



On the covering index of convex bodies

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Abstract. Covering a convex body by its homothets is a classical notion in discrete geometry that has resulted in a number of interesting and long-standing problems. Swanepoel introduced the covering parameter of a convex body as a means of quantifying its covering properties. In this paper, we introduce two relatives of the covering parameter called covering index and weak covering index, which upper bound well-studied quantities like the illumination number, the illumination parameter and the covering parameter of a convex body. Intuitively, the two indices measure how well a convex body can be covered by a relatively small number of homothets having the same relatively small homothety ratio. We show that the covering index is a lower semicontinuous functional on the Banach-Mazur space of convex bodies. We further show that the affine d -cubes minimize the covering index in any dimension d , while circular disks maximize it in the plane. Furthermore, the covering index satisfies a nice compatibility with the operations of direct vector sum and vector sum. In fact, we obtain an exact formula for the covering index of a direct vector sum of convex bodies that works in infinitely many instances. This together with a minimization property can be used to determine the covering index of infinitely many convex bodies. As the name suggests, the weak covering index loses some of the important properties of the covering index. Finally, we obtain upper bounds on the covering and weak covering index.

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1. Introduction

Let \mathbb{E}^d denote the d -dimensional Euclidean space with origin o . A d -dimensional convex body K is a compact convex subset of \mathbb{E}^d with nonempty interior. We denote the d -dimensional volume of K by $\text{vol}(K)$. Moreover, K is o -*symmetric* if $K = -K$. The *Minkowski sum* or simply the *vector sum* of two convex bodies $K, L \subseteq \mathbb{E}^d$ is defined by

$$K + L = \{k + l : k \in K, l \in L\}.$$

A *homothetic copy*, or simply a *homothet*, of K is a set of the form $M = \lambda K + x$, where λ is a nonzero real number and $x \in \mathbb{E}^d$. If $\lambda > 0$, then M is said to be a *positive homothet* and if in addition, $\lambda < 1$, we have a *smaller positive homothet* of K . Let C^d denote a d -dimensional cube, B^d a d -dimensional ball, Δ^d a d -simplex and ℓ a line segment (or more precisely, an affine image of any of these convex bodies). We use the symbol \mathcal{K}^d for the metric space of d -dimensional convex bodies under the (multiplicative) Banach-Mazur distance $d_{BM}(\cdot, \cdot)$. That is, for any $K, L \in \mathcal{K}^d$,

$$d_{BM}(K, L) = \inf \{ \delta \geq 1 : L - b \subseteq T(K - a) \subseteq \delta(L - b), a \in K, b \in L \},$$

where the infimum is taken over all invertible linear operators $T : \mathbb{E}^d \rightarrow \mathbb{E}^d$ [28].

The famous Hadwiger Covering Conjecture [12, 13, 20]—also called the Levi-Hadwiger Conjecture or the Gohberg-Markus-Hadwiger Conjecture—states that any $K \in \mathcal{K}^d$ can be covered by 2^d of its smaller positive homothetic copies with 2^d homothets needed only if K is an affine d -cube. This conjecture appears in several equivalent forms, one of which we discuss here. Boltyanski [6] and Hadwiger [14] introduced two notions of illumination of a convex body, the former being ‘*illumination by directions*’ while the latter being ‘*illumination by points*’. The two notions are actually equivalent [6] and K is said to be *illuminated* if all points on the boundary of K are illuminated (in either sense). The *illumination number* $I(K)$ of K is the smallest n for which K can be illuminated by n points (resp., directions). Furthermore, Boltyanski [6, 7] showed that $I(K) = n$ if and only if the smallest number of smaller positive homothets of K that can cover K is n . Thus the Hadwiger Covering Conjecture can be reformulated as the Boltyanski-Hadwiger Illumination Conjecture, which states that for any d -dimensional convex body K we have $I(K) \leq 2^d$, and $I(K) = 2^d$ only if K is an affine d -cube.

Despite the interest in these problems they have only been solved in general in two dimensions or for a few selected classes of convex bodies. We refer to [5, 9, 22] for detailed surveys of these and other related problems of homothetic covering and illumination. This apparent difficulty has recently led to the introduction of quantitative versions of illumination and covering problems. For instance, it can be seen that in the definition of illumination number $I(K)$, the light sources can be taken arbitrarily far from K . However, it seems natural to start with a relatively small number of light sources and quantify how far they need to be from K in order to illuminate it. This is the idea behind the illumination parameter $\text{ill}(K)$ of an o -symmetric convex body K defined by the first named author [3] as follows.

$$\text{ill}(K) = \inf \left\{ \sum_i \|p_i\|_K : \{p_i\} \text{ illuminates } K, p_i \in \mathbb{E}^d \right\},$$

where $\|x\|_K = \inf\{\lambda > 0 : x \in \lambda K\}$ is the norm of $x \in \mathbb{E}^d$ generated by the symmetric convex body K . Clearly, $I(K) \leq \text{ill}(K)$, for o -symmetric convex bodies. Several authors have investigated the illumination parameter of o -symmetric convex bodies [3, 5, 15, 22], determining exact values in several cases.

Inspired by the above quantification ideas, Swanepoel [29] introduced the covering parameter of a d -dimensional convex body to quantify its covering properties. This is given by

$$C(K) = \inf \left\{ \sum_i (1 - \lambda_i)^{-1} : K \subseteq \bigcup_i (\lambda_i K + t_i), \quad 0 < \lambda_i < 1, \quad t_i \in \mathbb{E}^d \right\}.$$

Thus large homothets are penalized in the same way as far away light sources are penalized in the definition of illumination parameter. Note here K is not assumed to have any symmetry as the definition of covering parameter does not make use of the norm $\|\cdot\|_K$. In the same paper, Swanepoel obtained the following Rogers-type upper bounds on $C(K)$ when $d \geq 2$.

$$C(K) < \begin{cases} e^{2^d d(d+1)(\ln d + \ln \ln d + 5)} = O(2^d d^2 \ln d), & \text{if } K \text{ is } o\text{-symmetric,} \\ e^{\binom{2d}{d} d(d+1)(\ln d + \ln \ln d + 5)} = O(4^d d^{3/2} \ln d), & \text{otherwise.} \end{cases} \tag{1}$$

He further showed that if K is o -symmetric, then

$$\text{ill}(K) \leq 2C(K). \tag{2}$$

Despite the usefulness of the covering parameter, not much is known about it. For instance, we do not know whether $C(\cdot)$ is lower or upper semicontinuous on \mathcal{K}^d and the only known exact value is $C(C^d) = 2^{d+1}$. The aim of this paper is to come up with a more refined quantification of covering in terms of the covering index with the Hadwiger Covering Conjecture as the eventual goal. We show that the covering index possesses a number of useful properties such as upper bounding several quantities associated with the covering and illumination of convex bodies, lower semicontinuity, compatibility with direct vector sum and Minkowski sum, a complete characterization of minimizers and the development of tools to compute its exact values for several convex bodies. Furthermore, the covering index gives rise to a number of open problems about the homothetic covering behavior of convex bodies in general, and d -dimensional balls and ball-polyhedra in particular. In Sect. 6, we discuss a variant of the covering index that is perhaps more natural, but possesses weaker properties. Finally, in Sect. 7, we obtain upper bounds on the covering and weak covering indices.

2. The covering index

Before formally defining the covering index, we describe two other related ideas that, in addition to the covering parameter, influence our definition of the covering index.

Given a positive integer m , Lassak [16] introduced the m -covering number of a convex body K as the minimal positive homothety ratio needed to cover K by m homothets. That is,

$$\gamma_m(K) = \inf \left\{ \lambda > 0 : K \subseteq \bigcup_{i=1}^m (\lambda K + t_i), \quad t_i \in \mathbb{E}^d, \quad i = 1, \dots, m \right\}.$$

Lassak showed that the m -covering number is well-defined and studied the special case $m = 4$ for planar convex bodies. Zong [31] studied $\gamma_m : \mathcal{K}^d \rightarrow \mathbb{R}$ as a functional and proved it to be uniformly continuous for all m and d . He did not use the term m -covering number for $\gamma_m(K)$ and simply referred to it as the smallest positive homothety ratio. Obviously, any $K \in \mathcal{K}^d$ can be covered by 2^d smaller positive homothets if and only if $\gamma_{2^d}(K) < 1$. Zong used these ideas to propose a possible computer-based approach to attack the Hadwiger Covering Conjecture [31].

Given convex bodies $K, L \in \mathcal{K}^d$, the covering number of K by L is denoted by $N(K, L)$ and is defined as the minimum number of translates of L needed to cover K . Among covering problems, the problem of covering the d -dimensional ball by smaller positive homothets has generated a lot of interest. One question that has been asked repeatedly is: what is the value of $N(B^d, \lambda B^d)$ [26, 30]? In particular, the case $\lambda = 1/2$ has attracted special attention. Verger-Gaugry [30] showed that

$$N \left(B^d, \frac{1}{2} B^d \right) = O(2^d d^{3/2} \ln d).$$

We can now present the formal definition of covering index.

Definition 1. Let K be a d -dimensional convex body. We define the covering index of K as

$$\text{coin}(K) = \inf \left\{ \frac{m}{1 - \gamma_m(K)} : \gamma_m(K) \leq 1/2, m \in \mathbb{N} \right\}.$$

Intuitively, $\text{coin}(K)$ measures how K can be covered by a relatively small number of positive homothets all corresponding to the same relatively small homothety ratio. We note that $\text{coin}(K)$ is an affine invariant quantity assigned to K , i.e., if $A : \mathbb{E}^d \rightarrow \mathbb{E}^d$ is an invertible linear map then $\text{coin}(A(K)) = \text{coin}(K)$.

We have the following relationship.

Proposition 2.1. *For any o-symmetric d -dimensional convex body K ,*

$$I(K) \leq \text{ill}(K) \leq 2C(K) \leq 2\text{coin}(K),$$

and in general for $K \in \mathcal{K}^d$,

$$I(K) \leq C(K) \leq \text{coin}(K).$$

Proposition 2.1 follows immediately from the definition of coin, the relation (2) and the observation

$$\begin{aligned} \text{coin}(K) &= \inf \left\{ \frac{m}{1 - \gamma_m(K)} : \gamma_m(K) \leq 1/2, m \in \mathbb{N} \right\} \\ &= \inf \left\{ \frac{m}{1 - \lambda} : K \subseteq \bigcup_{i=1}^m (\lambda K + t_i), 0 < \lambda \leq 1/2, t_i \in \mathbb{E}^d, m \in \mathbb{N} \right\} \\ &\geq C(K). \end{aligned}$$

We remark that the inequality $\text{ill}(K) \leq 2\text{coin}(K)$ can also be derived directly by suitably modifying the proof of Proposition 1 of Swanepoel [29].

2.1. Why $\gamma_m(K) \leq 1/2$?

The reader may be a bit surprised to see the restriction $\gamma_m(K) \leq 1/2$. One immediate consequence of this restriction is that for any $K \in \mathcal{K}^d$,

$$N \left(K, \frac{1}{2}K \right) \leq \text{coin}(K) \leq 2N \left(K, \frac{1}{2}K \right), \tag{3}$$

that is, $\text{coin}(K) = \Theta(N(K, \frac{1}{2}K))$. Therefore, $\text{coin}(B^d)$ (resp. $\text{coin}(K)$) can be used to estimate $N(B^d, \frac{1}{2}B^d)$ (resp. $N(K, \frac{1}{2}K)$), which is a quantity of special interest, and vice versa.

However, there are other more compelling reasons for choosing 1/2 as the threshold. To understand these better, we define

$$f_m(K) = \begin{cases} \frac{m}{1 - \gamma_m(K)}, & \text{if } 0 < \gamma_m(K) \leq \frac{1}{2}, \\ +\infty, & \text{if } \frac{1}{2} < \gamma_m(K) \leq 1. \end{cases}$$

Thus $\text{coin}(K) = \inf \{f_m(K) : m \in \mathbb{N}\}$. Later in Theorem 3.2, we show that for any $K, L \in \mathcal{K}^d$ and $m \in \mathbb{N}$ such that $\gamma_m(K) \leq 1/2$ and $\gamma_m(L) \leq 1/2$,

$$f_m(K) \leq d_{BM}(K, L)f_m(L), \tag{4}$$

and

$$f_m(K) \geq \frac{d_{BM}(K, L)}{2d_{BM}(K, L) - 1} f_m(L), \tag{5}$$

establishing a strong connection with the Banach-Mazur distance of convex bodies. The proofs of relations (4) and (5) make extensive use of homothety

ratios to be less than or equal to a half. This shows that the ‘half constraint’ in the definition of covering index results in a quantity with potentially nicer properties. In particular, relation (4) is important as for each m , it implies the Lipschitz continuity of f_m on the subspace

$$\mathcal{K}_m^d := \{K \in \mathcal{K}^d : \gamma_m(K) \leq 1/2\}, \tag{6}$$

which in turn leads to the continuity properties of coin discussed in Sect. 3. We remark that from the proof of Theorem 5.1, $\mathcal{K}_m^d \neq \emptyset$ if and only if $m \geq 2^d$.

In Sect. 6, we demonstrate what happens if we remove the restriction $\gamma_m(K) \leq 1/2$ from the definition of covering index. The resulting quantity, which we call the *weak covering index* loses some important properties satisfied by the covering index.

3. Continuity

In this section, we establish some important properties of coin. The first observation, though trivial, helps compute the exact values and upper estimates of coin for several convex bodies.

Lemma 3.1. (Minimization lemma) *For any d -dimensional convex body K , the inequality $f_l(K) > f_m(K)$ implies $m < f_l(K)$.*

This shows that the covering index of any convex body can be obtained by calculating a finite minimum, rather than the infimum of an infinite set. In particular, if $f_l(K) < \infty$ for some l , then $\text{coin}(K) = \min \{f_m(K) : m < f_l(K)\}$.

The next result summarizes what we know about the continuity of f_m and coin. Note that the restriction $\gamma_m(K) \leq 1/2$ plays a key role throughout the proof. We remark that without this constraint (or a constraint of the form $\gamma_m(K) \leq r$, where $0 < r \leq 1/2$), the proof of Theorem 3.2 would not hold.

Theorem 3.2. (Continuity) *Let d be any positive integer.*

- (i) *For any $K, L \in \mathcal{K}_m^d$, the relations (4) and (5) hold. Moreover, equality holds in (4) if and only if $d_{BM}(K, L) = 1$, i.e., L is an affine image of K and equality in (5) holds if and only if either $d_{BM}(K, L) = 1$ or $d_{BM}(K, L) > 1$ with*

$$\gamma_m(K) = \frac{\gamma_m(L)}{d_{BM}(K, L)} = \frac{1}{2d_{BM}(K, L)}.$$

- (ii) *The functional $f_m : \mathcal{K}_m^d \rightarrow \mathbb{R}$ is Lipschitz continuous with $\frac{d^2-1}{2 \ln d}$ as a Lipschitz constant and*

$$|f_m(K) - f_m(L)| \leq d_{BM}(K, L) - 1 \leq \frac{d^2 - 1}{2 \ln d} \ln(d_{BM}(K, L)),$$

for all $K, L \in \mathcal{K}_m^d$. On the other hand, $f_m : \mathcal{K}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, for all d and m .

(iii) Define $I_K = \{i : \gamma_i(K) \leq 1/2\} = \{i : K \in \mathcal{K}_i^d\}$, for any d -dimensional convex body K . If $I_L \subseteq I_K$, for some $K, L \in \mathcal{K}^d$, then

$$\text{coin}(K) \leq \frac{2d_{BM}(K, L) - 1}{d_{BM}(K, L)} \text{coin}(L) \leq d_{BM}(K, L) \text{coin}(L). \tag{7}$$

(iv) The functional $\text{coin} : \mathcal{K}^d \rightarrow \mathbb{R}$ is lower semicontinuous for all d .

(v) Define

$$\mathcal{K}^{d*} := \{K \in \mathcal{K}^d : \gamma_m(K) \neq 1/2, m \in \mathbb{N}\}.$$

Then the functional $\text{coin} : \mathcal{K}^{d*} \rightarrow \mathbb{R}$ is continuous for all d .

Proof. (i) We first show □

Proposition 3.3. For any $K, L \in \mathcal{K}^d$,

$$\gamma_m(K) \leq d_{BM}(K, L)\gamma_m(L) \tag{8}$$

holds and so γ_m is Lipschitz continuous on \mathcal{K}^d with $\frac{d^2-1}{2 \ln d}$ as a Lipschitz constant and

$$|\gamma_m(K) - \gamma_m(L)| \leq d_{BM}(K, L) - 1 \leq \frac{d^2 - 1}{2 \ln d} \ln(d_{BM}(K, L)),$$

for all $K, L \in \mathcal{K}^d$.

Proof. Let $\delta > 1$ be such that $d_{BM}(K, L) < \delta$. Now let $a \in K, b \in L$ and the invertible linear operator $T : \mathbb{E}^d \rightarrow \mathbb{E}^d$ satisfy $L - b \subseteq T(K - a) \subseteq \delta(L - b)$. Moreover, let $\{\lambda L + x_i : x_i \in \mathbb{E}^d, i = 1, \dots, m\}$ be a homothetic cover of L , having m homothets with homothety ratio $\lambda > 0$. Then

$$\begin{aligned} T(K - a) &\subseteq \delta(L - b) \subseteq \delta \left(\bigcup_{i=1}^m (\lambda L + x_i - b) \right) \\ &= \delta \left(\bigcup_{i=1}^m (\lambda(L - b) + x_i + (\lambda - 1)b) \right) \\ &\subseteq \delta \left(\bigcup_{i=1}^m (\lambda T(K - a) + x_i + (\lambda - 1)b) \right) \\ &= \bigcup_{i=1}^m (\delta \lambda T(K - a) + \delta x_i + \delta(\lambda - 1)b), \end{aligned}$$

which implies that there is a homothetic cover of $T(K - a)$ having m homothets with homothety ratio $\delta\lambda$. Hence there is a homothetic cover of K having m homothets with homothety ratio $\delta\lambda$. This implies that $\gamma_m(K) \leq \delta\gamma_m(L)$. Therefore, by taking $\inf \delta = d_{BM}(K, L)$, we get $\gamma_m(K) \leq d_{BM}(K, L)\gamma_m(L)$.

On the other hand, $\gamma_m(K) \leq 1, \gamma_m(L) \leq 1$ and (8) imply in a straightforward way that

$$|\gamma_m(K) - \gamma_m(L)| \leq d_{BM}(K, L) - 1.$$

If $d_{BM}(K, L) = 1$, we have nothing further to prove. Otherwise, recall John’s theorem ([28], page 587) implying $1 \leq d_{BM}(K, L) \leq d^2$. Thus using the concavity of $\ln(\cdot)$ one obtains $\frac{2 \ln d}{d^2 - 1} \leq \frac{\ln(d_{BM}(K, L))}{d_{BM}(K, L) - 1}$. This completes the proof of Proposition 3.3. \square

We now return to the main proof. To prove (4) let $K, L \in \mathcal{K}_m^d$. If $\gamma_m(K) \leq \gamma_m(L)$, then $f_m(K) \leq f_m(L) \leq d_{BM}(K, L)f_m(L)$, with equality if and only if $d_{BM}(K, L) = 1$. Therefore, we can assume without loss of generality that $\gamma_m(K) > \gamma_m(L)$. Note that this together with $\gamma_m(K) \leq 1/2$ and $\gamma_m(L) \leq 1/2$ implies

$$\gamma_m(K) - (\gamma_m(K))^2 > \gamma_m(L) - (\gamma_m(L))^2. \tag{9}$$

Thus by using (8),

$$\frac{f_m(K)}{f_m(L)} = \frac{1 - \gamma_m(L)}{1 - \gamma_m(K)} < \frac{\gamma_m(K)}{\gamma_m(L)} \leq d_{BM}(K, L),$$

which gives (4). In addition, equality never holds in this case. Thus equality in (4) holds if and only if $d_{BM}(K, L) = 1$.

Now to prove (5), we again use (8).

$$f_m(K) = \frac{m}{1 - \gamma_m(K)} \geq \frac{m}{1 - \frac{\gamma_m(L)}{d_{BM}(K, L)}} = \frac{d_{BM}(K, L)(1 - \gamma_m(L))}{d_{BM}(K, L) - \gamma_m(L)} f_m(L),$$

with equality if and only if $\gamma_m(K) = \frac{\gamma_m(L)}{d_{BM}(K, L)}$.

Since $\gamma_m(L) \leq 1/2$,

$$\frac{1 - \gamma_m(L)}{d_{BM}(K, L) - \gamma_m(L)} \geq \frac{1}{2d_{BM}(K, L) - 1},$$

with equality if and only if either $d_{BM}(K, L) = 1$ or $d_{BM}(K, L) > 1$ with $\gamma_m(L) = 1/2$. Thus (5) is satisfied and equality holds if and only if either $d_{BM}(K, L) = 1$ or $d_{BM}(K, L) > 1$ with $\gamma_m(K) = \frac{\gamma_m(L)}{d_{BM}(K, L)} = \frac{1}{2d_{BM}(K, L)}$.

(ii) The continuity on \mathcal{K}_m^d is immediate, since γ_m is continuous on \mathcal{K}^d , for all d and m [31]. The Lipschitz continuity follows from (4) in the same way as in Proposition 3.3.

For the lower semicontinuity on \mathcal{K}^d , we consider two cases.

Case 1: $f_m(K) = \frac{m}{1 - \gamma_m(K)}$, with $0 < \gamma_m(K) \leq \frac{1}{2}$.

We need to show that for every $\epsilon > 0$, there exists $\delta > 0$, such that $f_m(K') \geq f_m(K) - \epsilon$, for all K' with $1 \leq d_{BM}(K, K') \leq 1 + \delta$. Our proof of this claim is indirect:

Assume that there exist $\epsilon_0 > 0$, $\delta_1 > \delta_2 > \dots > \delta_n > \dots > 0$ with $\lim_{n \rightarrow +\infty} \delta_n = 0$, and $K_1, K_2, \dots, K_n, \dots \in \mathcal{K}^d$ such that $f_m(K_n) < f_m(K) - \epsilon_0$, where $1 \leq d_{BM}(K, K_n) \leq 1 + \delta_n$, $n = 1, 2, \dots$. Here

$$f_m(K_n) = \frac{m}{1 - \gamma_m(K_n)} < \frac{m}{1 - \gamma_m(K)} - \epsilon_0 = f_m(K) - \epsilon_0,$$

implying that

$$\gamma_m(K) > 1 - \frac{m}{\frac{m}{1-\gamma_m(K)} - \epsilon_0} > \gamma_m(K_n) > 0. \tag{10}$$

As $\lim_{n \rightarrow +\infty} d_{BM}(K, K_n) = 1$ and $\gamma_m : \mathcal{K}^d \rightarrow \mathbb{R}$ is continuous, $\lim_{n \rightarrow +\infty} \gamma_m(K_n) = \gamma_m(K)$, which together with (10) implies $\gamma_m(K) > \gamma_m(K)$, a contradiction.

Case 2: $f_m(K) = +\infty$, with $\frac{1}{2} < \gamma_m(K) \leq 1$.

Here we need to show that for any $K_1, K_2, \dots, K_n, \dots \in \mathcal{K}^d$ with $\lim_{n \rightarrow +\infty} d_{BM}(K, K_n) = 1$ we have that $\lim_{n \rightarrow +\infty} f_m(K_n) = +\infty$. Again, we show this via an indirect argument. First, recall that if $f_m(K_n) < +\infty$, then $m < f_m(K_n) = \frac{m}{1-\gamma_m(K_n)} \leq 2m$ with $0 < \gamma_m(K_n) \leq \frac{1}{2}$. Second, assume that for a subsequence $K_{i_1}, K_{i_2}, \dots, K_{i_n}, \dots \in \mathcal{K}^d$ with $\lim_{n \rightarrow +\infty} d_{BM}(K, K_{i_n}) = 1$ we have

$$\lim_{n \rightarrow +\infty} f_m(K_{i_n}) = \lim_{n \rightarrow +\infty} \frac{m}{1-\gamma_m(K_{i_n})} = \frac{m}{1-\gamma_m(K)} \leq 2m.$$

(Here, we have once again used the continuity of $\gamma_m : \mathcal{K}^d \rightarrow \mathbb{R}$.) Thus $\gamma_m(K) \leq \frac{1}{2}$ implying that $f_m(K) < +\infty$, a contradiction.

(iii) Note that $\text{coin}(K) = \inf\{f_m(K) : m \in I_K\}$. The result then follows from (4) and (5) and the fact that $I_L \subseteq I_K$.

(iv) Let $K \in \mathcal{K}^d$ and $h = 2^{d+1} \left(\binom{2d}{d}^{\frac{1}{2}} - \frac{1}{2} \right)^d d(\ln d + \ln \ln d + 5)$. From the proof of Lemma 3.1 and Corollary 7.3 (whose proof is completely independent of the results appearing in Sect. 1–6), $\text{coin}(K) = \min\{f_m(K) : m \leq h\}$. In fact, since it is volumetrically impossible to cover a convex body by fewer than 2^d of its positive homothets of homothety ratio less than or equal to $1/2$, $\text{coin}(K) = \min\{f_m(K) : 2^d \leq m \leq h\}$. Thus $\text{coin} : \mathcal{K}^d \rightarrow \mathbb{R}$ is the pointwise minimum of finitely many lower semicontinuous functions $f_m : \mathcal{K}^d \rightarrow \mathbb{R} \cup \{+\infty\}$, $2^d \leq m \leq h$, defined on the metric space \mathcal{K}^d . Since the minimum of finitely many lower semicontinuous functions defined on a metric space is lower semicontinuous, the result follows.

(v) It remains to establish the upper semicontinuity. Let $(K_n)_{n \in \mathbb{N}}$ be a sequence in \mathcal{K}^{d*} converging to $K \in \mathcal{K}^{d*}$. We prove that $\limsup \text{coin}(K_n) \leq \text{coin}(K)$. It suffices to show that for sufficiently large $n \in \mathbb{N}$, $I_K \subseteq I_{K_n}$ as, from (iii), this would imply $\text{coin}(K_n) \leq d_{BM}(K, K_n) \text{coin}(K)$.

Let $m \in I_K$, that is $\gamma_m(K) < 1/2$, as $K \in \mathcal{K}^{d*}$. Also note that since $K_n \in \mathcal{K}^{d*}$, either $\gamma_m(K_n) > 1/2$ or $\gamma_m(K_n) < 1/2$. Relation (8) now gives

$$\gamma_m(K_n) \leq d_{BM}(K, K_n) \gamma_m(K),$$

for any $n \in \mathbb{N}$. By choosing n sufficiently large we can ensure that $\gamma_m(K_n) < 1/2$ and so $m \in I_{K_n}$. □

We observe that $B^3 \in \mathcal{K}^{3*}$ (cf. Remark 5.4), so \mathcal{K}^{3*} is nonempty.

The lower semicontinuity of coin leads to some interesting consequences. On the one hand, it shows that there exists a d -dimensional convex body M such that $\text{coin}(M) = \inf \{ \text{coin}(K) : K \in \mathcal{K}^d \}$, for all d . Thus there exists a minimizer of coin over all d -dimensional convex bodies, for all d . On the other hand, although lower semicontinuity does not guarantee the existence of a coin-maximizer, it does show that $\sup \{ \text{coin}(K) : K \in \mathcal{K}^d \} = \sup \{ \text{coin}(P) : P \in \mathcal{P}^d \}$, where \mathcal{P}^d denotes the set of all d -dimensional convex polytopes, which is known to be dense in \mathcal{K}^d . Therefore, in trying to compute the supremum of coin one can restrict themselves to the class of polytopes. This is not true for the illumination number, which is known to be upper semicontinuous (see [5], pp. 23–24) but is not lower semicontinuous.

We do not know whether coin is continuous on \mathcal{K}^d or not. The argument used to prove the upper semicontinuity of coin on \mathcal{K}^{d*} does not seem to work in general. We, therefore, propose the following problem.

Problem 1. Either prove that coin is upper semicontinuous on \mathcal{K}^d or construct a counterexample.

It would be natural to ask whether analogues of inequalities (4) and (5) hold for coin. The answer is negative for both. One can look at the example of a circle B^2 and a square C^2 . It is well-known that $d(C^2, B^2) = \sqrt{2}$ and we will see in Sect. 5 that $\text{coin}(B^2) = 14$ and $\text{coin}(C^2) = 8$. But then $\text{coin}(B^2) > \sqrt{2} \text{coin}(C^2)$ and $\text{coin}(C^2) < \frac{\sqrt{2}}{2\sqrt{2}-1} \text{coin}(B^2)$.

4. Compatibility with vector sums

For the sake of brevity, we write $N_\lambda(K)$ instead of $N(K, \lambda K)$, for any d -dimensional convex body K and $0 < \lambda \leq 1$. Clearly, $N_1(K) = 1$,

$$N_{\gamma_m(K)}(K) \leq m \tag{11}$$

and

$$\gamma_{N_\lambda(K)}(K) \leq \lambda. \tag{12}$$

Moreover, either inequality can be strict. To see that (11) can be strict, consider the example of an affine regular convex hexagon H . Lassak [17] proved that $\gamma_7(K) = 1/2$ holds for any o -symmetric planar convex body K . Thus $\gamma_7(H) = 1/2$. On the other hand, from Fig. 1 and the monotonicity of $\gamma_m(K)$ in m [31] it follows that $1/2 = \gamma_7(H) \leq \gamma_6(H) \leq 1/2$. Thus $\gamma_6(H) = 1/2$ and $N_{\gamma_7(H)} = N_{1/2}(H) \leq 6$.

To see that (12) can be strict, note that it is possible to have $N_{\lambda_1}(K) = N_{\lambda_2}(K)$, for some $\lambda_1 < \lambda_2$. For instance, $N_{1/2}(C^d) = N_\lambda(C^d) = 2^d$, for any $1/2 < \lambda < 1$. Therefore, $\gamma_{N_\lambda(C^d)}(C^d) = \gamma_{2^d}(C^d) = 1/2 < \lambda$, for any $1/2 < \lambda < 1$. We use these ideas in the remainder of this section.

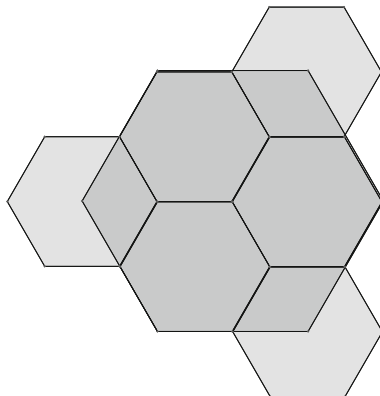


FIGURE 1. Covering H by six homothets with homothety ratio $\frac{1}{2}$

We now present some results showing that coin behaves very nicely with certain binary operations of convex bodies. The first five concern direct vector sums and will be used extensively in computing the exact values and estimates of coin for higher dimensional convex bodies from the covering indices of lower dimensional convex bodies. To state these results, we introduce the notion of tightly covered convex bodies.

Definition 2. We say that a convex body $K \in \mathcal{K}^d$ is *tightly covered* if for any $0 < \lambda < 1$, K contains $N_\lambda(K)$ points, no two of which belong to the same homothet of K with homothety ratio λ .

For instance, $\ell \in \mathcal{K}^1$ is tightly covered since for any $0 < \lambda < 1$, the line segment ℓ contains $N_\lambda(\ell) = \lceil \lambda^{-1} \rceil$ points, no two of which can be covered by the same homothet of the form $\lambda\ell + t$, $t \in \mathbb{E}^1$. Later (see Corollary 4.3) we will see that for any $d \geq 2$, the d -dimensional cube C^d is also tightly covered. We do not know of any other examples of tightly covered convex bodies. Could it be that, for $d \geq 2$, d -cubes are the only such convex bodies?

Problem 2. For some $d \geq 2$, find a tightly covered convex body $K \in \mathcal{K}^d$ other than C^d or show that no such convex body exists.

We do, however, know that not all convex bodies are tightly covered as will be seen through the example of the circle B^2 . (See the discussion right after the proof of Corollary 4.3.)

Theorem 4.1. Let $\mathbb{E}^d = \mathbb{L}_1 \oplus \dots \oplus \mathbb{L}_n$ be a decomposition of \mathbb{E}^d into the direct vector sum of its linear subspaces \mathbb{L}_i and let $K_i \subseteq \mathbb{L}_i$ be convex bodies such that $\text{coin}(K_i) = f_{m_i}(K_i)$, $i = 1, \dots, n$, and

$$\Gamma = \max\{\gamma_{m_i}(K_i) : 1 \leq i \leq n\}.$$

If some $n - 1$ of the K_i 's are tightly covered, then

$$\max\{\text{coin}(K_i) : 1 \leq i \leq n\} \leq$$

$$\begin{aligned} \text{coin}(K_1 \oplus \cdots \oplus K_n) &= \inf_{\lambda \leq \frac{1}{2}} \frac{\prod_{i=1}^n N_\lambda(K_i)}{1 - \lambda} \\ &\leq \frac{\prod_{i=1}^n N_\Gamma(K_i)}{1 - \Gamma} \leq \frac{\prod_{i=1}^n m_i}{1 - \Gamma} < \prod_{i=1}^n \text{coin}(K_i), \end{aligned} \tag{13}$$

where $K_1 \oplus \cdots \oplus K_n$ stands for the direct sum of the convex bodies $K_1 \subseteq \mathbb{L}_1, \dots, K_n \subseteq \mathbb{L}_n$. Moreover, the first two upper bounds in (13) are tight.

Proof. First, we prove the lower bound for $\text{coin}(K_1 \oplus \cdots \oplus K_n)$. Let $P_{\mathbb{L}_i} : \mathbb{E}^d \rightarrow \mathbb{L}_i$ denote the projection of \mathbb{E}^d onto \mathbb{L}_i parallel to the linear subspace $\mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_{i-1} \oplus \mathbb{L}_{i+1} \oplus \cdots \oplus \mathbb{L}_n$, $i = 1, \dots, n$. Let $\{\lambda K + x_j : x_j \in \mathbb{E}^d, j = 1, \dots, m\}$ be a homothetic covering of $K = K_1 \oplus \cdots \oplus K_n \subseteq \mathbb{E}^d$ with homothety ratio $0 < \lambda \leq 1/2$. As $\{P_{\mathbb{L}_i}(\lambda K + x_j) = \lambda K_i + P_{\mathbb{L}_i}(x_j) : x_j \in \mathbb{E}^d, j = 1, \dots, m\}$ is a homothetic covering of K_i with homothety ratio λ in \mathbb{L}_i , $1 \leq i \leq n$, the lower bound follows.

Second, we prove the formula and the upper bounds on $\text{coin}(K_1 \oplus \cdots \oplus K_n)$.

Proposition 4.2. *If some $n - 1$ of the K_i 's are tightly covered, then for all $0 < \lambda < 1$,*

$$N_\lambda(K_1 \oplus \cdots \oplus K_n) = \prod_{i=1}^n N_\lambda(K_i). \tag{14}$$

Proof. Let $N_i = N_\lambda(K_i)$, $i = 1, \dots, n$, and let $\{\lambda K_i + t_{ij_i} : t_{ij_i} \in \mathbb{L}_i, j_i = 1, \dots, N_i\}$ be a homothetic covering of K_i with homothety ratio λ in \mathbb{L}_i , for $i = 1, \dots, n$.

Clearly,

$$\begin{aligned} &\{(\lambda K_1 + t_{1j_1}) \oplus \cdots \oplus (\lambda K_n + t_{nj_n}) : t_{ij_i} \in \mathbb{L}_i, i = 1, \dots, n, j_i = 1, \dots, N_i\} \\ &= \{\lambda(K_1 \oplus \cdots \oplus K_n) + t_{1j_1} + \cdots + t_{nj_n} : i = 1, \dots, n, j_i = 1, \dots, N_i\} \end{aligned}$$

is a homothetic covering of $K_1 \oplus \cdots \oplus K_n$ with homothety ratio λ in \mathbb{E}^d having cardinality $\prod_{i=1}^n N_i$. Thus $N_\lambda(K_1 \oplus \cdots \oplus K_n) \leq \prod_{i=1}^n N_\lambda(K_i)$.

Next, let $\mathcal{C} = \{\lambda(K_1 \oplus \cdots \oplus K_n) + t_j : t_j \in \mathbb{E}^d, j = 1, \dots, N\}$ be a minimal cardinality homothetic covering of $K_1 \oplus \cdots \oplus K_n$ with homothety ratio λ in \mathbb{E}^d . Let us assume without loss of generality that K_1, \dots, K_{n-1} are tightly covered. So, for $i = 1, \dots, n - 1$ and $j_i = 1, \dots, N_\lambda(K_i)$, there exist points $x_{ij_i} \in K_i$ such that for any fixed i and $1 \leq j_i \neq j'_i \leq N_\lambda(K_i)$, x_{ij_i} and $x_{ij'_i}$ cannot both be contained in a homothet of K_i with homothety ratio λ . Therefore, no homothet in \mathcal{C} intersects any two of the $\prod_{i=1}^{n-1} N_\lambda(K_i)$ cross sections $x_{1j_1} + \cdots + x_{(n-1)j_{n-1}} + K_n$ of $K_1 \oplus \cdots \oplus K_n$. In order to cover each such cross section, we require at least $N_\lambda(K_n)$ homothets from \mathcal{C} . Thus $N_\lambda(K_1 \oplus \cdots \oplus K_n) = N \geq \prod_{i=1}^n N_\lambda(K_i)$. \square

Hence, for any $0 < \lambda < 1$,

$$\frac{N_\lambda(K_1 \oplus \cdots \oplus K_n)}{1 - \lambda} = \frac{\prod_{i=1}^n N_\lambda(K_i)}{1 - \lambda}.$$

Thus,

$$\begin{aligned} \text{coin}(K_1 \oplus \cdots \oplus K_n) &= \inf_{m \in \mathbb{N}} \left\{ \frac{m}{1 - \gamma_m(K_1 \oplus \cdots \oplus K_n)} : \gamma_m(K_1 \oplus \cdots \oplus K_n) \leq \frac{1}{2} \right\} \\ &= \inf_{\lambda \leq \frac{1}{2}} \frac{N_\lambda(K_1 \oplus \cdots \oplus K_n)}{1 - \lambda} \\ &= \inf_{\lambda \leq \frac{1}{2}} \frac{\prod_{i=1}^n N_\lambda(K_i)}{1 - \lambda}, \end{aligned}$$

completing the proof of the equality appearing in (13).

The upper bounds in (13) now follow from the definition of Γ and m_i , $i = 1, \dots, n$. Moreover, the example of d -cubes, considered as direct vector sums of d 1-dimensional line segments, shows that the first two upper bounds in (13) are tight (cf. Theorem 5.1). □

We have the following immediate corollary of Proposition 4.2, which shows that d -cubes are tightly covered.

Corollary 4.3. *Let $\mathbb{E}^d = \mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_n$ be a decomposition of \mathbb{E}^d into the direct vector sum of its linear subspaces \mathbb{L}_i and let $K_i \subseteq \mathbb{L}_i$, $i = 1, \dots, n$, be tightly covered convex bodies. Then $K_1 \oplus \cdots \oplus K_n$ is tightly covered.*

Proof. For any $0 < \lambda < 1$, allowing K_n to be tightly covered in the proof of Proposition 4.2 yields $\prod_{i=1}^n N_\lambda(K_i) = N_\lambda(K_1 \oplus \cdots \oplus K_n)$ points in the convex body $K_1 \oplus \cdots \oplus K_n$, no two of which belong to the same homothet of $K_1 \oplus \cdots \oplus K_n$ with homothety ratio λ . □

Boltyanski and Martini [8] showed that $I(K_1 \oplus \cdots \oplus K_n) \leq \prod_{j=1}^n I(K_j)$, but that the equality does not hold in general since $I(B^2 \oplus B^2) = 7 < 9 = (I(B^2))^2$. Thus there exists $\lambda < 1$ such that $N_\lambda(B^2 \oplus B^2) = 7$, whereas $N_\lambda(B^2) = 3$. Hence, relation (14) does not hold and by Proposition 4.2, B^2 is not tightly covered.

Although the inequality $N_\lambda(K_1 \oplus \cdots \oplus K_n) \leq \prod_{i=1}^n N_\lambda(K_i)$ always holds, the example of $B^2 \oplus B^2$ shows that the equality (14) is not satisfied in general. We have the following general result on the covering index of direct vector sums of convex bodies.

Corollary 4.4. *Let $\mathbb{E}^d = \mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_n$ be a decomposition of \mathbb{E}^d into the direct vector sum of its linear subspaces \mathbb{L}_i and let $K_i \subseteq \mathbb{L}_i$ be convex bodies such that $\text{coin}(K_i) = f_{m_i}(K_i)$, $i = 1, \dots, n$, and $\Gamma = \max\{\gamma_{m_i}(K_i) : 1 \leq i \leq n\}$. Then*

$$\max\{\text{coin}(K_i) : 1 \leq i \leq n\} \leq$$

$$\begin{aligned} \text{coin}(K_1 \oplus \cdots \oplus K_n) &\leq \inf_{\lambda \leq \frac{1}{2}} \frac{\prod_{i=1}^n N_\lambda(K_i)}{1 - \lambda} \\ &\leq \frac{\prod_{i=1}^n N_\Gamma(K_i)}{1 - \Gamma} \leq \frac{\prod_{i=1}^n m_i}{1 - \Gamma} < \prod_{i=1}^n \text{coin}(K_i). \end{aligned} \tag{15}$$

Moreover, the first three upper bounds in (15) are tight.

Let

$$K \subseteq \mathbb{E}^{d-k} \subseteq \mathbb{E}^{d-k} \oplus \underbrace{\mathbb{E}^1 \oplus \cdots \oplus \mathbb{E}^1}_k = \mathbb{E}^d$$

be a $(d - k)$ -dimensional convex body and $\ell \subseteq \mathbb{E}^1 \subseteq \mathbb{E}^d$ denote a line segment that can be optimally covered (in the sense of coin) by two homothets of homothety ratio $1/2$. We say that the d -dimensional convex body

$$K \oplus \underbrace{\ell \oplus \cdots \oplus \ell}_k \subseteq \mathbb{E}^d$$

is a (bounded) k -codimensional cylinder. We have seen that the covering index behaves nicely with direct vector sums. We now show that in the case of 1-codimensional cylinders it behaves even nicer.

Corollary 4.5. *For any 1-codimensional d -dimensional cylinder $K \oplus \ell$, the first two upper bounds in (13) become equalities and*

$$\text{coin}(K \oplus \ell) = 4N_{1/2}(K).$$

Proof. First note that since ℓ is tightly covered, Theorem 4.1 is applicable. From (13),

$$\begin{aligned} \text{coin}(K \oplus \ell) &= \inf_{\lambda \leq \frac{1}{2}} \frac{N_\lambda(K)N_\lambda(\ell)}{1 - \lambda} = \inf_{\lambda \leq \frac{1}{2}} \frac{N_\lambda(K) \lceil \lambda^{-1} \rceil}{1 - \lambda} \\ &\leq \frac{N_{1/2}(K)N_{1/2}(\ell)}{1 - \frac{1}{2}} = 4N_{1/2}(K). \end{aligned}$$

Suppose for some $0 < \lambda < 1/2$, $\frac{N_\lambda(K) \lceil \lambda^{-1} \rceil}{1 - \lambda} < 4N_{1/2}(K)$. Then

$$\lceil \lambda^{-1} \rceil \frac{N_\lambda(K)}{N_{1/2}(K)} < 4(1 - \lambda),$$

which is impossible, since, for $0 < \lambda < 1/2$, $\lceil \lambda^{-1} \rceil \geq 4(1 - \lambda)$ and $N_\lambda(K) \geq N_{1/2}(K)$.

Thus

$$\text{coin}(K \oplus \ell) = 4N_{1/2}(K).$$

□

In addition to the direct vector sum, coin displays a compatibility with the Minkowski sum (or simply vector sum) of convex bodies. We note that the upper bounds appearing here are the same as in Corollary 4.4.

Theorem 4.6. *Let the convex body K be the vector sum of the convex bodies K_1, \dots, K_n in \mathbb{E}^d , i.e., let $K = K_1 + \dots + K_n$ such that $\text{coin}(K_i) = f_{m_i}(K_i)$, $i = 1, \dots, n$, and $\Gamma = \max\{\gamma_{m_i}(K_i) : 1 \leq i \leq n\}$. Then*

$$\text{coin}(K) \leq \inf_{\lambda \leq \frac{1}{2}} \frac{\prod_{i=1}^n N_\lambda(K_i)}{1 - \lambda} \leq \frac{\prod_{i=1}^n N_\Gamma(K_i)}{1 - \Gamma} \leq \frac{\prod_{i=1}^n m_i}{1 - \Gamma} < \prod_{i=1}^n \text{coin}(K_i). \tag{16}$$

Moreover, equality in (16) does not hold in general.

Proof. Given homothetic coverings of K_i , $i = 1, \dots, n$, with homothety ratio $0 < \lambda \leq 1/2$, one can construct a homothetic covering of $K = K_1 + \dots + K_n$ with the same homothety ratio λ in a natural way. The proof of the upper bounds follows the same lines as in Theorem 4.1 and Corollary 4.4.

Furthermore, to show that equality in (16) does not hold in general, we consider the example of an affine regular convex hexagon $H = \Delta^2 + (-\Delta^2)$ and the corresponding triangle Δ^2 .

Belousov [1] showed that $\gamma_6(\Delta^2) = 1/2$ and $\gamma_m(\Delta^2) > 1/2$, for $1 \leq m < 6$. By Lemma 3.1, $\text{coin}(\Delta^2) = \inf\{f_m(\Delta^2) : 6 \leq m < 12\} \leq f_6(\Delta^2) = 12$. But Fudali [11] determined $\gamma_m(\Delta^2)$, for $7 \leq m \leq 15$, and routine calculations show that the corresponding f_m 's satisfy $f_m(\Delta^2) > 12$. Thus $\text{coin}(\Delta^2) = 12$. Now, Fig. 1 shows that H can be covered by 6 half-sized homothets. Thus $\text{coin}(H) \leq 12 = \text{coin}(\Delta^2)$. □

It is, in fact, easy to show that $\text{coin}(H) = 12$. First, observe that any translate of $\frac{1}{2}H$ can cover at most one-sixth of the boundary of H . Therefore, $\gamma_m(H) > 1/2$, for $m = 1, \dots, 5$. Thus, as in the case of Δ^2 , $\text{coin}(H) = \inf\{f_m(H) : 6 \leq m < 12\} \leq 12$. If $f_m(H) < 12$, for some $7 \leq m \leq 11$, then by the definition of $f_m(\cdot)$, $\gamma_m(H) < \frac{12-m}{12}$, and by the definition of covering, $m\gamma_m(H)^2 \text{vol}(H) \geq \text{vol}(H)$. Therefore, $m \left(\frac{12-m}{12}\right)^2 > 1$, which is impossible for $8 \leq m \leq 11$. This only leaves the case $m = 7$, but it is known [17] that (cf. the remarks immediately following (12)) $\gamma_7(H) = 1/2$ and as a result, $f_7(H) = 14$. We conclude that $\text{coin}(H) = 12$. This kind of ‘volumetric’ argument will remain useful throughout the next section in determining covering index values for convex bodies. Also Lemma 3.1 plays an important role, reducing the problem to finding the minimum of a finite set.

We now present an application of Theorem 4.6 to the difference body $K - K = K + (-K)$ of a convex body K . The result is quite useful for non-symmetric convex bodies. Once again, from the example of an affine regular convex hexagon and a triangle we note that equality does not hold in general.

Corollary 4.7. *If K is any d -dimensional convex body, such that $\text{coin}(K) = f_m(K)$. Then*

$$\text{coin}(K - K) \leq \frac{(N_{\gamma_m(K)}(K))^2}{1 - \gamma_m(K)} \leq \frac{m^2}{1 - \gamma_m(K)} < (\text{coin}(K))^2. \tag{17}$$

Moreover, equality in (17) does not hold in general.

Since the upper bounds given in relations (16) and (17) match the upper bounds in (15), it is natural to ask if the same is true for the lower bounds. However, the arguments used in the proof of Theorems 4.1 and 4.6 do not seem to settle this question.

Problem 3. Let K_1, \dots, K_n be d -dimensional convex bodies, for some $d \geq 2$. Then prove (disprove) that

$$\max\{\text{coin}(K_i) : i = 1, \dots, n\} \leq \text{coin}(K_1 + \dots + K_n). \tag{18}$$

If this does not hold, one can try to prove the following weaker lower bound.

$$\min\{\text{coin}(K_i) : i = 1, \dots, n\} \leq \text{coin}(K_1 + \dots + K_n). \tag{19}$$

The example of a triangle and a hexagon considered above indicates that either lower bound, if it holds, would be tight. The conjectured relations (18) and (19) both lead to interesting consequences, which we discuss below.

If the weaker result (19) is satisfied, combining it with Corollary 4.7 would give $\text{coin}(K) \leq \text{coin}(K - K)$. This would show that for any convex body K , the o -symmetric convex body $K - K$ has a covering index at least as large as $\text{coin}(K)$. This, in turn, would imply that in computing the supremum of $\text{coin}(K)$ over all d -dimensional convex bodies one could restrict themselves to the class of o -symmetric convex polytopes.

If the stronger result (18) holds, we would be able to say even more. It is known that any nonempty intersection of translates of B^d is a Minkowski summand of B^d (see [28], Theorem 3.2.5). This includes the class of all d -dimensional ball-polyhedra [4], which are nonempty intersections of finitely many translates of B^d . Result (18) would imply that $\text{coin}(B^d)$ upper bounds the covering indices of ball-polyhedra, or more generally of nonempty intersections of translates of B^d .

5. Extremal bodies

The aim of this section is to characterize the convex bodies that maximize or minimize the covering index among all d -dimensional convex bodies. In addition, we compute exact values and estimates of the covering index for a number of convex bodies.

Since coin is a lower semicontinuous functional defined on the compact space [21] \mathcal{K}^d , it is guaranteed to achieve its infimum over \mathcal{K}^d , that is, there

exists $M \in \mathcal{K}^d$ such that $\text{coin}(M) = \inf \{ \text{coin}(K) : K \in \mathcal{K}^d \}$. We have the following assertion about the minimizers of coin .

Theorem 5.1. *Let d be any positive integer and $K \in \mathcal{K}^d$. Then $\text{coin}(C^d) = 2^{d+1} \leq \text{coin}(K)$ and thus (affine) d -cubes minimize the covering index in all dimensions.*

Proof. Clearly, C^d can be covered by 2^d homothets of homothety ratio $1/2$, and cannot be covered by fewer homothets. Therefore, $\text{coin}(C^d) \leq f_{2^d}(C^d) = 2^{d+1}$. Let p be a positive integer. If there exists a homothetic covering of C^d by $m = 2^d + p$ homothets giving $f_m(C^d) < 2^{d+1}$, then

$$\gamma_m(C^d) < \frac{1}{2} - \frac{p}{2^{d+1}}.$$

However,

$$\begin{aligned} m \text{vol}(\gamma_m(C^d)C^d) &= m\gamma_m(C^d)^d \text{vol}(C^d) \\ &< (2^d + p) \left[\frac{1}{2} - \frac{p}{2^{d+1}} \right]^d \text{vol}(C^d) < \text{vol}(C^d), \end{aligned}$$

a contradiction, showing that $\text{coin}(C^d) = 2^{d+1}$.

Now consider an arbitrary d -dimensional convex body K . By repeating the above calculations for K we see that for $m > 2^d$, $f_m(K)$ cannot be smaller than 2^{d+1} . A similar volumetric argument shows that K cannot be covered by 2^d homothets having homothety ratio less than $1/2$. Likewise, it is impossible to cover K by fewer than 2^d homothets if the homothety ratio does not exceed $1/2$. Thus $\text{coin}(K) \geq 2^{d+1}$. \square

It is known that $C(C^d) = 2^{d+1}$ [29]. Thus $\text{coin}(C^d) = C(C^d)$. Do affine d -cubes also minimize the covering parameter? The answer is negative in general and open for $d = 2, 3$. An affine regular d -simplex Δ^d can be covered by $d + 1$ homothetic copies each with homothety ratio $d/(d + 1)$. Thus $C(\Delta^d) \leq (d + 1)^2$, which is less than $C(C^d)$ for $d > 3$. The question which convex bodies minimize (or maximize) the covering parameter is wide open, even in the plane. Restricting the homothety ratio to not exceed half plays a crucial role in determining the optimizers of the covering index.

The case of coin -maximizers is more involved. Indeed, since we have not established the upper semicontinuity of coin , it may be the case that for some d , $\sup \{ \text{coin}(K) : K \in \mathcal{K}^d \}$ is not achieved by any d -dimensional convex body. However, this is not the case for $d = 2$.

Theorem 5.2. *If K is a planar convex body then $\text{coin}(K) \leq \text{coin}(B^2) = 14$.*

Proof. First, we show that $\text{coin}(B^2) = 14$. It is rather trivial that $\gamma_1(B^2) = \gamma_2(B^2) = 1$, $\gamma_3(B^2) = \sqrt{3}/2 = 0.866\dots$, and $\gamma_4(B^2) = 1/\sqrt{2} = 0.707\dots$. Hence, $f_1(B^2) = f_2(B^2) = f_3(B^2) = f_4(B^2) = +\infty$. Moreover, the first named author [2] showed that $\gamma_5(B^2) = 0.609\dots$ and $\gamma_6(B^2) = 0.555\dots$,

implying that $f_5(B^2) = f_6(B^2) = +\infty$. On the other hand, it is easy to see that $\gamma_7(B^2) = 1/2$ and therefore $f_7(B^2) = 14$. Hence Lemma 3.1 implies that $\text{coin}(B^2) = \min \{f_m(B^2) : 7 \leq m < 14\}$.

Next, recall Fejes Tóth’s result [10] according to which $\gamma_8(B^2) = 0.445\dots$ and $\gamma_9(B^2) = 1/(1 + \sqrt{2}) = 0.414\dots$. This implies $f_8(B^2) = 14.420\dots > 14$ and $f_9(B^2) = 15.363\dots > 14$.

We claim that $f_m(B^2) > 14$, for all $10 \leq m < 14$. Suppose for some $10 \leq m < 14$, $f_m(B^2) \leq 14$. In this case, we must have $\gamma_m(B^2) \leq \frac{14-m}{14}$ and $m \text{ vol}(\gamma_m(B^2)B^2) > \text{vol}(B^2)$. This implies $m \left(\frac{14-m}{14}\right)^2 > 1$. But, routine calculations show that the latter inequality fails to hold for all $10 \leq m \leq 13$. Thus $\text{coin}(B^2) = 14$.

Levi [19] showed that any planar convex body K can be covered by 7 homothets of homothety ratio $1/2$. Thus $\text{coin}(K) \leq 14$, proving that the circle maximizes the covering index in the plane. \square

Although the question of maximizers is open in general, we can use Corollary 4.5 and Theorem 5.2 to determine a maximizer among 1-codimensional cylinders in \mathcal{K}^3 . In addition, we determine the covering indices of several 1-codimensional cylinders.

Corollary 5.3. *We have the following:*

- (i) $\text{coin}(\Delta^2 \oplus \ell) = 24$.
- (ii) $\text{coin}(H \oplus \ell) = 24$.
- (iii) $\text{coin}(B^2 \oplus \ell) = 28$.
- (iv) *If $K \oplus \ell$ is a 1-codimensional cylinder in \mathcal{K}^3 , then $\text{coin}(K \oplus \ell) \leq 28$, that is $B^2 \oplus \ell$ maximizes coin among 3-dimensional 1-codimensional cylinders.*

Proof. The assertions (i)–(iii) follow immediately from Corollary 4.5 and the values of $\text{coin}(\Delta^2)$, $\text{coin}(H)$ and $\text{coin}(B^2)$ determined earlier. For (iv), recall that [19] for a planar convex body K , $\max N_{1/2}(K) = 7$. \square

We remark that the process can be continued in higher dimensions to obtain exact values or estimates of the covering index of convex bodies that are vector sums or direct vector sums of lower dimensional convex bodies.

So far, we have computed the covering index mostly for planar convex bodies. Since in higher dimensions very little is known about $\gamma_m(K)$, it is a lot harder to determine exact values of coin. In some cases it is possible to derive upper bounds. For instance, we make the following observation for d -dimensional balls.

Remark 5.4. Recently, O’Rourke [23] raised the question as to what is the minimum number of homothets of homothety ratio $1/2$ needed to cover B^3 . Using spherical cap coverings, Wynn [23] showed this number to be 21. Thus $N_{1/2}(B^3) = 21$. In fact, Wynn also demonstrated that if the homothety ratio is decreased to 0.49439, we can still cover B^3 by 21 homothets. Figure 2 illustrates

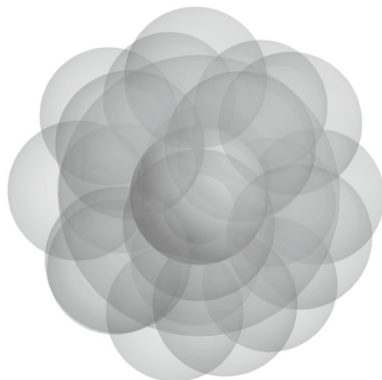


FIGURE 2. A covering of B^3 by 21 homothets of homothety ratio 0.49439. One homothet is centered at the center of B^3 , while the centers of the other 20 homothets lie at a distance of 0.8595 from the center of B^3 (see Remark 5.4)

such a covering. (On request one can obtain the Mathematica code to generate this covering from the second named author.) Therefore, $\text{coin}(B^3) \leq f_{21}(B^3) \leq 41.5339886473764$. Moreover, by applying Corollary 4.5, $\text{coin}(B^3 \oplus \ell) = 84$.

In general, Verger-Gaugry [30] showed that in any dimension $d \geq 2$ one can cover a ball of radius $1/2 < r \leq 1$ with $O((2r)^{d-1} d^{3/2} \ln d)$ balls of radius $1/2$. Substituting $r = 1$ and performing the standard covering index calculations shows that $\text{coin}(B^d) = O(2^d d^{3/2} \ln d)$.

The above remark is interesting for three different reasons. First, we observed that for B^2 , C^2 and Δ^2 , the value of covering index is associated with the homothety ratio $1/2$. Remark 5.4 provides us an example, namely B^3 , where the covering index is associated with a homothety ratio strictly less than $1/2$. Thus half-sized homothets do not always correspond to the covering index values. Second, Remark 5.4 provides another example of a situation when inequality (12) is strict, as $\gamma_{N_{1/2}(B^3)}(B^3) = \gamma_{21}(B^3) < 1/2$. Finally, since B^2 maximizes the covering index in the plane, it can be asked if the same is true for B^d in higher dimensions.

Problem 4. For any d -dimensional convex body K , prove or disprove that $\text{coin}(K) \leq \text{coin}(B^d)$ holds.

An affirmative answer to Problem 4 would considerably improve the known general (Rogers-type) upper bound on the illumination number. It is known (e.g., see [5]) that for any d -dimensional convex body K , in general

$$I(K) \leq \binom{2d}{d} d(\ln d + \ln \ln d + 5) = O(4^d \sqrt{d} \ln d), \tag{20}$$

TABLE 1. Known values (or estimates) of coin. The table can be extended indefinitely by including values (or estimates) of $\text{coin}(K \oplus L)$ and by including upper bounds on $\text{coin}(K+L)$, for any convex bodies K and L appearing in the table

K	m	$\gamma_m(K)$	$\text{coin}(K)$
ℓ	2	1/2	4
H	6	1/2	12
Δ^2	6	1/2	12
B^2	7	1/2	14
B^3	≥ 21	≤ 0.49439	$\leq 41.53398 \dots$
B^d	$O(2^d d^{3/2} \ln d)$	$\leq 1/2$	$O(2^d d^{3/2} \ln d)$
C^d	2^d	1/2	2^{d+1}
$H \oplus \ell$	12	1/2	24
$\Delta^2 \oplus \ell$	12	1/2	24
$B^2 \oplus \ell$	14	1/2	28
\vdots	\vdots	\vdots	\vdots

and if, in addition, K is o -symmetric, then

$$I(K) \leq 2^d d(\ln d + \ln \ln d + 5) = O(2^d d \ln d). \tag{21}$$

If B^d maximizes the covering index, then the general asymptotic bound in (20) would improve to within a factor \sqrt{d} of the bound (21) in the o -symmetric case.

We conclude by listing some of the known values (or estimates) of the covering index. We remark that Table 1 can be continued indefinitely by using the operations of direct vector addition and the Minkowski addition, resulting in infinitely many convex bodies for which we know exact values of coin, and infinitely many convex bodies for which we can estimate coin.

6. The weak covering index

In this section, we introduce a variant of the covering index, which we call the *weak covering index*.

Definition 3. Let K be a d -dimensional convex body. We define the *weak covering index* of K as

$$\text{coin}_w(K) = \inf \left\{ \frac{m}{1 - \gamma_m(K)} : \gamma_m(K) < 1, m \in \mathbb{N} \right\}.$$

TABLE 2. Known values (or estimates) of $\text{coin}_w(\cdot)$ together with the corresponding m and $\gamma_m(\cdot)$

K	m	$\gamma_m(K)$	$\text{coin}_w(K)$
ℓ	2	1/2	4
H	3	2/3	9
Δ^2	3	2/3	9
B^2	5	0.609...	12.800...
C^d	2^d	1/2	2^{d+1}
Δ^d	$\geq d + 1$	$\leq \frac{d}{d+1}$	$\leq (d + 1)^2$
$H \oplus \ell$	≥ 6	$\leq 2/3$	≤ 18
$\Delta^2 \oplus \ell$	≥ 6	$\leq 2/3$	≤ 18
$B^2 \oplus \ell$	≥ 10	$\leq 0.609...$	$\leq 25.60...$
\vdots	\vdots	\vdots	\vdots

Let us define

$$g_m(K) = \begin{cases} \frac{m}{1 - \gamma_m(K)}, & \text{if } 0 < \gamma_m(K) < 1, \\ +\infty, & \text{if } \gamma_m(K) = 1. \end{cases}$$

Then $\text{coin}_w(K) = \inf \{g_m(K) : m \in \mathbb{N}\}$.

Some properties of the weak covering index naturally mirror the corresponding properties of the covering index. These include Proposition 2.1, Lemma 3.1, Theorem 4.1, Corollary 4.4 and Theorem 4.6. The corresponding statements can be obtained by replacing coin with coin_w and f_m by g_m throughout.

However, no suitable analogue of Corollary 4.5 exists for coin_w . As a result, we can only estimate the weak covering index of 1-codimensional cylinders in Table 2. Also the discussed aspects of continuity of the covering index (Sect. 3) seem to be lost for the weak covering index.

More importantly, the problem of finding the maximizers and minimizers of coin_w seems a lot harder than the corresponding problem for coin . We only know a minimizer for $d = 2$.

Theorem 6.1. *Let $K \in \mathcal{K}^2$, then $\text{coin}_w(K) \geq \text{coin}_w(C^2) = 8$. Thus the (affine) square minimizes the weak covering index in the plane.*

Proof. If K is such that $\text{coin}_w(K) = g_m(K) < 8$, then from the proof of Theorem 5.1, $m < 4$. Since any convex body in \mathcal{K}^2 requires at least 3 smaller positive homothets to cover it, we only need to consider the case $m = 3$. But Belousov [1] showed that

$$\min_{K \in \mathcal{K}^2} \gamma_3(K) = \frac{2}{3}$$

and so, $g_3(K) \geq 9 > \text{coin}_w(C^2)$, a contradiction. □

It is worth noting that for $d \geq 3$, the simplex Δ^d gives a smaller value ($\leq (d + 1)^2$) of coin_w than the d -cube C^d . Thus coin_w possibly has different minimizers in different dimensions.

7. Bounds on the covering indices

In this section, we obtain upper bounds on the covering and weak covering index in the spirit of Rogers’ bounds on covering numbers. The main ingredients include Rogers’ estimate [24] of the infimum $\theta(K)$ of the covering density of \mathbb{E}^d by translates of the convex body K , namely, for $d \geq 2$,

$$\theta(K) \leq d(\ln d + \ln \ln d + 5),$$

the Rogers-Shephard inequality [25]

$$\text{vol}(K - K) \leq \binom{2d}{d} \text{vol}(K)$$

on the volume of the difference body, and a well-known result of Rogers and Zong [27], which states that for d -dimensional convex bodies K and L , $d \geq 2$,

$$N(K, L) \leq \frac{\text{vol}(K - L)}{\text{vol}(L)} \theta(L), \tag{22}$$

with $K - L = K + (-L)$.

The above inequalities yield the well-known upper bounds (20) and (21) on the illumination number. In addition, we mention Lassak’s general upper bound [18] on the illumination number

$$I(K) \leq (d + 1)d^{d-1} - (d - 2)(d - 1)^{d-1}, \tag{23}$$

which is sharper than (20) for small d , although we do not use it here.

Theorem 7.1. *Given $K \in \mathcal{K}^d$, $d \geq 2$ and a real number $0 < \lambda < 1$, we have*

$$\begin{aligned} \text{coin}_w(K) &\leq \frac{N_\lambda(K)}{1 - \lambda} \\ &\leq \begin{cases} \frac{(1 + \lambda)^d}{\lambda^d(1 - \lambda)} d(\ln d + \ln \ln d + 5), & \text{if } K \text{ is } o\text{-symmetric,} \\ \frac{1}{\lambda^d(1 - \lambda)} \left(\left(\binom{2d}{d} \right)^{1/d} - 1 + \lambda \right)^d d(\ln d + \ln \ln d + 5), & \text{otherwise.} \end{cases} \end{aligned} \tag{24}$$

Proof. Consider a minimal cardinality covering of K by homothets $\lambda K + t_i$, for some $t_i \in \mathbb{E}^d$, $i = 1, \dots, N_\lambda(K)$. By (22), we have

$$\begin{aligned} N_\lambda(K) &\leq \frac{\text{vol}(K - \lambda K)}{\text{vol}(\lambda K)} \theta(\lambda K) = \frac{\text{vol}(K - \lambda K)}{\text{vol}(\lambda K)} \theta(K) \\ &= \frac{\text{vol}(K - \lambda K)}{\text{vol}(\lambda K)} d(\ln d + \ln \ln d + 5). \end{aligned} \tag{25}$$

If K is o -symmetric, then $\text{vol}(K - \lambda K) = \text{vol}((1 + \lambda)K) = \frac{(1+\lambda)^d}{\lambda^d} \text{vol}(\lambda K)$ and so, (25) implies

$$N_\lambda(K) \leq \frac{(1 + \lambda)^d}{\lambda^d} d(\ln d + \ln \ln d + 5).$$

In the general case, applying the Brunn-Minkowski inequality gives

$$\begin{aligned} \lambda^{-1} \text{vol}(K - K)^{1/d} &= \text{vol}((\lambda^{-1}K - K) + (-\lambda^{-1} - 1)K)^{1/d} \\ &\geq \text{vol}(\lambda^{-1}K - K)^{1/d} + \text{vol}((\lambda^{-1} - 1)K)^{1/d} \\ &= \lambda^{-1} \text{vol}(K - \lambda K)^{1/d} + (\lambda^{-1} - 1) \text{vol}(K)^{1/d}, \end{aligned}$$

which gives

$$\text{vol}(K - \lambda K)^{1/d} \leq \text{vol}(K - K)^{1/d} - (\lambda^{-1} - 1)\lambda \text{vol}(K)^{1/d}.$$

By the Rogers-Shephard inequality, we have

$$\begin{aligned} \text{vol}(K - \lambda K)^{1/d} &\leq \binom{2d}{d}^{1/d} \text{vol}(K)^{1/d} - (1 - \lambda) \text{vol}(K)^{1/d} \\ &= \lambda^{-1} \left(\binom{2d}{d}^{1/d} - 1 + \lambda \right) \text{vol}(\lambda K)^{1/d}. \end{aligned}$$

Substituting for $\text{vol}(K - \lambda K)$ in (25) gives

$$N_\lambda(K) \leq \lambda^{-d} \left(\binom{2d}{d}^{1/d} - 1 + \lambda \right)^d d(\ln d + \ln \ln d + 5).$$

Finally, note that clearly $\text{coin}_w(K) \leq \frac{N_\lambda(K)}{1-\lambda}$. The upper bounds in (24) follow. □

For $\lambda = \frac{d}{d+1}$, Theorem 7.1 gives the following upper bounds on the weak covering index.

Corollary 7.2. *Let $K \in \mathcal{K}^d$, $d \geq 2$. Then*

$$\text{coin}_w(K) < \begin{cases} 2^d \sqrt{e}(d+1)d(\ln d + \ln \ln d + 5) = O(2^d d^2 \ln d), & \text{if } K \text{ is } o\text{-symmetric,} \\ e(d+1) \left(\binom{2d}{d}^{1/d} - 1 + \frac{d}{d+1} \right)^d d(\ln d + \ln \ln d + 5) \\ = O(4^d d^{3/2} \ln d), & \text{otherwise.} \end{cases}$$

Finally, in order to determine an upper bound on coin, one only needs to apply (24) with $\lambda = 1/2$.

Corollary 7.3. *Given $K \in \mathcal{K}^d$, $d \geq 2$, we have*

$$\text{coin}(K) \leq 2N_{1/2}(K) \leq \begin{cases} 3^d(2d)(\ln d + \ln \ln d + 5) = O(3^d d \ln d), & \text{if } K \text{ is } o\text{-symmetric,} \\ 2^{d+1} \left(\binom{2d}{d}^{\frac{1}{d}} - \frac{1}{2} \right)^d d(\ln d + \ln \ln d + 5) \\ = O(7^d \sqrt{d} \ln d), & \text{otherwise.} \end{cases}$$

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