \Rightarrow \sigma W$ in $C(\mathbb{R}_+)$.

In order to prove the next lemma on the weak convergence of U_η^M, let's introduce the following assumptions, where assumption 3.12 (resp. 3.10) below is similar but weaker than assumption 2.9 (resp. 2.20) and assumption 3.14 is linked to uniform integrability and will be needed to apply lemma 3.9.

Assumption 3.10. *We say that a function $h : \mathbb{R}_+ \to \mathbb{R}$ satisfies this assumption if:*

$$h \in \bigcap_{i\in\mathbb{J}} L^1(F(i,\cdot)) \text{ and } \sup_{\substack{t\in\mathbb{R}_+\\ i\in\mathbb{J}}} F_t(i,\cdot)|h| < \infty, \tag{3.55}$$

$$\lim_{n\to\infty} F^{(n)}\phi'_h = \mathbb{E}[\phi'_h(T_n)] = 0, \tag{3.56}$$

where $\phi'_h(t) := \sup_{s\geq t, i\in\mathbb{J}} |F_s(i,\cdot)h - F(i,\cdot)h|$. As before, if the domain of h is $\mathbb{J}^n \times \mathbb{R}_+$ for some n, then we will simply let $\phi'_h(t) := \max_{i\in\mathbb{J}^n} \phi'_{h(i,\cdot)}(t)$.

Remark 3.11. *As mentioned in a previous remark, if the Markov renewal process is regular, i.e. $T_n \to \infty$ a.e., then the second condition in assumption 3.10 is true if $\lim_{t\to\infty} \phi'_h(t) = 0$, since ϕ'_h is non increasing. Sufficient conditions for the latter are $h(X(i,t)) \Rightarrow h(X(i))$ as $t \to \infty$ for every $i \in \mathbb{J}$ - where the random variables $X(i,t), X(i)$ have respective laws $F_t(i,\cdot), F(i,\cdot)$ - and the collection of random variables $\{h(X(i,t)) : i \in \mathbb{J}, t \in \mathbb{R}_+\}$ are uniformly integrable (see [7], appendix, proposition 2.3).*

Assumption 3.12. *We say that a function $h : \mathbb{R}_+ \to \mathbb{R}$ satisfies this assumption if:*

$$h \in \bigcap_{i,j\in\mathbb{J}} L^1(H(i,j,\cdot)) \text{ and } \sup_{\substack{t\in\mathbb{R}_+\\ i,j\in\mathbb{J}}} H_t(i,j,\cdot)|h| < \infty, \tag{3.57}$$

$$\lim_{n\to\infty} F^{(n)}\phi_h = \mathbb{E}[\phi_h(T_n)] = 0, \tag{3.58}$$

where $\phi_h(t) := \sup_{s\geq t, i,j\in\mathbb{J}} |H_s(i,j,\cdot)h - H(i,j,\cdot)h|$. *As before, if the domain of* h *is* $\mathbb{J}^n \times \mathbb{R}_+$ *for some* n, *then we will simply let* $\phi_h(t) := \max_{i\in\mathbb{J}^n} \phi_{h(i,\cdot)}(t)$.

Remark 3.13. *Again, if the Markov renewal process is regular, i.e.* $T_n \to \infty$ *a.e., then the second condition in assumption 3.12 is true if* $\lim_{t\to\infty} \phi_h(t) = 0$, *since* ϕ_h *is non increasing. Sufficient conditions for the latter are* $h(X(i,j,t)) \Rightarrow h(X(i,j))$ *as* $t \to \infty$ *for every* $i, j \in \mathbb{J}$ *- where the random variables* $X(i,j,t)$, $X(i,j)$ *have respective laws* $H_t(i,j,\cdot), H(i,j,\cdot)$ *- and the collection of random variables* $\{h(X(i,j,t)) : i,j \in \mathbb{J}, t \in \mathbb{R}_+\}$ *are uniformly integrable.*

Assumption 3.14. *We say that a function* $h : \mathbb{R}_+ \to \mathbb{R}$ *satisfies this assumption if there exists an increasing convex function* φ *on* $\mathbb{R}_+$ *such that* $\lim_{x\to\infty} x^{-1}\varphi(x) = +\infty$ *and:*

$$\sup_{\substack{t\in\mathbb{R}_+\\ i,j\in\mathbb{J}}} H_t(i,j,\cdot)(\varphi \circ |h|) = \sup_{\substack{t\in\mathbb{R}_+\\ i,j\in\mathbb{J}}} \int_0^\infty \varphi(|h(u)|)H_t(i,j,du) < \infty. \tag{3.59}$$

Remark 3.15. *Assumption 3.14 is by [7] (Appendix, proposition 2.2) equivalent to the uniform integrability of the collection of random variables* $\{h(X(i,j,t)) : i,j \in \mathbb{J}, t \in \mathbb{R}_+\}$, *where* $X(i,j,t) \sim H_t(i,j,\cdot)$. *Typically we will have* $\phi(x) = x^p$ *for* $p > 1$.

Now we can prove the following result:

Lemma 3.16. *Assume that* $f : \mathbb{J} \times \mathbb{J} \times \mathbb{R}_+ \to \mathbb{R}$, *that assumptions 2.2, 3.2, 3.6 hold true, that the functions* $t \to t$, $t \to t^2$ *satisfy assumption 3.10, that the functions* $t \to t$, $t \to f(i,j,t)$, $t \to f^2(i,j,t)$, $t \to tf(i,j,t)$ $(i,j \in \mathbb{J})$ *satisfy assumption 3.12 and that the functions* $t \to t^2$, $t \to f^2(i,j,t)$ $(i,j \in \mathbb{J})$ *satisfy assumption 3.14. If* $(M_n, \mathcal{F}_n)$ *is the martingale defined in lemma 3.8, we have* $U^M_\eta \Rightarrow \sqrt{\Pi\sigma^2}W$ *in* $C(\mathbb{R}_+)$, *where* W *is a standard Brownian motion,* $\Pi\sigma^2 := \sum_i \pi(i)\sigma^2(i)$ *and for* $i \in \mathbb{J}$*:*

$$\begin{aligned}\sigma^2(i) &= \sum_{j\in\mathbb{J}} \int_0^\infty (f(i,j,u) - \alpha_f u)^2 Q(i,j,du) + \sum_{j\in\mathbb{J}} (g(j) - g(i))^2 P(i,j) \\ &- 2\sum_{j\in\mathbb{J}} (g(j) - g(i))P(i,j) \int_0^\infty (f(i,j,u) - \alpha_f u)H(i,j,du).\end{aligned} \tag{3.60}$$

Proof. We will use lemma 3.9. Recall that:

$$\begin{aligned}M_n = \sum_{k=1}^n & f(J_{k-1}, J_k, X_k) - \alpha_f X_k - (\alpha_f(T_{k-1}) - \alpha_f)m_{T_{k-1}}(J_{k-1}) \\ & - g(J_k, T_{k-1}) + g(J_{k-1}, T_{k-1}).\end{aligned} \tag{3.61}$$

Therefore we have, denoting $D_n := M_n - M_{n-1}$:

$$D_{n+1} = \underbrace{f(J_n, J_{n+1}, X_{n+1}) - \alpha_f X_{n+1}}_{\gamma(J_n,J_{n+1},X_{n+1})} - \underbrace{(\alpha_f(T_n) - \alpha_f)m_{T_n}(J_n)}_{\beta(J_n,T_n)} - \underbrace{[g(J_{n+1}, T_n) - g(J_n, T_n)]}_{\delta(J_n,J_{n+1},T_n)}. \tag{3.62}$$

And so:

$$\begin{aligned} D_{n+1}^2 = {} & \gamma^2(J_n, J_{n+1}, X_{n+1}) + \beta^2(J_n, T_n) + \delta^2(J_n, J_{n+1}, T_n) + 2\beta(J_n, T_n)\delta(J_n, J_{n+1}, T_n) \\ & - 2\gamma(J_n, J_{n+1}, X_{n+1})(\beta(J_n, T_n) + \delta(J_n, J_{n+1}, T_n)). \end{aligned} \tag{3.63}$$

Therefore we have:

$$\begin{aligned} \mathbb{E}[D_{n+1}^2 | T_n = t, J_n = i] = {} & \sum_{j \in \mathbb{J}} Q_t(i, j, \cdot)\gamma^2(i, j, \cdot) + \beta^2(i, t) + P_t\delta^2(i, \cdot, t)(i) + 2\beta(i, t)P_t\delta(i, \cdot, t)(i) \\ & - 2\beta(i, t) \sum_{j \in \mathbb{J}} Q_t(i, j, \cdot)\gamma(i, j, \cdot) - 2P_t[\delta(i, \cdot, t)H_t(i, \cdot, \cdot)\gamma(i, \cdot, \cdot)](i). \end{aligned} \tag{3.64}$$

To show that M_n is square integrable, it is enough to show that D_n is square integrable. To this end, it is enough to show that:

$$\mathbb{E}(D_n^2) \leq \sup_{i,t} \mathbb{E}[D_n^2 | T_{n-1} = t, J_{n-1} = i] < \infty. \tag{3.65}$$

Assumptions 3.10, 3.12, 3.6 together with (3.64) show us that it is the case because:

$$\sup_t |\alpha_f(t)| \lesssim [\inf_t \Pi_t m_t]^{-1} \cdot \sup_{i,j,t} H_t(i, j, \cdot)|f(i, j, \cdot)| < \infty, \tag{3.66}$$

$$\sup_{i,t} |\beta(i, t)| \leq \sup_{i,t} m_t(i) \cdot (|\alpha_f| + \sup_t |\alpha_f(t)|) < \infty, \tag{3.67}$$

$$\sup_{i,j,t} H_t(i, j, \cdot)|\gamma^2(i, j, \cdot)| < \infty, \tag{3.68}$$

$$\begin{aligned} & \sup_{i,j,t} |\delta(i, j, t)| \leq 2 \sup_{i,t} |g(i, t)| \leq 2 \sup_t ||(P_t + \Pi_t - I)^{-1}|| \sup_{i,t} |b(i, t)| \\ & \lesssim 2 \sup_t ||(P_t + \Pi_t - I)^{-1}|| \cdot [\sup_{i,t} m_t(i) \sup_t |\alpha_f(t)| + \sup_{i,j,t} H_t(i, j, \cdot)|f(i, j, \cdot)|] < \infty. \end{aligned} \tag{3.69}$$

Now, assumptions 3.10, 3.12 together with lemma 3.7 guarantee that the limit in L^1 of $\mathbb{E}[D_{n+1}^2|T_n, J_n = i]$ as $n \to \infty$ is equal to $\sigma^2(i)$ defined above. We have therefore:

$$\lim_{n\to\infty} \mathbb{E} \max_{i\in\mathbb{J}} |\mathbb{E}[D_{n+1}^2|T_n, J_n = i] - \sigma^2(i)| = 0 \tag{3.70}$$

$$\Rightarrow \lim_{n\to\infty} \mathbb{E} n^{-1} \sum_{k=1}^{n} \mathbb{E}[D_k^2|\mathcal{F}_{k-1}] = \lim_{n\to\infty} \mathbb{E} n^{-1} \sum_{k=1}^{n} \sigma^2(J_{k-1}) \tag{3.71}$$

$$= \lim_{n\to\infty} \mathbb{E} \sum_{i\in\mathbb{J}} \sigma^2(i) n^{-1} \sum_{k=1}^{n} 1_{\{J_{k-1}=i\}}. \tag{3.72}$$

By lemma 2.19 we have:

$$\lim_{n\to\infty} \sum_{i\in\mathbb{J}} \sigma^2(i) n^{-1} \sum_{k=1}^{n} 1_{\{J_{k-1}=i\}} = \Pi\sigma^2 \text{ a.e.} \tag{3.73}$$

By the dominated convergence theorem, the previous convergence holds in L^1, and therefore in probability. Finally, we need to show that:

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \mathbb{E}[D_k^2 I(D_k^2 \geq \epsilon n)] = 0, \quad \forall \epsilon > 0. \tag{3.74}$$

By (3.62), (3.66)-(3.69), we have for some constant C:

$$|D_n| \leq |f(J_{n-1}, J_n, X_n) - \alpha_f X_n| + C \text{ a.e.} \tag{3.75}$$

The latter estimation, together with the fact that the functions $t \to t^2$, $t \to f^2(i,j,t)$ $(i,j \in \mathbb{J})$ satisfy assumption 3.14 ensures that the collection of random variables $\{D_k^2\}$ is uniformly integrable, and therefore that (3.74) holds.

□

Now we can prove that $U_\eta \Rightarrow \sqrt{\Pi\sigma^2}W$ in $C(\mathbb{R}_+)$.

Theorem 3.17. *Assume that $f : \mathbb{J} \times \mathbb{J} \times \mathbb{R}_+ \to \mathbb{R}$, that assumptions 2.2, 3.2, 3.6 hold true, that the functions $t \to t$, $t \to t^2$ satisfy assumption 3.10, that the functions $t \to t$, $t \to f(i,j,t)$, $t \to f^2(i,j,t)$, $t \to tf(i,j,t)$ $(i,j \in \mathbb{J})$ satisfy assumption 3.12, that the functions $t \to t^2$, $t \to f^2(i,j,t)$ $(i,j \in \mathbb{J})$ satisfy assumption 3.14 and that:*

$$\sum_{k=1}^{\infty} k^{-1/2} F^{(k)}\gamma < \infty, \tag{3.76}$$

where $\gamma(t) := \phi_f(t) + ||m_t - m|| + \psi'(t) + ||\Pi_t - \Pi||$ and where we recall that $F^{(k)}\gamma = \mathbb{E}[\gamma(T_k)]$. Then we have:

$$U_\eta \Rightarrow \sqrt{\Pi\sigma^2}W \;\; in \; C(\mathbb{R}_+), \tag{3.77}$$

where W is a standard Brownian motion, $\Pi\sigma^2 := \sum_i \pi(i)\sigma^2(i)$, and for $i \in \mathbb{J}$:

$$\begin{aligned}\sigma^2(i) &= \sum_{j\in\mathbb{J}} \int_0^\infty (f(i,j,u)-\alpha_f u)^2 Q(i,j,du) + \sum_{j\in\mathbb{J}} (g(j)-g(i))^2 P(i,j) \\ &- 2\sum_{j\in\mathbb{J}} (g(j)-g(i))P(i,j)\int_0^\infty (f(i,j,u)-\alpha_f u)H(i,j,du).\end{aligned} \tag{3.78}$$

Remark 3.18. *By lemma 3.4, we have $||\Pi_t - \Pi|| \leq n\psi'(t) + \delta_P^n$ for any integer n. Therefore, in particular, the integrability condition (3.76) holds true if there exists an integer valued function $n(t)$ such that:*

$$\sum_{k=1}^{\infty} k^{-1/2} F^{(k)}\widetilde{\gamma} < \infty, \tag{3.79}$$

$$where \quad \widetilde{\gamma}(t) := \phi_f(t) + ||m_t - m|| + n(t)\psi'(t) + \delta_P^{n(t)}. \tag{3.80}$$

Proof. We need to show that for every $T \in \mathbb{R}_+$:

$$\sup_{t\in[0,T]} |U_\eta(t) - U_\eta^M(t)| \Rightarrow 0. \tag{3.81}$$

We have:

$$|U_\eta(t) - U_\eta^M(t)| \leq \eta^{-1/2}\left[|R_{\lfloor \eta t\rfloor} - M_{\lfloor \eta t\rfloor}| + |R_{\lfloor \eta t\rfloor+1} - M_{\lfloor \eta t\rfloor+1}|\right] \tag{3.82}$$

Using lemma 3.8 we get:

$$|R_{\lfloor \eta t\rfloor} - M_{\lfloor \eta t\rfloor}| \leq |\sum_{k=1}^{\lfloor \eta t\rfloor} g(J_k, T_{k-1}) - g(J_{k-1}, T_{k-1})| + |\sum_{k=1}^{\lfloor \eta t\rfloor} (\alpha_f(T_{k-1}) - \alpha_f) m_{T_{k-1}}(J_{k-1})| \tag{3.83}$$

We have:

$$|\sum_{k=1}^{\lfloor \eta t\rfloor} g(J_k, T_{k-1}) - g(J_{k-1}, T_{k-1})| \tag{3.84}$$

$$\leq |\sum_{k=1}^{\lfloor \eta t\rfloor} g(J_k, T_{k-1}) - g(J_k)| + |\sum_{k=1}^{\lfloor \eta t\rfloor} g(J_k) - g(J_{k-1})| + |\sum_{k=1}^{\lfloor \eta t\rfloor} g(J_{k-1}) - g(J_{k-1}, T_{k-1})| \tag{3.85}$$

The middle term (scaled by $\eta^{-1/2}$) is equal to $\eta^{-1/2}|g(J_{\lfloor \eta t \rfloor}) - g(J_0)|$, and therefore goes to 0 a.e. uniformly in $t \in [0, T]$, since $\mathbb{J}$ is finite. The first and third term will be treated similarly. Let's start with the first term. We have for $t \in [0, T]$:

$$|\sum_{k=1}^{\lfloor \eta t \rfloor} g(J_k, T_{k-1}) - g(J_k)| \leq \sum_{k=1}^{\lfloor \eta T \rfloor} |g(J_k, T_{k-1}) - g(J_k)|. \tag{3.86}$$

Using lemma 3.7 we have for all $t \in \mathbb{R}_+$, $i \in \mathbb{J}$:

$$||g(i, t) - g(i)|| \lesssim \phi_f(t) + ||m_t - m|| + \psi'(t) + ||\Pi_t - \Pi|| =: \gamma(t) \tag{3.87}$$

And so:

$$\sum_{k=1}^{\lfloor \eta T \rfloor} |g(J_k, T_{k-1}) - g(J_k)| \leq \sum_{k=1}^{\lfloor \eta T \rfloor} \gamma(T_{k-1}). \tag{3.88}$$

Therefore a sufficient condition to have $\eta^{-1/2} \sum_{k=1}^{\lfloor \eta T \rfloor} |g(J_k, T_{k-1}) - g(J_k)| \Rightarrow 0$ is, by Kronecker's lemma:

$$\sum_{k=1}^{\infty} k^{-1/2} \mathbb{E}[\gamma(T_{k-1})] = \sum_{k=1}^{\infty} k^{-1/2} F^{(k-1)} \gamma < \infty. \tag{3.89}$$

Now, it remains to study the term:

$$\eta^{-1/2} \sum_{k=1}^{\lfloor \eta T \rfloor} |\alpha_f(T_{k-1}) - \alpha_f| m_{T_{k-1}}(J_{k-1}). \tag{3.90}$$

Since $\sup_{i,t} m_t(i) < \infty$ by assumption, the latter converges weakly to 0 if (again, using Kronecker's lemma):

$$\sum_{k=1}^{\infty} k^{-1/2} \mathbb{E}[|\alpha_f(T_{k-1}) - \alpha_f|] < \infty. \tag{3.91}$$

But in the proof of lemma 3.7, we showed that:

$$|\alpha_f(t) - \alpha_f| \lesssim \phi_f(t) + ||\Pi_t - \Pi|| + ||m_t - m|| + \psi'(t). \tag{3.92}$$

This brings us back to the previous case, i.e. $\sum_{k=1}^{\infty} k^{-1/2} F^{(k-1)}\gamma < \infty$. Therefore, the proof is completed. □

Now, we can get the following corollary:

Corollary 3.19. *Under the setting of theorem 3.17, and assuming that the Markov renewal process is regular and that theorem 2.21 holds true for the function $t \to t$, we have $\widetilde{U}_\eta \Rightarrow \sqrt{\frac{\Pi\sigma^2}{\Pi m}} W$ in $C(\mathbb{R}_+)$, where σ was defined in theorem 3.17 and:*

$$\widetilde{U}_\eta(t) := \eta^{-1/2}(W_f(N(\eta t)) - \alpha_f \eta t). \tag{3.93}$$

Proof. Given theorem 3.17, the proof is the same as in the homogeneous case and comes from a change of time (see [14], lemma 4.1 or [9], proposition 2). Indeed, we have $\lim_{t\to\infty} \frac{t}{N(t)} = \Pi m$ a.e., and:

$$|\widetilde{U}_\eta(t) - U_\eta(\eta^{-1}N(\eta t))| = \eta^{-1/2}|\alpha_f||\eta t - T_{N(\eta t)}| \to 0 \text{ a.s. in } C(\mathbb{R}_+). \tag{3.94}$$

□

The results corresponding to the case $W'_f(n) = \sum_{k=1}^{n} f(J_{k-1}, X_k)$ can be obtained as corollaries from the case $W_f(n)$. They read:

Theorem 3.20. *Assume that $f : \mathbb{J} \times \mathbb{R}_+ \to \mathbb{R}$, that assumptions 2.2, 3.2, 3.6 hold true, that the functions $t \to t$, $t \to t^2$, $t \to f(i,t)$, $t \to f^2(i,t)$, $t \to tf(i,t)$ $(i \in \mathbb{J})$ satisfy assumption 3.10, that the functions $t \to t$, $t \to f(i,t)$ $(i \in \mathbb{J})$ satisfy assumption 3.12, that the functions $t \to t^2$, $t \to f^2(i,t)$ $(i \in \mathbb{J})$ satisfy assumption 3.14 and that:*

$$\sum_{k=1}^{\infty} k^{-1/2} F^{(k)}\gamma' < \infty, \tag{3.95}$$

where $\gamma'(t) := \phi'_f(t) + ||m_t - m|| + \psi'(t) + ||\Pi_t - \Pi||$ and where we recall that $F^{(k)}\gamma' = \mathbb{E}[\gamma'(T_k)]$. Then we have:

$$U_\eta \Rightarrow \sqrt{\Pi\sigma^2} W \text{ in } C(\mathbb{R}_+), \tag{3.96}$$

where W is a standard Brownian motion, $\Pi\sigma^2 := \sum_i \pi(i)\sigma^2(i)$, and for $i \in \mathbb{J}$:

$$\begin{aligned}\sigma^2(i) &= \int_0^\infty (f(i,u) - \alpha_f u)^2 F(i,du) + \sum_{j\in\mathbb{J}} (g(j) - g(i))^2 P(i,j) \\ &- 2\sum_{j\in\mathbb{J}} (g(j) - g(i))P(i,j) \int_0^\infty (f(i,u) - \alpha_f u) H(i,j,du).\end{aligned} \tag{3.97}$$

Corollary 3.21. *Under the setting of theorem 3.20, and assuming that the Markov renewal process is regular and that theorem 2.21 holds true for the function $t \to t$, we have $\widetilde{U}_\eta \Rightarrow \sqrt{\frac{\Pi\sigma^2}{\Pi m}} W$ in $C(\mathbb{R}_+)$, where σ was defined in theorem 3.20 and:*

$$\widetilde{U}_\eta(t) := \eta^{-1/2}(W_f(N(\eta t)) - \alpha_f \eta t). \tag{3.98}$$

4. Application: Risk processes based on a inhomogeneous semi-Markov version of the continuous-time Markov chain

A classical model used in insurance is the following: the insurance company is assumed to receive a constant premium at a rate $c > 0$ per unit of time, representing the guaranteed revenue from the customers. After an amount of time of t, the company will have received a total amount of ct. On the other hand, the company receives random claims $\{Y_n\}$ at random times $\{T_n\}$ from the customers, where Y_n is a non negative random variable representing the amount of the claim that has to be payed by the insurance company at time T_n. The wealth process S_t of the company at time t can be represented as:

$$S_t = ct - \sum_{n=1}^{N(t)} Y_n, \tag{4.1}$$

where $N(t) := \sup\{n : T_n \leq t\}$. For simplicity, assume that the random variables $\{Y_n\}$ are independent identically distributed with finite mean $\mathbb{E}(Y)$, independent of the T_n's, and that the T_n's are modeled by the occurrence times of a inhomogeneous Markov renewal process (J_n, T_n). Think of the state process $\{J_n\}$ the following way: assume we have three states: a "bad" state where the company has to pay a lot of claims (given that the state $J_n =$"bad", the next payment time T_{n+1} is expected to be close to T_n), and similarly: a "moderate" state and a "good" state. We justify the choice of the inhomogeneous model the following way: in a various number of situations, the risk exhibits seasonality. For example, the insurer of a ski resort will need a model which allows to have more "bad" states in the winter than in the summer, namely that the transition matrix P_s between the states is different whether $s \in Summer$ or $s \in Winter$. Therefore, we could choose the following inhomogeneous Markov renewal model for the arrival times T_n:

$$P_s = \widetilde{P_1} \text{ if } s \in Winter, \tag{4.2}$$

$$P_s = \widetilde{P_2} \text{ if } s \in Summer, \tag{4.3}$$

where $\widetilde{P_1}$ and $\widetilde{P_2}$ are given Markov kernels. The insurance company might be interested in computing the expected value of its wealth at time t, $\mathbb{E}(S_t)$, which is given by:

$$\mathbb{E}(S_t) = ct - \mathbb{E}(Y)\mathbb{E}(N(t)), \tag{4.4}$$

using the independence assumption. Corollary 2.22 allows us to compute almost surely the quantity $\lim_{t\to\infty} t^{-1}N(t) = (\Pi m)^{-1}$. By the dominated convergence theorem for uniformly integrable random variables (provided we have proved the uniform integrability), one could conclude that a rough approximation for $\mathbb{E}(N(t))$ is $(\Pi m)^{-1}t$, and thus approximate the expected value of the wealth process $\mathbb{E}(S_t)$. To fix the ideas and check in practice the assumptions of corollary 2.22, let's introduce the following model, which is a inhomogeneous semi-Markov version of the continuous time Markov chain model. The latter is a popular model which has had various applications in the literature, see for example [2], [10], [21] for interesting applications in finance. Remember that a continuous-time Markov chain is a semi-Markov process associated to a kernel of the type:

$$Q(i,j,t) = P(i,j)(1 - e^{-\alpha(i)t}), \quad \alpha(i) > 0 \ \ \forall i, \tag{4.5}$$

where P is a Markov kernel. From the latter expression, we can consider the following time-inhomogeneous version by introducing the inhomogeneous semi-Markov kernel:

$$Q_s(i,j,t) := P_s(i,j)(1 - e^{-\alpha_s(i)t}), \quad \alpha_s(i) > 0 \ \ \forall i, s. \tag{4.6}$$

$$\text{which implies } F_s(i,t) = H_s(i,j,t) = 1 - e^{-\alpha_s(i)t} \text{ (independent of } j). \tag{4.7}$$

Here, α_s governs the distribution F_s of the sojourn times and is allowed to be time-dependent. Recall also that P_s has been defined in (4.2)-(4.3). An important observation is that the inhomogeneous semi-Markov process $J_{N(t)}$ associated to the latter kernel Q_s is not Markov: indeed, because of the time-inhomogeneity, the distribution of $J_{N(t)}$ given $\mathcal{F}_s$ will depend on both $J_{N(s)}$ and $T_{N(s)}$: introducing the time-inhomogeneity on the kernel P breaks the Markov property of the continuous-time Markov chain (4.5).

Before checking the assumptions of corollary 2.22, we let $\underline{\alpha} := \inf_{i,s} \alpha_s(i)$, $\overline{\alpha} := \sup_{i,s} \alpha_s(i)$ and we assume $0 < \underline{\alpha} \leq \overline{\alpha} < \infty$. Further, we assume that $\lim_{t\to\infty} \alpha_t(i) = \alpha(i)$ for all i. We can notice that:

$$F_s(i,t) \leq 1 - e^{-\overline{\alpha}t}, \tag{4.8}$$

and therefore by proposition 2.16 that the Markov renewal process is regular. The proof of the latter proposition allows us to establish that the counting function $N(t)$ of our inhomogeneous Markov renewal process can be dominated by the one of a standard renewal process and therefore that the random variables $\{t^{-1}N(t) : t \in \mathbb{R}_+\}$ are uniformly integrable, which justifies the approximation $\mathbb{E}(N(t)) \approx (\Pi m)^{-1}t$ that we discussed above.

Going back to corollary 2.22, first, we need to check assumption 2.2. To illustrate our situation, define $\widetilde{P_1}$ and $\widetilde{P_2}$ the following way:

$$\widetilde{P_1} := \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\widetilde{P_2} := \begin{pmatrix} 0.1 & 0.45 & 0.45 \\ 0.45 & 0.45 & 0.1 \\ 0.45 & 0.1 & 0.45 \end{pmatrix}$$

We observe that $\widetilde{P_1}$ and $\widetilde{P_2}$ have common invariant probability measure $\pi = \frac{1}{3}(1\,1\,1)$, and therefore we are in the setting of [19]. As in [19], denote $\sigma := \max_{k=1,2} \sigma_1(\widetilde{P_k})$, where $\sigma_1(\widetilde{P_k})$ is the singular value of the kernel $\widetilde{P_k}$, that is the square-root of the second largest eigenvalue of the operator $\widetilde{P_k}\widetilde{P_k^*}$ (see [19], definition 3.1). By the proof of [19], theorem 3.4, together with our Proposition 2.4, we know that the convergence of $P_s^{n,m}$ to Π is dominated by σ^{m-n}. Therefore, if $\sigma < 1$, assumption 2.2 will be satisfied. By [19], the adjoint $\widetilde{P_k^*}$ is defined by $\widetilde{P_k^*}(i,j) = \pi(i)^{-1}\pi(j)\widetilde{P_k}(j,i)$. Therefore we get for $k = 1, 2$, in our case:

$$\widetilde{P_k^*} = \widetilde{P_k}^T = \widetilde{P_k}.$$

One can check that we have $\sigma \approx \max\{0, \sqrt{0.1225}\} < 1$, which is what was desired.

It remains to check assumption 2.20 for the function $g(t) = t$. We first observe that the following integrability condition is satisfied:

$$\sup_{s,i} F_s(i,\cdot)|g|^q \leq \int_0^\infty t^q \overline{\alpha} e^{-\underline{\alpha} t} dt < \infty, \quad \forall q \in \mathbb{N}. \tag{4.9}$$

This means that we can take for example $(p, q) = (2, 2)$ in assumption 2.20. It remains to check the integrability condition:

$$\sum_{n\geq 1} \frac{[\mathbb{E}\phi_g^2(T_n)]^{1/2}}{n} < \infty, \tag{4.10}$$

which requires a knowledge of the law $F^{(n)}$ of T_n. We let φ be a function that will be fixed later on. Partitioning Ω on whether or not $T_n \leq \varphi(n)$ we get, since ϕ_g is uniformly bounded (by (4.9)), and non increasing:

$$[\mathbb{E}\phi_g^2(T_n)]^{1/2} \lesssim \mathbb{P}[T_n \leq \varphi(n)]^{1/2} + \phi_g(\varphi(n)). \tag{4.11}$$

On the other hand, using (2.47)-(2.49), we can check by recursion that:

$$\mathbb{P}[T_n \leq t] \leq e^{-\overline{\alpha} t} \sum_{k=n}^{\infty} \frac{\overline{\alpha}^k t^k}{k!} \leq \frac{\overline{\alpha}^n t^n}{n!}, \tag{4.12}$$

the last inequality coming from Taylor-Lagrange. Therefore we need to choose φ such that:

$$\sum_{n\geq 1} n^{-1} \left(\frac{\overline{\alpha}^n \varphi(n)^n}{n!} \right)^{1/2} < \infty \text{ and } \sum_{n\geq 1} n^{-1} \phi_g(\varphi(n)) < \infty. \tag{4.13}$$

We can check directly, denoting F the cumulative distribution function associated to α:

$$|F_s(i,\cdot)g - F(i,\cdot)g| = \left| \int_0^\infty t\alpha_s(i)e^{-\alpha_s(i)t}dt - \int_0^\infty t\alpha(i)e^{-\alpha(i)t}dt \right| \tag{4.14}$$

$$\Rightarrow \phi_g(s) \lesssim \sup_{\substack{i\in\mathbb{J} \\ t\geq s}} |\alpha_t(i) - \alpha(i)|. \tag{4.15}$$

Taking for example $\varphi(n) = \delta \ln(n)$ for some $\delta > 0$ and choosing α_s such that $\max_i |\alpha_s(i) - \alpha(i)| \lesssim e^{-s}$ yields the desired result. Of course the latter analysis is specific to the semi-Markov kernel that we have considered.

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