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THE KATO PROBLEM FOR OPERATORS WITH WEIGHTED ELLIPTICITY

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ABSTRACT. We consider second order operators $\mathcal{L}_w = -w^{-1}\operatorname{div}\mathbf{A}_w\nabla$ with ellipticity controlled by a Muckemphout A_2 weight w. We prove that the Kato square root estimate $\left\|\mathcal{L}_w^{1/2}f\right\|_{L^2(w)} \approx \|\nabla f\|_{L^2(w)}$ holds in the weighted space $L^2(w)$.

1. Introduction

The Kato problem has a long history. Originally conjectured in 1961 by T. Kato [13] in the context of evolution equations for maximally accretive operators on Hilbert spaces, it developed into its present formulation for differential operators after counterexamples were found for general operators on Hilbert spaces and for operators arising from sesquilinear forms [15,16]. See [14] for a clear exposition and further references on the rich history of this problem and its solution. The Kato square root problem, as it is known today [16], consisted in showing that if $\mathbf{A}$ is a complex-valued $n\times n$ elliptic matrix, then the square root $\mathcal{L}^{1/2}$ of the second order operator $\mathcal{L} = -\operatorname{div}\mathbf{A}\nabla$ has domain H^1 and it satisfies

$$\left\|\mathcal{L}^{1/2}f\right\|_{L^2} \approx \|\nabla f\|_{L^2},$$

where $\approx$ means that the related quantities lie within two multiple constants of each other, with constants independent of the function f.

Given a weight w in the Muckenhoupt class A_2, and $0<\lambda\le\Lambda<\infty$, denote by $\mathcal{E}_n(\lambda,\Lambda,w)$ the class of complex-valued $n\times n$ matrices $\mathbf{A}_w$ such that

$$\begin{cases} \lambda w(x)|\xi|^2 & \le \operatorname{Re}\langle\mathbf{A}_w(x)\xi,\xi\rangle = \operatorname{Re}\sum_{i,j=1}^n A_{ij}(x)\xi_i\overline{\xi_j}, \\ \Lambda w(x)|\xi||\eta| & \ge |\langle\mathbf{A}_w(x)\xi,\eta\rangle| \end{cases} \tag{1.1}$$

for all $\xi,\eta\in\mathbb{C}^n$. That is, $\mathbf{A}_w\in\mathcal{E}_n(w,\lambda,\Lambda)$ if and only if $w^{-1}\mathbf{A}_w$ is complex elliptic with constants λ and Λ. For $\mathbf{A}_w\in\mathcal{E}_n(w,\lambda,\Lambda)$, we consider weighted elliptic operators

$$\mathcal{L}_w = -w^{-1}\operatorname{div}\mathbf{A}_w\nabla \qquad \text{on } H^1(w). \tag{1.2}$$

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Our main result is the following extension of the Kato problem to A_2-weighted spaces (see Section 2.1 for the definitions of $L^2(w)$ and $H^1(w)$):

Theorem 1.1. *Given $w \in A_2$ and $\mathbf{A}_w \in \mathcal{E}_n(w, \lambda, \Lambda)$, there exists a positive constant $C = C\left(n, \lambda, \Lambda, [w]_{A_2}\right)$ such that for all $f \in H^1(w)$,*

$$C^{-1} \|\nabla f\|_{L^2(w)} \leq \left\|\mathcal{L}_w^{1/2} f\right\| \leq C \|\nabla f\|_{L^2(w)}.$$

Weighted operators arise naturally as pull-back operators under (smooth enough) changes of variables. Let F be a one-to-one invertible transformation $\mathbb{R}^n \mapsto \mathbb{R}^n$ such that F and $F^{-1} \in W^{1,n}_{\text{loc}}(\mathbb{R}^n)$. The transformation F is quasiconformal if for some constant $K \geq 1$, $\|F_y\|_\infty \leq K (\det F_y)^{1/n}$. Let $w = |\det F_y|^{1-2/n}$, and for an elliptic matrix $\widetilde{\mathbf{A}}(x)$ let $\mathbf{A}_w(y) = \mathbf{M}^t \widetilde{\mathbf{A}}(F(y)) \mathbf{M}$, with $\mathbf{M} = |\det(F_y)|^{1/2} \left((F_y)^{-1}\right)^t$. Then $\mathcal{L}_w = -w^{-1} \operatorname{div} \mathbf{A}_w \nabla = -\operatorname{div} \widetilde{\mathbf{A}} \nabla v = \mathcal{L}$, in the sense that

$$\langle \mathcal{L}_w u, \psi \rangle_w = \int \mathbf{A}_w(y) \nabla u(y) \cdot \overline{\nabla \psi(y)}\, dy = \int \widetilde{\mathbf{A}}(x) \nabla v(x) \cdot \overline{\nabla \varphi(x)}\, dx = \langle \mathcal{L} v, \varphi \rangle,$$

where $u = v \circ F$, $\psi = \varphi \circ F$. It was shown in [11] (see also Section 3 in [9]) that when F is quasiconformal the weight $w = |\det(F_y)|^{1-2/n}$ belongs to the Muckenhoupt class A_∞. In this work we consider operators $\mathcal{L}_w$ with weights lying in the class $A_2 \subset A_\infty$.

Operators of the form (1.2) have been studied in the case that $\mathbf{A}_w$ is real and symmetric. In [5] it was proved that solutions to the associated parabolic equation satisfy the parabolic Harnack inequality. In [4] similar parabolic operators were considered with quasiconformal weights. In [6, 7] we considered operators (1.2) under extra conditions on the coefficients (real symmetric) or under the assumption that the associated semigroup satisfies Gaussian upper bounds.

To prove Theorem 1.1 it suffices to prove the second inequality, as the first one follows from it by duality (see [7] for details). Our proof follows the same broad outline as the original proof in [1] for uniformly elliptic operators but it differs significantly in detail. The usual resources of integration by parts and Fourier transforms used in the original proof are not available in the weighted case. We circumvent these obstacles using techniques from the theory of weighted norm inequalities and also from interpolation theory. The proof is lengthy, but it divides naturally into three parts. First, we reduce the problem to an inequality for a square function based on the resolvent of $\mathcal{L}_w$. The second step is to prove that the square function estimate follows from a Carleson measure estimate. The last step is to prove a Tb theorem for the square root operator that yields the Carleson measure bound. Weighted norm inequalities play a central role in all three parts of the proof. However, we want to highlight one interesting feature of the proof. In [7] we were able to work exclusively with one-weight inequalities; here, however, these are not adequate. At the heart of the reduction to a Carleson measure estimate, to get the fine control required we need an (elementary) two-weight inequality (Theorem 2.7).

We must warn the reader that the Gaussian bounds obtained in [6] are mistaken. The techniques in that paper in fact yield a representation of the semigroup $e^{-t\mathcal{L}_w}$ with a kernel K_t acting on $L^1(w)$, and with upper bounds of the form $|K_t(x,y)| \leq C\overline{w}_t(x)^{-1} \exp\left(-c|x-y|^2/t\right)$, where $\overline{w}_t(x) = w\left(B_{\sqrt{t}}(x)\right)$. These bounds are equivalent to $L^1(w)$-$L^\infty(w)$ off-diagonal estimates on balls, as defined in [2]; see

Proposition 2.2 there. In turn, as a consequence of erroneous bounds in [6], the Gaussian bounds hypotheses in [7] are too restrictive, only allowing weights that are basically constant. However, the square function estimates obtained in that second paper, which do not rely on Gaussian bounds, are useful to prove the general Kato estimate in the present work.

The remainder of this paper is organized as follows. In Section 2 we gather a number of preliminary results on the weighted spaces involved and the weighted norm inequalities we will use. Much of what we need was developed in detail in [6, 7], so here we will only sketch the details. In this section we also include several weighted estimates for the resolvent of $\mathcal{L}_w$, some of which are included without proofs since they are standard estimates for operators arising from sesquilinear forms. The last three sections are devoted to the proof of Theorem 1.1; each section corresponds to one of the parts described above.

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2. Preliminaries

A function w is in the Muckenhoupt class A_2 if $w \geq 0$, $w \in L^1_{\text{loc}}$, and there exists a constant C such that $\left(\fint_B w\, dx\right)\left(\fint_B w^{-1}\, dx\right) \leq C$ for every ball $B \subset \mathbb{R}^n$. We denote the best constant of w in the A_2 condition by $[w]_{A_2}$. Throughout this paper C, c will denote constants whose value may change at each appearance. Unless we specify otherwise, the constants may depend on dimension n, the ellipticity constants λ, Λ, and the A_2 constant $[w]_{A_2}$ of the weight w.

2.1. **Weighted Sobolev spaces and weighted operators.** For $1 \leq p < \infty$, the space $L^p(w)$ is defined as all locally integrable f in $\mathbb{R}^n$ such that

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f|^p \;\, w dx\right)^{1/p} < \infty.$$

When $p = 2$, this space is a Hilbert space with inner product $\langle f, g\rangle_w = \int_{\mathbb{R}^n} f\overline{g}\, w dx$.

The domain of $\mathcal{L}_w$ is a dense subspace of $L^2(w)$ contained in the Hilbert space $H^1(w)$, with inner product

$$\langle f, g\rangle_{H^1(w)} = \int_{\mathbb{R}^n} f\overline{g}\, w dx + \int_{\mathbb{R}^n} \nabla f \cdot \overline{\nabla g}\, w dx. \tag{2.1}$$

Notice that $\lambda \|\nabla f\|^2_{L^2(w)} \leq \left|\int_{\mathbb{R}^n} \nabla f \mathbf{A}_w \overline{\nabla f}\, dx\right| \leq \Lambda \|\nabla f\|^2_{L^2(w)}$.

When w is a weight in the Muckenhoupt class A_2, the space $H^1(w)$ was first studied in [9], where local weighted Poincaré and Sobolev inequalities were proved. There it is also shown that $H^1(w)$ may be obtained as the closure of smooth compactly supported functions under the norm associated with the inner product (2.1). N. Miller proved in [18] (Theorem 3.3) that $H^1(w) = \Lambda^{-1}\left(L^2(w)\right)$, where Λ^s is the pseudodifferential operator with symbol $\left(1 + 4\pi^2 |\xi|^2\right)^{s/2}$ acting on tempered distributions. Note that every $f \in L^2(w)$ is a tempered distribution since it is locally integrable.

The properties and definition of the operator $\mathcal{L}_w$ and its associated semigroup $e^{-t\mathcal{L}_w}$ may be found in detail in [6]; $\mathcal{L}_w$ arises as the operator associated with the sesquilinear form

$$\mathfrak{a}(f,g) = \int_{\mathbb{R}^n} \mathbf{A}_w \nabla f \cdot \overline{\nabla g}\, dx = \int_{\mathbb{R}^n} \mathbf{A}\nabla f \cdot \overline{\nabla g}\, wdx, \qquad \text{where } \mathbf{A} = \mathbf{A}_w/w,$$

defined for $f,g \in H^1(w)$. That is, given f in the domain of $\mathcal{L}_w$, $\mathcal{L}_w f$ is the unique element of $L^2(w)$ such that

$$\langle \mathcal{L}_w f, g\rangle_w = \mathfrak{a}(f,g) \qquad \text{for all } g \in H^1(w).$$

2.2. **Weighted square function estimates.** Here we gather a number of one-weight inequalities for square functions.

Lemma 2.1 (Lemma 4.1 in [7]). *Suppose $|\phi(x)| \le \Phi(x)$ with $\Phi \in L^1(\mathbb{R}^n)$ a radial function. Then for any $w \in A_2$ the operators $f \mapsto \phi_t * f$ are uniformly bounded on $L^2(w)$. Moreover,*

$$\sup_{t>0} \|\phi_t * f\|_{\mathcal{B}(L^2(w))} \le C\left(n, [w]_{A_2}\right) \|\Phi\|_{L^1}.$$

Lemma 2.2 (Lemma 4.5 in [7]). *Let $w \in A_2$ and suppose that $\gamma_t(x)$ satisfies*

$$\|\gamma_t\|_{C,w} = \sup_Q \frac{1}{w(Q)} \int_Q \int_0^{\ell(Q)} |\gamma_t(x)|^2 \frac{dt}{t}\, wdx < \infty,$$

where the supremum is taken over all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes. Let $p \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be nonnegative, support $(p) \subset B_1(0)$, $\int p dx = 1$, *and denote $P_t f = p_t * f$, where $p_t(x) = t^{-n} p(x/t)$. Then for all $f \in L^2(w)$,*

$$\int_0^\infty \int_{\mathbb{R}^n} |P_t f(x)|^2 |\gamma_t(x)|^2\, wdx\, \frac{dt}{t} \le C\, \|\gamma_t\|_{C,w} \|f\|^2_{L^2(w)}.$$

Proposition 2.3 (Proposition 4.7 in [7]). *Let $w \in A_2$ and let ψ be a radial Schwartz function such that $\widehat{\psi}(0) = 0$ and*

$$\int_0^\infty \widehat{\psi}(t)^2 \frac{dt}{t} = 1.$$

Let $Q_t f(x) = \psi_t * f(x)$. Assume given a family of sublinear operators $\{R_t\}$ with each R_t individually bounded on $L^2(w)$ and such that for all $t, s > 0$ the composition $R_t Q_s$ verifies

$$\|R_t Q_s\|_{\mathcal{B}(L^2(w))} \le K \left(\min\left\{\frac{t}{s}, \frac{s}{t}\right\}\right)^\alpha$$

for some $K, \alpha > 0$. Then $\{R_t\}$ satisfies the quadratic estimate

$$\int_0^\infty \int_{\mathbb{R}^n} |R_t f(x)|^2\, wdx\, \frac{dt}{t} \le K\, C\left(n, \psi, \alpha, [w]_{A_2}\right) \|f\|^2_{L^2(w)}.$$

Remark 2.4. The radial function ψ in the previous proposition may be chosen to also satisfy support $(\psi) \subset B_1(0)$. Moreover, given an integer $d \ge 0$, there exists ψ that also satisfies $\int_{\mathbb{R}^n} p(x)\psi(x)\, dx = 0$ for all polynomials of degree at most d. See Lemma 1.1 in [10].

Lemma 2.5 (Lemma 4.10 in [7]). *Suppose that a sublinear operator T is bounded on $L^2(w)$ for all $w \in A_2$, with $\|T\|_{\mathcal{B}(L^2(w))}$ depending only on $[w]_{A_2}$ and the dimension n. Then for any $w \in A_2$, there exists $0<\theta<1$ depending on $[w]_{A_2}$ such that*

$$\|T\|_{\mathcal{B}(L^2(w))} \le C\left(n,[w]_{A_2}\right)\|T\|^{1-\theta}_{\mathcal{B}(L^2(w^s))}\|T\|^{\theta}_{\mathcal{B}(L^2)},$$

where $s>1$ is such that $w^s \in A_2(\mathbb{R}^n)$.

2.3. **A local two-weight estimate.** Given $f \in L^1_{\mathrm{loc}}$, for each radius $r>0$ we define the averaging operator

$$\Theta_r f(x) = \fint_{B_r(x)} |f(y)|\ dy := \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y)|\ dy. \tag{2.2}$$

In the proof of the reduction of our main result to a Carleson measure condition we will use a two-weight estimate for the operator Θ_r. We define a radius-restricted two-weight A^r_p class:

Definition 2.6. We say that a pair of weights (u,v) satisfies the A^r_p-condition if there exists $C>0$ such that for all balls $B_r \subset \mathbb{R}^n$ of radius $r>0$:

$$\left(\fint_{B_r} u(x)\ dx\right)\left(\fint_{B_r} v(x)^{1-p'}\ dx\right)^{p-1} \le C. \tag{2.3}$$

The infimum of such constants will be denoted by $[u,v]_{A^r_p}$.

Note that the classic two-weight A_p class consists of pairs of weights (u,v) such that $\sup_{r>0}[u,v]_{A^r_p}<\infty$.

Theorem 2.7. *Suppose $(u,v) \in A^r_p$ and that for some $\delta>0$ the weight $v^{1-p'}$ satisfies a reverse Hölder condition on ball of radii r:*

$$\fint_{B_r} v(x)^{1-p'}\ dx \le C\left(\fint_{B_r} v(x)^{\frac{1-p'}{1+\delta}}\ dx\right)^{1+\delta}. \tag{2.4}$$

Then there exists $\varepsilon = \varepsilon(\delta,[u,v]_{A^r_p})>0$ such that Θ_r is bounded from $L^q(v)$ to $L^q(u)$ for all $p-\varepsilon<q<\infty$. More precisely, there exists $C_q = C\left(q,n,[u,v]_{A^r_p},\delta\right)$ such that

$$\|\Theta_r f\|_{L^q(u)} \le C_q\|f\|_{L^q(v)}.$$

Proof. By a standard approximation argument we may assume $f \in L^p(v)$ is nonnegative, bounded, and has compact support. We will first show that Θ_r satisfies the weak-type inequality $\|\Theta_r f\|_{L^{p,\infty}(u)} \le C\|f\|_{L^p(v)}$ whenever $(u,v)\in A^r_p$. For $\lambda>0$ let $E_\lambda = \{x\in\mathbb{R}^n : \Theta_r f(x)>\lambda\}$. Then E_λ is bounded, so by the Besicovitch covering lemma, there exist points $\{x_j\}\in E_\lambda$ such that $E_\lambda \subset \bigcup_{j\in\mathbb{N}} B_r(x_j)$ and the collection $\{B_r(x_j)\}_{j\in\mathbb{N}}$ has finite overlap. Then

$$\begin{aligned}
u(E_\lambda) &\le \sum_{j\in\mathbb{N}} u(B_r(x_j))\\
&\le \frac{1}{\lambda^p}\sum_{j\in\mathbb{N}} u(B_r(x_j))\left(\fint_{B_r(x_j)}|f(y)|\,dy\right)^p\\
&\le \sum_{j\in\mathbb{N}}\frac{1}{\lambda^p}\left(\int_{B_r(x_j)}|f(y)|^p\ v dy\right)\left(\fint_{B_r(x_j)} u dx\right)\left(\fint_{B_r(x_j)} v^{1-p'}dy\right)^{p-1}.
\end{aligned}$$

The two-weight A_p^r condition (2.3) and the finite overlap property of $\{B_r(x_j)\}_{j\in\mathbb{N}}$ imply

$$u(E_\lambda) \leq [u,v]_{A_p^r} \frac{1}{\lambda^p} \sum_{j\in\mathbb{N}} \int_{B_r(x_j)} |f(y)|^p \ v dy \leq \frac{C(n)[u,v]_{A_p^r}}{\lambda^p} \int_{\mathbb{R}^n} |f(y)|^p \ v dy.$$

This shows that Θ_r is of weak type (p,p).

Now it follows from (2.4) that $(u,v) \in A_{p-\varepsilon}^r$ for some $\varepsilon > 0$. Moreover, by Hölder's inequality, $(u,v) \in A_q^r$ for all $q > p$. Therefore, by the Marcinkiewicz interpolation, for $p-\varepsilon < q < \infty$,

$$\left(\int_{\mathbb{R}^n} |\Theta_r f|^q \ u(x)\, dx\right)^{\frac{1}{q}} \leq C_q \left(\int_{\mathbb{R}^n} |f(x)|^q \ v(x)\, dx\right)^{\frac{1}{q}}.$$

□

2.4. **Estimates for the resolvent.** Henceforth we adopt the following notation: $\arg z$ denotes the argument in $(-\pi,\pi]$ for $z \in \mathbb{C}$; for $\theta \in (-\pi,\pi]$ we let $\Sigma_\theta = \{z \in \mathbb{C} : |\arg z| < \theta\}$. The following lemma follows by standard techniques of the theory of operators arising from accretive sesquilinear forms (see [19] for details).

Lemma 2.8. *For all $z \in \Sigma\left(\frac{\pi}{2}\right)$ we have that the operators $(\mathcal{L}_w + z)^{-1}$, $\nabla(\mathcal{L}_w + z)^{-1}$, and $(\mathcal{L}_w + z)^{-1} \frac{1}{w} \operatorname{div} w$ are bounded in $L^2(w)$. Moreover, for all $f \in L^2(w)$ and $\mathbf{f} \in \left(L^2(w)\right)^n$,*

$$\left\|(1+z\mathcal{L}_w)^{-1} f\right\|_{L^2(w)} \leq \frac{|z|}{\operatorname{Re} z} \|f\|_{L^2(w)}, \tag{2.5}$$

$$\left\|\nabla(1+z\mathcal{L}_w)^{-1} f\right\|_{L^2(w)} \leq \frac{1}{2(\lambda \operatorname{Re} z)^{\frac{1}{2}}} \|f\|_{L^2(w)}, \tag{2.6}$$

$$\left\|(1+z\mathcal{L}_w)^{-1} \frac{1}{w} \operatorname{div} w\mathbf{f}\right\|_{L^2(w)} \leq \frac{1}{2(\lambda \operatorname{Re} z)^{\frac{1}{2}}} \|\mathbf{f}(x)\|_{L^2(w)}. \tag{2.7}$$

Moreover, if $z \in \mathbb{C}\backslash\Sigma((\mu) \bigcup -\Sigma(\mu))$ for some μ satisfying $0 < \omega < \mu < \frac{\pi}{2}$, where $\omega = \arctan\left(\left(\frac{\Lambda^2}{\lambda^2} - 1\right)^{\frac{1}{2}}\right)$, then

$$\left\|\nabla(1+z\mathcal{L}_w)^{-1} \frac{1}{w} \operatorname{div} w\mathbf{f}\right\|_{L^2(w)} \leq \frac{|z|}{c} \|\mathbf{f}\|_{L^2(w)}, \tag{2.8}$$

where $c = c(\omega,\mu) > 0$.

The corollary below follows from the above resolvent bounds as in the proof of Lemma 2.2 in [1].

Corollary 2.9. *For any $\mathcal{C}^{0,1}$ function f and $t > 0$,*

$$\left\|\left[\left(1+t^2\mathcal{L}_w\right)^{-1}, f\right]\right\|_{\mathcal{B}(L^2(w))} \leq Ct \|\nabla f\|_\infty$$

and

$$\left\|\nabla\left[\left(1+t^2\mathcal{L}_w\right)^{-1}, f\right]\right\|_{\mathcal{B}(L^2(w))} \leq C \|\nabla f\|_\infty.$$

2.5. **Off-diagonal estimates.**

Lemma 2.10. *Let E and F be two closed sets in $\mathbb{R}^n$ and let $d = \operatorname{dist}(E,F)$. Let $\tau = \arctan \frac{\lambda}{\sqrt{\Lambda^2-\lambda^2}}$; fix ν, $0<\nu<\frac{\pi}{4}+\frac{\tau}{2}$ and $z \in \Sigma(\nu)$. Then there exist positive constants C and c depending only on n, Λ, λ, and ν and such that for all $f \in L^2(w)$, $\mathbf{f} \in \left(L^2(w)\right)^n$, with support in E,*

$$\int_F \left|\left(I+z^2\mathcal{L}_w\right)^{-1} f\right|^2 \, wdx \quad \leq \quad Ce^{-c\frac{d}{|z|}} \int_E |f|^2 \, wdx, \tag{2.9}$$

$$\int_F \left|z\nabla\left(I+z^2\mathcal{L}_w\right)^{-1} f\right|^2 \, wdx \quad \leq \quad Ce^{-c\frac{d}{|z|}} \int_E |f|^2 \, wdx, \tag{2.10}$$

$$\int_F \left|\left(I+z^2\mathcal{L}_w\right)^{-1} z\frac{1}{w}\operatorname{div} w\mathbf{f}\right|^2 \, wdx \quad \leq \quad Ce^{-c\frac{d}{|z|}} \int_E |\mathbf{f}|^2 \, wdx. \tag{2.11}$$

Proof. Notice that by uniform boundedness in $L^2(w)$ of the operators $\left(I+z^2\mathcal{L}_w\right)^{-1}$, $z\nabla\left(I+z^2\mathcal{L}_w\right)^{-1}$, and $\left(I+z^2\mathcal{L}_w\right)^{-1} z \operatorname{div}$, it is enough to check the case $d \geq |z| > 0$. The first inequality (2.9) is Lemma 4.1 in [6]. Let η be a nonnegative cutoff function such that $\eta \equiv 1$ in F, $\eta(x) \equiv 0$ for $\operatorname{dist}(x,E) \geq \frac{d}{2}$, and $\|\eta\|_\infty + d\|\nabla\eta\| \leq C$ for some universal constant. By the ellipticity of $\mathbf{A}_w(x)$ and integration by parts we have

$$\begin{aligned}
&\int_F \left|z\nabla\left(I+z^2\mathcal{L}_w\right)^{-1} f(x)\right|^2 wdx \\
\leq\ & \int_{\mathbb{R}^n} \left|z\nabla\eta\left(I+z^2\mathcal{L}_w\right)^{-1} f(x)\right|^2 wdx \\
\lesssim\ & |z|^2 \operatorname{Re}\int_{\mathbb{R}^n} \overline{\left(z\nabla\eta\left(I+z^2\mathcal{L}_w\right)^{-1} f(x)\right)} \cdot \mathbf{A}_w(x)\left(z\nabla\eta\left(I+z^2\mathcal{L}_w\right)^{-1} f(x)\right) dx \\
\leq\ & \left|\int_{\mathbb{R}^n} \overline{\left(\eta\left(I+z^2\mathcal{L}_w\right)^{-1} f(x)\right)}\, z^2\mathcal{L}_w\left(\eta\left(I+z^2\mathcal{L}_w\right)^{-1} f(x)\right) \, wdx\right| \\
\leq\ & \left|\int_{\mathbb{R}^n} \overline{\left(\eta\left(I+z^2\mathcal{L}_w\right)^{-1} f(x)\right)}\, z^2\left[\mathcal{L}_w,\eta\right]\left(\left(I+z^2\mathcal{L}_w\right)^{-1} f(x)\right) \, wdx\right| \\
&+ \int_{\mathbb{R}^n} \left|\eta\left(I+z^2\mathcal{L}_w\right)^{-1} f(x)\right|^2 \, wdx,
\end{aligned}$$

where in the last inequality we used that f and η have disjoint supports.

From the identity $\left[\mathcal{L}_w,\eta\right] = -\frac{1}{w}(\nabla\eta)\cdot\mathbf{A}_w\nabla - \frac{1}{w}\operatorname{div}\mathbf{A}_w(\nabla\eta)$, we have that the first integral on the right is bounded by

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^n} \left|z\nabla\eta\left(I+z^2\mathcal{L}_w\right)^{-1} f(x)\right|^2 \, w(x)\,dx \\
&+C|z|^2\int_{\mathbb{R}^n} |\nabla\eta|^2 \left|\left(I+z^2\mathcal{L}_w\right)^{-1} f(x)\right|^2 \, wdx.
\end{aligned}$$

If we substitute this above and rearrange terms we get that

$$\begin{aligned}
&\int_F \left|z\nabla\left(I+z^2\mathcal{L}_w\right)^{-1} f(x)\right|^2 wdx \leq 2\int_{\mathbb{R}^n} \left|\eta\left(I+z^2\mathcal{L}_w\right)^{-1} f(x)\right|^2 \, wdx \\
&+C|z|^2\int_{\mathbb{R}^n} |\nabla\eta|^2 \left|\left(I+z^2\mathcal{L}_w\right)^{-1} f(x)\right|^2 \, wdx.
\end{aligned}$$

Let $\tilde{F} = \text{support}(\varphi)$; then $F \Subset \tilde{F}$, and $\text{dist}\left(E, \tilde{F}\right) \geq d/2$. Therefore, by (2.9),

$$\begin{aligned}
&\int_F \left| z\nabla \left(I + z^2\mathcal{L}_w\right)^{-1} f(x)\right|^2 w dx \\
\leq\ & C\left(2 + |z|^2/d^2\right) \int_{\tilde{F}} \left| \left(I + z^2\mathcal{L}_w\right)^{-1} f(x)\right|^2\ w dx \\
\leq\ & C\int_{\tilde{F}} \left| \left(I + z^2\mathcal{L}_w\right)^{-1} f(x)\right|^2\ w dx \leq Ce^{-c\frac{d}{|z|}} \int_E |f(x)|^2\ w(x)\, dx.
\end{aligned}$$

This proves (2.10). The third inequality follows from the second one considering adjoint operators. Indeed, given $g \in L^2(w)$,

$$\begin{aligned}
&\int_F \left(\left(I + z^2\mathcal{L}_w\right)^{-1} z\, \text{div}\, \mathbf{f}(x)\right)\ \eta\overline{g}(x)\ \ w dx \\
=\ & \int_{\mathbb{R}^n} \mathbf{f}(x) \cdot z\left(\frac{1}{w}\nabla w\left(\left(I + z^2\mathcal{L}_w\right)^{-1}\right)^* \eta\overline{g}(x)\right)\ w dx \\
\leq\ & \left(\int_E |\mathbf{f}(x)|^2\ \ w dx\right)^{\frac{1}{2}} \left(\int_E \left| z\nabla\left(\left(I + z^2\mathcal{L}_w\right)^{-1}\right)^* \eta\overline{g}(x)\right|^2\ \ w dx\right)^{\frac{1}{2}} \\
\leq\ & \left(\int_E |\mathbf{f}(x)|^2\ \ w dx\right)^{\frac{1}{2}} \left(Ce^{-c\frac{d}{|z|}} \int_{\tilde{F}} |\eta\overline{g}(x)|^2\ \ w dx\right)^{\frac{1}{2}} \\
\leq\ & \left(Ce^{-c\frac{d}{|z|}} \int_E |\mathbf{f}(x)|^2\ \ w dx\right)^{\frac{1}{2}} \|g\|_{L^2(w)}.
\end{aligned}$$

Then (2.11) follows by taking supremum over all $g \in L^2(w)$ with $\|g\|_{L^2(w)} = 1$. □

Corollary 2.11. *Suppose* $\mathbf{f} \in (L^\infty)^n$. *Then for all* $y \in \mathbb{R}^n$ *and* $t > 0$,

$$\frac{1}{w(B_t(y))} \int_{B_{4t}(y)} \left| t\left(1 + t^2\mathcal{L}_w\right)^{-1} \frac{1}{w}\, \text{div}\, \mathbf{A}_w \mathbf{f}\right|^2\ \ w dx \leq C\, \|\mathbf{f}\|_\infty^2.$$

Proof. Let $g \in L^2(w)$ with $\text{support}(g) \subset B_{4t}(y)$ and $\|g\|_{L^2(w)} = 1$. Then

$$\begin{aligned}
&\int_{B_{4t}(y)} \left(t\left(1 + t^2\mathcal{L}_w\right)^{-1} \frac{1}{w}\, \text{div}\, \mathbf{A}_w\mathbf{f}\right) \overline{g}\ w dx \\
=\ & \int_{\mathbb{R}^n} (\mathbf{A}_w\mathbf{f}) \cdot t\left(\nabla\left(1 + t^2\mathcal{L}_w\right)^{-1}\right)^* \overline{g}\ dx \\
\leq\ & \Lambda \int_{\mathbb{R}^n} |\mathbf{f}| \left| t\left(\nabla\left(1 + t^2\mathcal{L}_w\right)^{-1}\right)^* \overline{g}\right|\ w dx \\
=\ & \Lambda \sum_{j=0}^{\infty} \int_{R_j} |\mathbf{f}| \left| t\left(\nabla\left(1 + t^2\mathcal{L}_w\right)^{-1}\right)^* \overline{g}\right|\ w dx \\
\leq\ & \Lambda\, \|\mathbf{f}\|_\infty \sum_{j=0}^{\infty} \left(w\left(B_{2^{j+2}t}(y)\right) \int_{R_j} \left| t\left(\nabla\left(1 + t^2\mathcal{L}_w\right)^{-1}\right)^* \overline{g}\right|^2\ \ w dx\right)^{\frac{1}{2}},
\end{aligned}$$

where $R_0 = B_{4t}(y)$ and $R_j = B_{2^{j+2}t}(t) \setminus B_{2^{j+1}t}(t)$, $j \geq 1$. By the doubling property of w, we have $w\left(B_{2^{j+2}t}(y)\right) \leq (D_w)^{j+2}\, w(B_t(y))$ for some fixed constant D_w.

Then, by (2.10) in Lemma 2.10,

$$\begin{aligned}&\int_{B_{4t}(y)}\left(t\left(1+t^2\mathcal{L}_w\right)^{-1}\frac{1}{w}\operatorname{div}\mathbf{A}_w\mathbf{f}\right)\overline{g}\;wdx\\ \leq\quad & C\left\|\mathbf{f}\right\|_\infty\left(w\left(B_t\left(y\right)\right)\right)^{\frac{1}{2}}\sum_{j=0}^{\infty}\left(\left(D_w\right)^j e^{-c2^j}\right)^{\frac{1}{2}}\leq C\left\|\mathbf{f}\right\|_\infty\left(w\left(B_t\left(y\right)\right)\right)^{\frac{1}{2}}.\end{aligned}$$

The corollary follows taking supremum over all g as above. □

Lemma 2.12 (Conservation property of the resolvent)**.** *We have the identity*

$$\left(1+t^2\mathcal{L}_w\right)^{-1}1=1 \tag{2.12}$$

in the sense that $\left(1+t^2\mathcal{L}_w\right)^{-1}\eta_R\to 1$ *in* $L^2_{\text{loc}}(w)$*, where* $\eta_R=\eta\left(\frac{\cdot}{R}\right)$ *and* η *is a smooth bump function with* $\eta\equiv 1$ *near the origin. That is, for every* $N>0$*,*

$$\int_{Q_N}\left|\left(1+t^2\mathcal{L}_w\right)^{-1}\eta_R-1\right|^2 wdx\to 0\qquad \text{as } R\to\infty,$$

where Q_N *is a cube centered at the origin with side length* $2N$*.*

Proof. Indeed, fix η as above, and let $\eta_R\left(x\right)=\eta\left(x/R\right)$. Then for any $N>0$,

$$\begin{aligned}&\lim_{R\to\infty}\int_{Q_N}\left|\left(I+t^2\mathcal{L}_w\right)^{-1}\eta_R-1\right|^2\;wdx\\ =\quad&\lim_{R\to\infty}\int_{Q_N}\left|\left(I+t^2\mathcal{L}_w\right)^{-1}\eta_R-\eta_R\right|^2\;wdx\\ =\quad&\lim_{R\to\infty}\int_{Q_N}\left|\left(I+t^2\mathcal{L}_w\right)^{-1}\left(\eta_R-\left(I+t^2\mathcal{L}_w\right)\eta_R\right)\right|^2\;wdx\\ =\quad&\lim_{R\to\infty}\int_{Q_N}\left|\left(I+t^2\mathcal{L}_w\right)^{-1}t^2\mathcal{L}_w\eta_R\right|^2\;wdx.\end{aligned}$$

Let $H_R=\text{support}\left(\nabla\eta_R\right)$, and $d\left(N,R\right)=\text{dist}\left(Q_N,H_R\right)$. Then $d\left(N,R\right)\to\infty$ as $R\to\infty$ and by (2.10) from Lemma 2.10 we have

$$\begin{aligned}&\lim_{R\to\infty}\int_{B_N}\left|\left(I+t^2\mathcal{L}_w\right)^{-1}\eta_R-1\right|^2 wdx\\ \leq\quad&\lim_{R\to\infty}t^4\lim_{R\to\infty}\int_{Q_N}\left|\left(I+t^2\mathcal{L}_w\right)^{-1}\frac{1}{w}\operatorname{div}w\frac{\mathbf{A}_w}{w}\nabla\eta_R\right|^2\;wdx\\ \leq\quad&\lim_{R\to\infty}e^{-c\frac{d(N,R)}{t}}\int_{H_R}\left|\frac{\mathbf{A}_w}{w}\nabla\eta_R\right|^2\;wdx\\ \leq\quad&\lim_{R\to\infty}e^{-c\frac{d(N,R)}{t}}\left\|\nabla\eta\right\|_\infty R^{-1}\Lambda\;w\left(H_R\right)=0.\end{aligned}$$

Here we used that since w is a doubling measure, $w\left(H_R\right)\leq w\left(2^M\,\text{support}\,\eta\right)\leq D_w^M w\left(\text{support}\,\eta\right)$, where $M\approx\ln R/\ln 2$ and D_w is the doubling constant. Hence the limit on the right vanishes. □

The next result is proved exactly as Lemma 2.3 in [1] using Corollary 2.9 and Lemma 2.12.

Corollary 2.13. *There exists $C>0$ depending only on n, λ, Λ, and $[w]_{A_2}$ such that for any cube $Q\subset\mathbb{R}^n$, $t\leq \ell(Q)$, and f Lipschitz in $\mathbb{R}^n$,*

$$\begin{aligned}\int_{5Q}\left|\left(1+t^2\mathcal{L}_w\right)^{-1}f-f\right|^2\,wdx &\leq Ct^2\left\|\nabla f\right\|_\infty^2 w(Q), \qquad \textit{and}\\ \int_{5Q}\left|\nabla\left(\left(1+t^2\mathcal{L}_w\right)^{-1}f-f\right)\right|^2\,wdx &\leq C\left\|\nabla f\right\|_\infty^2 w(Q).\end{aligned}$$

3. Reduction to a square function estimate

In this section we show that to prove Theorem 1.1 it suffices to establish the square function estimate

$$\int_0^\infty\left\|\left(1+t^2\mathcal{L}_w\right)^{-1}t\mathcal{L}_wf\right\|_{L^2(w)}^2\frac{dt}{t}\leq C\left\|\nabla f\right\|_{L^2(w)}^2. \tag{3.1}$$

We start with the square root representation

$$a\mathcal{L}_w^{1/2}f(x)=\int_0^\infty\left(1+t^2\mathcal{L}_w\right)^{-3}t^3\mathcal{L}_w^2f(x)\,\frac{dt}{t}, \tag{3.2}$$

where $a=\int_0^\infty\left(1+u^2\right)^{-3}u^2\,du$. This formula makes sense for $f\in\mathcal{D}\left(\mathcal{L}_w^2\right)$, which is a dense subset of $H^1(w)$. Moreover, by (2.5) in Lemma 2.8 it follows that this integral converges absolutely whenever $f\in\mathcal{D}\left(\mathcal{L}_w^2\right)$.

Now, by duality and the Cauchy-Schwarz's inequality,

$$\begin{aligned}a\left|\left\langle\mathcal{L}_w^{1/2}f,g\right\rangle_w\right|^2 &= \left|\int_{\mathbb{R}^n}\left(a\mathcal{L}_w^{1/2}f\right)\overline{g}\,wdx\right|^2\\ &= \left|\int_{\mathbb{R}^n}\left(\int_0^\infty\left(1+t^2\mathcal{L}_w\right)^{-3}t^3\mathcal{L}_w^2f(x)\,\frac{dt}{t}\right)\overline{g}\,wdx\right|^2\\ &\leq \left(\int_0^\infty\left\|\left(1+t^2\mathcal{L}_w\right)^{-1}t\mathcal{L}_wf\right\|_{L^2(w)}^2\frac{dt}{t}\right)\left(\int_0^\infty\left\|V_tg\right\|_{L^2(w)}^2\frac{dt}{t}\right),\end{aligned} \tag{3.3}$$

where $V_tg=\left(\left(1+t^2\mathcal{L}_w\right)^{-2}\right)^*t^2\mathcal{L}_w^*g$. Thus, to establish (3.1) it suffices to prove that V_t satisfies the quadratic estimate

$$\int_0^\infty\left\|V_tg\right\|_{L^2(w)}^2\frac{dt}{t}\leq C\left\|g\right\|_{L^2(w)}^2. \tag{3.4}$$

This follows by the boundedness of the functional calculus of $\mathcal{L}_w$ [17]; see Theorem 2.22 in [3] for an exposition of this abstract approach for divergence form operators (the case when $w=1$). We sketch the proof, based on §7.3.1 in [12], for completeness.

We have

$$\begin{aligned}\int_0^\infty\left\|V_tg\right\|_{L^2(w)}^2\frac{dt}{t} &= \sum_{k=-\infty}^{\infty}\int_{2^k}^{2^{k+1}}\left\|V_tg\right\|_{L^2(w)}^2\frac{dt}{t}\\ &= \sum_{k=-\infty}^{\infty}\int_1^2\left\|V_{2^kt}g\right\|_{L^2(w)}^2\frac{dt}{t}.\end{aligned}$$

Denote by $\{r_k\}_{k\in\mathbb{Z}}$ the orthonormal Rademacher basis in $L^2([0,1])$, where $r_k(x)\in\{-1,1\}$, and r_k is constant on dyadic subsets of $[0,1]$ small enough depending on

k. Then, since $\int_0^1 r_k(s)\, r_j(s)\; ds = \delta_{kj}$, for each $t > 0$ and positive integer N,

$$\begin{aligned}\sum_{k=-N}^{N} \|V_{2^k t} g(x)\|_{L^2(w)}^2 &= \int_{\mathbb{R}^n} \sum_{k=-N}^{N} \sum_{j=-N}^{N} \delta_{kj}\; (V_{2^k t} g)(V_{2^j t} g)\; w dx \\ &= \int_0^1 \left\| \sum_{k=-N}^{N} r_k(s)\, V_{2^k t} g \right\|_{L^2(w)}^2 ds.\end{aligned}$$

In the functional calculus frame, the operator V_t is given by

$$V_t f = \phi\left(t^2 \mathcal{L}_w^*\right) f, \qquad \text{where } \phi(z) = (1+z)^{-2} z.$$

Hence,

$$\begin{aligned}\sum_{k=-N}^{N} \|V_{2^k t} g\|_{L^2(w)}^2 &= \int_0^1 \left\| \left(\sum_{k=-N}^{N} r_k(s)\, \phi\left(2^{2k} t^2 \mathcal{L}_w^*\right) \right) g \right\|_{L^2(w)}^2 ds \\ &\le C \left(\int_0^1 \sup_{z \in S_\varphi} \left| \sum_{k=-N}^{N} r_k(s)\; \phi\left(2^{2k} t^2 z\right) \right|^2 ds \right) \|g\|_{L^2(w)}^2 \\ &\le C \left(\sup_{z \in S_\varphi} \sum_{k \in \mathbb{Z}} \left| \phi\left(2^{2k} t^2 z\right) \right| \right)^2 \|g\|_{L^2(w)}^2 .\end{aligned}$$

Here we used that, since $\mathcal{L}_w^*$ has a bounded functional calculus, $\|\eta(\mathcal{L}_w^*)\|_{\mathcal{B}(L^2(w))} \le C \|\eta\|_{L^\infty(S_\omega)}$ for all $\eta \in H_0^\infty(S_\omega)$, where $S_\omega = \{z \in \mathbb{C} : |\arg z| < \omega\}$, $\omega = \omega(\mathcal{L}_w) \in (0, \pi/2)$, and $H_0^\infty(S_\omega)$ is the set of bounded holomorphic functions on S_ω that decay polynomially at 0 and at ∞. Now, since for $z \in S_\omega$, $|\phi(z)| \le \min\left\{|z|, |z|^{-1}\right\}$, it follows that

$$\begin{aligned}\sup_{1 \le t \le 2} \sup_{z \in S_\varphi} \sum_{k \in \mathbb{Z}} \left| \phi\left(2^{2k} t^2 z\right) \right| &\le \sup_{1 \le t \le 2} \sup_{z \in S_\varphi} \sum_{k \in \mathbb{Z}} \min\left\{4^k t^2 |z|, 4^{-k} t^{-2} |z|^{-1}\right\} \\ &\le \sup_{1 \le t \le 2} \sum_{k \in \mathbb{Z}} \min\left\{4^k t^2, 4^{-k} t^{-2}\right\} \le \frac{16}{3}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{2^{-N}}^{2^{N+1}} \|V_t g\|_{L^2(w)}^2 \frac{dt}{t} &= \int_1^2 \sum_{k=-N}^{N} \|V_{2^k t} g\|_{L^2(w)}^2 \frac{dt}{t} \\ &\le C \|g\|_{L^2(w)}^2 \int_1^2 \frac{dt}{t} \le C \|g\|_{L^2(w)}^2 ,\end{aligned}$$

and (3.4) follows taking the limit as $N \to \infty$.

Note that since this reduction is based on the existence of a bounded functional calculus, it also works for more abstract operators. However, the general Kato problem for abstract operators with a bounded functional calculus is not true [15, 16]. The next two sections are where harmonic analysis techniques are actually needed for the proof, and where the previous abstract approach would not suffice.

4. Reduction to a Carleson measure estimate

In this section we will prove that the square function estimate (3.1) holds provided we have a weighted Carleson estimate. Following the original approach in [3], for $\mathbf{f} \in \left(L^2(w)\right)^n$ we let

$$\theta_t \mathbf{f} = -t\left(1+t^2\mathcal{L}_w\right)^{-1}\frac{1}{w}\operatorname{div}\mathbf{A}_w\mathbf{f}.$$

We claim that $\theta_t \in \mathcal{B}\left(\left(L^2(w)\right)^n, L^2(w)\right)$. Indeed, by (2.6) in Lemma 2.8, given $g \in L^2(w)$ with $\|g\|_{L^2(w)} = 1$, we have

$$\begin{aligned}
\langle \theta_t \mathbf{f}, g\rangle_w &= \int_{\mathbb{R}^n}\left(-t\left(1+t^2\mathcal{L}_w\right)^{-1}\frac{1}{w}\operatorname{div}\mathbf{A}_w\mathbf{f}\right)\overline{g}\;wdx \\
&= \int_{\mathbb{R}^n}\frac{1}{w}\mathbf{A}_w\mathbf{f}\cdot t\nabla\left(\left(1+t^2\mathcal{L}_w\right)^{-1}\right)^*\overline{g}\;wdx \\
&\leq \Lambda\|\mathbf{f}\|_{L^2(w)}\left\|t\nabla\left(\left(1+t^2\mathcal{L}_w\right)^{-1}\right)^*\overline{g}\right\|_{L^2(w)} \leq \frac{\Lambda}{2\lambda^{\frac{1}{2}}}\|\mathbf{f}\|_{L^2(w)}. \qquad (4.1)
\end{aligned}$$

Within this context, (3.1) is written as

$$\int_0^\infty \|\theta_t\nabla f\|^2_{L^2(w)}\frac{dt}{t} \leq C\|\nabla f\|^2_{L^2(w)}. \tag{4.2}$$

To prove this we first introduce an extra smoothing operator. Let $P_t g(x) = p_t * g(x)$, where, as before, $p_t(x) = t^{-n}p(x/t)$ and

$$p \text{ is radial},\quad p\in\mathcal{C}_0^\infty(B_1(0)) \quad\text{and}\quad \textstyle\int p\,dx = 1. \tag{4.3}$$

Now, writing the identity operator as a convolution with the delta distribution,

$$(\widehat{\delta - P_t^2})\,f(\xi) = t\left(\frac{1-\widehat{p}(t\xi)^2}{|t\xi|^2}it\xi\right)\cdot\widehat{\nabla f}(\xi) = t\widehat{R_t\nabla f}(\xi),$$

where $R_t\mathbf{f} = \mathbf{r}_t * \mathbf{f} = ((r_t)_1 * f_1, \ldots, (r_t)_n * f_n)$, and $\mathbf{r}(x)$ is a vector of smooth functions such that $|\widehat{\mathbf{r}}(\xi)| \leq C\min\left\{|\xi|, |\xi|^{-1}\right\}$ (notice that the hypotheses on p imply $\widehat{p}(0) = 1$ and $\nabla\widehat{p}(0) = \mathbf{0}$). Then, taking $Q_s f = \psi_s * f$ with ψ as in Proposition 2.3, that is, ψ is a radial Schwartz function such that $\widehat{\psi}(0) = 0$ and $\int_0^\infty \widehat{\psi}(t)^2\frac{dt}{t} = 1$, it readily follows that

$$|\widehat{\mathbf{r}}(t\xi)|\left|\widehat{\psi}(s\xi)\right| \leq C\min\left\{\frac{t}{s}, \frac{s}{t}\right\},$$

so $\|R_tQ_s\|_{\mathcal{B}(L^2)} \leq C\min\left\{\frac{t}{s}, \frac{s}{t}\right\}$. On the other hand, since $p \in \mathcal{C}_0^\infty(B_1(0))$ is radial, it satisfies the vanishing moments conditions $\int x_i p\,dx = 0$, for $i = 1, \ldots, n$. Then, by equation (35), chapter 4, in [3], $\mathbf{r}(x)$ satisfies the bounds

$$|\mathbf{r}(x)| \leq \frac{C}{|x|^{n-1}(1+|x|)^2}.$$

Since the right hand side is integrable, it follows from Lemma 2.1 that $\|R_t\|_{\mathcal{B}(L^2(w))}$ is uniformly bounded in t, with bounds depending only on n, $[w]_{A_2}$, and the L^1

norm of $|\mathbf{r}|$, and similar bounds apply to $\|R_t Q_s\|_{\mathcal{B}(L^2(w))}$. Then Lemma 2.5 implies that for some constant $\beta > 0$,

$$\|R_t Q_s\|_{\mathcal{B}(L^2(w))} \leq C \min\left\{\frac{t}{s}, \frac{s}{t}\right\}^\beta . \tag{4.4}$$

Now, following the reduction in [1], since by the functional calculus

$$\|\theta_t \nabla\|_{\mathcal{B}(L^2(w))} = \left\| t\left(1 + t^2 \mathcal{L}_w\right)^{-1} \mathcal{L}_w \right\|_{\mathcal{B}(L^2(w))} \leq C/t,$$

and since derivatives commute with convolutions, we have that

$$\begin{aligned}\int_0^\infty \left\|\theta_t \left(1 - P_t^2\right) \nabla f\right\|_{L^2(w)}^2 \frac{dt}{t} &\leq C \int_0^\infty \left\|\left(1 - P_t^2\right) f\right\|_{L^2(w)}^2 \frac{dt}{t^3} \\ &\leq C \int_0^\infty \|R_t \nabla f\|_{L^2(w)}^2 \frac{dt}{t}.\end{aligned}$$

By (4.4) and Proposition 2.3 we then have that

$$\int_0^\infty \left\|\theta_t \left(1 - P_t^2\right) \nabla f\right\|_{L^2(w)}^2 \frac{dt}{t} \leq C \|\nabla f\|_{L^2(w)}^2 . \tag{4.5}$$

Hence, to prove (4.2) it suffices to establish

$$\int_0^\infty \left\|\theta_t P_t^2 \nabla f\right\|_{L^2(w)}^2 \frac{dt}{t} \leq C \|\nabla f\|_{L^2(w)}^2 . \tag{4.6}$$

To state our reduction to a Carleson measure estimate, let

$$\gamma_t (x) = \theta_t \mathbf{I} (x) = -t \left(1 + t^2 \mathcal{L}_w\right)^{-1} \frac{1}{w} \operatorname{div} \mathbf{A}_w \mathbf{I} (x) ,$$

where $\mathbf{I}(x)$ denotes the $n \times n$ identity matrix. We will use the notation

$$⨍_E f \; dw := \frac{1}{w(E)} \int_E f(x) \; w dx$$

for the weighted average over a set E.

Lemma 4.1 (Reduction to a Carleson measure estimate). *For $f \in L^2(w)$*

$$\int_0^\infty \left\|\gamma_t (x) \cdot P_t^2 \nabla f - \theta_t P_t^2 \nabla f\right\|_{L^2(w)}^2 \frac{dt}{t} \leq C \|\nabla f\|_{L^2(w)}^2 . \tag{4.7}$$

Moreover, if

$$\|\gamma_t\|_{C,w}^2 = \sup_Q ⨍_Q \int_0^{\ell(Q)} |\gamma_t (x)|^2 \frac{dt}{t} \; dw < \infty, \tag{4.8}$$

where the supremum is taken among all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes, then inequality (4.6) *holds with constant* $C\left(1 + \|\gamma_t\|_{C,w}^2\right)$.

We first prove the last conclusion of the lemma. The left side of (4.6) is bounded by

$$\int_0^\infty \left\|\gamma_t (x) \cdot P_t^2 \nabla f - \theta_t P_t^2 \nabla f\right\|_{L^2(w)}^2 \frac{dt}{t} + \int_0^\infty \left\|\gamma_t (x) \cdot P_t^2 \nabla f\right\|_{L^2(w)}^2 \frac{dt}{t}.$$

If (4.7) and (4.8) hold, then (4.7) applied to the first term above and Lemma 2.2 applied to the second term yields (4.6) with constant $C\left(1 + \|\gamma_t\|_{C,w}^2\right)$.

The remainder of this section is devoted to the proof of (4.7). For $t>0$ define the family of operators $U_t\mathbf{f}=\gamma_t(x)\cdot P_t\mathbf{f}-\theta_t P_t\mathbf{f}$. Then (4.7) is re-written as

$$\int_0^\infty \|U_tP_t\nabla f\|_{L^2(w)}^2\frac{dt}{t}\le C\|\nabla f\|_{L^2(w)}^2. \tag{4.9}$$

To show this inequality, by Proposition 2.3, it is sufficient to show that

$$\|U_tP_t\|_{\mathcal{B}(L^2(w))}\ \le\ C,\qquad \text{and} \tag{4.10}$$

$$\|U_tP_tQ_s\|_{\mathcal{B}(L^2(w))}\ \le\ C\min\left\{\frac{s}{t},\frac{t}{s}\right\}^\alpha\qquad\text{for some } 0<\alpha\le 1, \tag{4.11}$$

where Q_s is defined as above. Note that we should write $\|\cdot\|_{\mathcal{B}((L^2(w))^n)}$ instead, but we adopt above and henceforth this abuse of notation for simplicity. We first prove (4.10), right after the following two lemmas.

Lemma 4.2. *For $t>0$, the operator U_t is given by a kernel $U_t(x,y)$ such that*

$$U_t\mathbf{f}(x)=\int_{\mathbb{R}^n}U_t(x,y)\mathbf{f}(y)\ dy.$$

Moreover, U_t is bounded in L^1 and there exists $C=C(n,\lambda,\Lambda,w)>0$, such that the kernel $U_t(x,y)$ satisfies

$$\sup_{y\in\mathbb{R}^n}\int_{\mathbb{R}^n}|U_t(x,y)|\ dx\le C.$$

Proof. Fix $t>0$ and let $\{Q_k^t\}_{k\in\mathbb{Z}^n}$ denote the lattice of cubes of sidelength t with vertices on $t\mathbb{Z}^n$. Consider $\mathbf{f}\in[\mathcal{C}_0^\infty(\mathbb{R}^n)]^n$ and let $\mathbf{f}^k(x)=\chi_{Q_k^t}\mathbf{f}(x)$. Since $|p_t(x-y)|\le C/t^n\chi_{B_t(x)}(y)$, and $B_t(x)\subset 3Q_k^t$ whenever $y\in Q_k^t$ and $|x-y|\le t$, we have that

$$\left|P_t\mathbf{f}^k(x)\right|=\left|\int_{B_t(x)}p_t(x-y)\chi_{Q_k^t}\mathbf{f}(y)\ dy\right|\le\frac{C}{t^n}\left\|\chi_{Q_k^t}\mathbf{f}\right\|_{L^1}\chi_{3Q_k^t}(x). \tag{4.12}$$

By Corollary 2.11 (which holds for cubes as well as for balls), for all cubes $Q\subset\mathbb{R}^n$ we have the uniform bound

$$\frac{1}{w(Q)}\int_Q|\gamma_t(x)|^2\ wdx\le C. \tag{4.13}$$

Then

$$\int_{3Q_k^t}|\gamma_t(x)|\ dx\le\left(\int_{3Q_k^t}|\gamma_t(x)|^2\ wdx\right)^{\frac12}\left(\int_{3Q_k^t}w^{-1}dx\right)^{\frac12}\le Ct^n. \tag{4.14}$$

Now, by (4.12),

$$\begin{aligned}\int_{\mathbb{R}^n}|\gamma_t(x)\cdot P_t\mathbf{f}(x)|\ dx &= \int_{\mathbb{R}^n}\left|\sum_{k\in\mathbb{Z}^n}\gamma_t(x)\cdot P_t\mathbf{f}_k(x)\right|\ dx\\ &\le \int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}^n}|\gamma_t(x)|\frac{C}{t^n}\left\|\chi_{Q_k^t}\mathbf{f}\right\|_{L^1}\chi_{3Q_k^t}(x)\ dx.\end{aligned}$$

Note that we have the inequality

$$\sum_{k\in\mathbb{Z}^n}|\gamma_t(x)|\left\|\chi_{Q_k^t}\mathbf{f}\right\|_{L^1}\chi_{3Q_k^t}(x)\le 3^n\sum_{k\in\mathbb{Z}^n}|\gamma_t(x)|\left\|\chi_{3Q_k^t}\mathbf{f}\right\|_{L^1}\chi_{Q_k^t}(x).$$

This, together with the fact that $\{Q_k^t\}_{k\in\mathbb{Z}}$ are pairwise disjoint, yield

$$\sum_{k\in\mathbb{Z}^n} |\gamma_t(x)| \left\|\chi_{Q_k^t}\mathbf{f}\right\|_{L^1} \chi_{3Q_k^t}(x) \le 3^n \sum_{k\in\mathbb{Z}^n} |\gamma_t(x)| \left\|\chi_{3Q_k^t}\mathbf{f}\right\|_{L^1} \chi_{Q_k^t}(x).$$

Then, by (4.14) it follows that

$$\begin{aligned}\int_{\mathbb{R}^n} |\gamma_t(x)\cdot P_t\mathbf{f}(x)|\, dx &\le \frac{C}{t^n}\sum_{k\in\mathbb{Z}^n}\left\|\chi_{3Q_k^t}\mathbf{f}\right\|_{L^1}\int_{Q_k^t}|\gamma_t(x)|\, dx\\ &\le C\sum_{k\in\mathbb{Z}^n}\left\|\chi_{Q_k^t}\mathbf{f}\right\|_{L^1} = C\|\mathbf{f}\|_{L^1}.\end{aligned} \tag{4.15}$$

On the other hand, by (2.11) in Lemma 2.10, (4.14), and the doubling property of w, for all j and k,

$$\begin{aligned}&\textstyle\int_{Q_j^t}\left|\theta_t P_t\mathbf{f}^k(x)\right|^2\, wdx = \int_{Q_j^t}\left|t\left(1+t^2\mathcal{L}_w\right)^{-1}\frac{1}{w}\operatorname{div} w\frac{\mathbf{A}_w}{w}P_t\mathbf{f}^k(x)\right|^2\, wdx\\ &\le Ce^{-c|k-j|}\textstyle\int_{3Q_k^t}\left|P_t\mathbf{f}^k(x)\right|^2\, wdx \le C\frac{e^{-c|k-j|}}{t^{2n}}\left\|\chi_{Q_k^t}\mathbf{f}\right\|_{L^1}^2 w\left(3Q_j^t\right)\\ &\le C\frac{e^{-c|k-j|}}{t^{2n}}w\left(Q_j^t\right)\left\|\chi_{Q_k^t}\mathbf{f}\right\|_{L^1}^2.\end{aligned}$$

Hence,

$$\begin{aligned}&\textstyle\int_{Q_j^t}\left|\theta_t P_t\mathbf{f}^k(x)\right|\, dx \le \left(\int_{Q_j^t}\left|\theta_t P_t\mathbf{f}^k(x)\right|^2\, wdx\right)^{\frac12}\left(\int_{Q_j^t} w^{-1}dx\right)^{\frac12}\\ &\le C\frac{e^{-\frac{c}{2}|k-j|}}{t^n}\left\|\chi_{Q_k^t}\mathbf{f}\right\|_{L^1}\left(\textstyle\int_{Q_j^t} w\, dx\right)^{\frac12}\left(\int_{Q_j^t} w^{-1}dx\right)^{\frac12}\\ &\le Ce^{-\frac{c}{2}|k-j|}\left\|\chi_{Q_k^t}\mathbf{f}\right\|_{L^1}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^n}|\theta_t P_t\mathbf{f}|\, dx &\le \sum_{k\in\mathbb{Z}^n}\sum_{j\in\mathbb{Z}^n}\int_{Q_j^t}\left|\theta_t P_t\mathbf{f}^k\right|\, dx\\ &\le C\sum_{k\in\mathbb{Z}^n}\sum_{j\in\mathbb{Z}^n} e^{-\frac{c}{2}|k-j|}\left\|\chi_{Q_k^t}\mathbf{f}\right\|_{L^1} \le C\|\mathbf{f}\|_{L^1}.\end{aligned}$$

We conclude that $U_t : L^1 \to L^1$, and it is bounded. Hence by a classical result from [8] the operator U_t is given by a locally integrable kernel $U_t(x,y)$ satisfying

$$\sup_{y\in\mathbb{R}^n}\int_{\mathbb{R}^n}|U_t(x,y)|\, dx \le C.$$

□

Lemma 4.3. *The kernel of the operator* $U_t\mathbf{f} = \gamma_t(x)\cdot P_t\mathbf{f} - \theta_t P_t\mathbf{f}$ *satisfies*

$$\int_{\mathbb{R}^n} e^{\frac{c}{4}\frac{|x-y|}{t}}|U_t(x,y)|^2\, w(x)\,dx \le C\frac{w(B_t(y))}{t^{2n}},$$

where $C>0$ *is the constant in Lemma* 2.10.

Proof. Let $\varphi \in \mathcal{C}_0^\infty(B_1(0))$, $\varphi \ge 0$ and $\int\varphi = 1$. For $\varepsilon > 0$ and $y \in \mathbb{R}^n$, write $\varphi_{\varepsilon,y}(x) = \varphi_\varepsilon(x-y) = \varepsilon^{-n}\varphi\left(\frac{x-y}{\varepsilon}\right)$. Since by Lemma 4.2 we know that $U(\cdot,y)\in L^1(\mathbb{R}^n)$ uniformly, we have that

$$U(\cdot,y) * \varphi_\varepsilon(x) \to U(x,y) \quad \text{as } \varepsilon\to 0 \tag{4.16}$$

pointwise a.e. and in L^1 for all $y\in\mathbb{R}^n$.

Let $R_0(y) = B_{2t}(y)$ and $R_j(y) = B_{2^{j+1}t}(y) \backslash B_{2^j t}(y)$ for $j = 1, 2, \ldots$; we have

$$\int_{\mathbb{R}^n} e^{\frac{c}{4}\frac{|x-y|}{t}} |U_t(\varphi_{\varepsilon,y}\mathbf{I})(x)|^2 \; wdx \leq \sum_{j=0}^{\infty} e^{\frac{c}{4}2^{j+1}} \int_{R_j(y)} |U_t(\varphi_{\varepsilon,y}\mathbf{I})(x)|^2 \; wdx. \tag{4.17}$$

Note that a priori we do not know that either side of (4.17) is finite. Consider first the case $j = 0$:

$$\begin{aligned} \int_{R_0(y)} |U_t(\varphi_{\varepsilon,y}\mathbf{I})(x)|^2 \; wdx &\leq 2\int_{B_{2t}(y)} |\gamma_t(x) \cdot P_t(\varphi_{\varepsilon,y}\mathbf{I})(x)|^2 \; wdx \\ &\quad +2\int_{B_{2t}(y)} |\theta_t P_t(\varphi_{\varepsilon,y}\mathbf{I})(x)|^2 \; wdx. \end{aligned}$$

Since for $0 < \varepsilon \leq 1$, $|P_t\varphi_{\varepsilon,y}(x)| \leq Ct^{-n}\chi_{B_{2t}(y)}(x)$; by (4.13), Corollary 2.11 and the doubling property of w, the right hand side is bounded by $Ct^{-2n}w(B_t(y))$. When $j \geq 1$,

$$\begin{aligned} &\textstyle\int_{R_j(y)} |U_t(\varphi_{\varepsilon,y}\mathbf{I})(x)|^2 \; wdx \\ &= \textstyle\int_{B_{2^{j+1}t}(y)\backslash B_{2^j t}(y)} |\gamma_t(x) \cdot P_t(\varphi_{\varepsilon,y}\mathbf{I})(x) - \theta_t P_t(\varphi_{\varepsilon,y}\mathbf{I})(x)|^2 \; wdx \\ &= \textstyle\int_{B_{2^{j+1}t}(y)\backslash B_{2^j t}(y)} \left| t\left(1 + t^2\mathcal{L}_w\right)^{-1} \frac{1}{w} \operatorname{div} \mathbf{A}_w P_t(\varphi_{\varepsilon,y}\mathbf{I})(x)\right|^2 \; wdx, \end{aligned}$$

where we used that support $P_t(\varphi_{\varepsilon,y}\mathbf{I}) \cap B_{2^{j+1}t}(y) = \emptyset$ for $j \geq 1$. Then, by (2.9) in Lemma 2.10,

$$\begin{aligned} &\textstyle\int_{R_j(y)} |U_t(\varphi_{\varepsilon,y}\mathbf{I})(x)|^2 \; wdx \\ &\leq \textstyle\int_{B_{2^{j+1}t}(y)\backslash B_{2^j t}(y)} \left| t\left(1 + t^2\mathcal{L}_w\right)^{-1} \frac{1}{w} \operatorname{div} w\frac{\mathbf{A}_w}{w} P_t(\varphi_{\varepsilon,y}\mathbf{I})(x)\right|^2 \; wdx \\ &\quad \leq Ce^{-c2^j} \textstyle\int_{B_{2t}(y)} |P_t(\varphi_{\varepsilon,y}\mathbf{I})(x)|^2 \; wdx \leq Ct^{-2n}e^{-c2^j} w(B_{2t}(y)). \end{aligned}$$

If we plug these two estimates on the right of (4.17), by the doubling property of w we obtain

$$\int_{\mathbb{R}^n} e^{\frac{c}{4}\frac{|x-y|}{t}} |U_t(\varphi_{\varepsilon,y}\mathbf{I})(x)|^2 \; wdx \leq Ct^{-2n}w(B_t(y)) \sum_{j=0}^{\infty} e^{-\frac{c}{2}2^j} \leq Ct^{-2n}w(B_t(y)).$$

Then, by Fatou's lemma and (4.16) we get that

$$\begin{aligned} \int_{\mathbb{R}^n} e^{\frac{c}{4}\frac{|x-y|}{t}} |U_t(x,y)|^2 \; w(x)\,dx &\leq \liminf_{\varepsilon \to o^+} \int_{\mathbb{R}^n} e^{\frac{c}{4}\frac{|x-y|}{t}} |U_t(\varphi_{\varepsilon,y}\mathbf{I})(x)|^2 \; wdx \\ &\leq Ct^{-2n}w(B_t(y)). \end{aligned}$$

□

We will now prove (4.10). Since P_t and θ_t are uniformly bounded in $L^2(w)$ because of Lemma 2.1 and (4.1), respectively, to establish the $L^2(w)$-boundedness of $U_t = \gamma_t(x) \cdot P_t\mathbf{f} - \theta_t P_t\mathbf{f}$ it is enough to consider $\gamma_t(x) \cdot P_t$. Now,

$$\begin{aligned} \int_{\mathbb{R}^n} |\gamma_t(x) \cdot P_t f|^2 \; wdx &= \sum_{k\in\mathbb{Z}^n} \int_{Q_k^t} |\gamma_t(x) \cdot P_t f|^2 \; wdx \\ &= \sum_{k\in\mathbb{Z}^n} \int_{Q_k^t} \left|\gamma_t(x) \cdot P_t\left(f\chi_{3Q_k^t}\right)\right|^2 \; wdx, \end{aligned}$$

where the lattice $\{Q_k^t\}_{k\in\mathbb{Z}^n}$ is an in the proof of Lemma 4.2. Since, for $x \in Q_k^t$

$$\left|P_t\left(f\chi_{3Q_k^t}\right)(x)\right|^2 \le C\left(\fint_{3Q_t^k}|f|\ dx\right)^2 \le C\,[w]_{A_2}\frac{1}{w(3Q)}\int_{3Q_t^k}|f|^2\ wdx,$$

using the doubling property of w and the finite overlapping of the dilated grid, we obtain

$$\int_{\mathbb{R}^n}|\gamma_t(x)\cdot P_tf|^2\ wdx \le C\sum_{k\in\mathbb{Z}^n}\int_{3Q_t^k}|f|^2\ wdx \le C\,\|f\|^2_{L^2(w)}.$$

This proves that U_t is uniformly bounded in $\left(L^2(w)\right)^n$. Since P_t is uniformly bounded in $L^2(w)$, this proves (4.10).

To prove (4.11), and hence finish the proof of the reduction to a Carleson measure estimate, Lemma 4.1, we consider as before $Q_sf(x) = \psi_s * f(x)$ as above. Since by standard Fourier analysis we have $\|P_tQ_s\|_{\mathcal{B}(L^2)} \le C\left(\frac{s}{t}\right)^\beta$ for some $\beta > 0$, by (4.10) and Lemma 2.5 we get

$$\|U_tP_tQ_s\|_{\mathcal{B}(L^2(w))} \le C\,\|P_tQ_s\|_{\mathcal{B}(L^2(w))} \le C\left(\|P_tQ_s\|_{\mathcal{B}(L^2)}\right)^\theta \le C\left(\frac{s}{t}\right)^\alpha \tag{4.18}$$

for some $\alpha > 0$. Hence to prove (4.11) it is enough to establish:

Lemma 4.4. *We have*

$$\|U_tP_tQ_s\|_{\mathcal{B}(L^2(w))} \le C\frac{t}{s}.$$

Lemma 4.4 will follow in a standard way from the kernel representation for U_t, some further kernel estimates (Corollary 4.6 below), and the following conservation property of U_t :

Lemma 4.5. $U_t(\mathbf{I}) = 0$ *in the sense that* $\langle U_t(\chi_R\mathbf{I}), \mathbf{g}\rangle_w$ *converges to* 0 *as* $R \to \infty$ *for each* $\mathbf{g} \in L^2(w)$*, where* χ_R *is the indicator of the ball* $B_R = B_R(0)$.

Proof. Recall that $U_t = \gamma_t(x)\cdot P_t - \theta_tP_t$, where

$$\gamma_t(x) = \theta_t\mathbf{I}(x) = -t\left(1+t^2\mathcal{L}_w\right)^{-1}\frac{1}{w}\operatorname{div}\mathbf{A}_w\mathbf{I}(x).$$

Let $\mathbf{g} = (g_1,\ldots,g_n) \in L^2(w)$. We write $\mathbf{g} = \chi_{R/4}\mathbf{g} + \left(1-\chi_{R/4}\right)\mathbf{g}$. Since for R big enough $1 - P_t\chi_R \equiv 0$ on support $\chi_{R/4}\mathbf{g}$, we have

$$\begin{aligned}
&\left\langle U_t(\chi_R\mathbf{I}), \chi_{R/4}\mathbf{g}\right\rangle_w \\
=\ & -\int_{\mathbb{R}^n} t\left(1+t^2\mathcal{L}_w\right)^{-1}\frac{1}{w}\operatorname{div}\mathbf{A}_w\mathbf{I}(x)\cdot\left(P_t\chi_R\mathbf{I}(x)\right)\overline{\chi_{R/4}\mathbf{g}}(x)\ wdx \\
& + \int_{\mathbb{R}^n} t\left(1+t^2\mathcal{L}_w\right)^{-1}\frac{1}{w}\operatorname{div}\mathbf{A}_w\left(P_t\chi_R\mathbf{I}\right)(x)\cdot\overline{\chi_{R/4}\mathbf{g}}(x)\ wdx \\
=\ & -\int_{\mathbb{R}^n} t\left(1+t^2\mathcal{L}_w\right)^{-1}\frac{1}{w}\operatorname{div}\mathbf{A}_w\left(\left(1-P_t\chi_R\right)\mathbf{I}\right)(x)\cdot\overline{\chi_{R/4}\mathbf{g}}(x)\ wdx \\
=\ & \int_{\mathbb{R}^n}\mathbf{A}_w\left(\left(1-P_t\chi_R\right)\mathbf{I}\right)(x)\cdot t\nabla\left(\left(1+t^2\mathcal{L}_w\right)^{-1}\right)^*\overline{\chi_{R/4}\mathbf{g}}(x)\ dx.
\end{aligned}$$

Assuming also that $R > 4t$, it follows that

$$\begin{aligned}
&\left|\left\langle U_t\left(\chi_R \mathbf{I}\right), \chi_{R/4}\mathbf{g}\right\rangle_w\right| \\
= &\left|\int_{\mathbb{R}^n\setminus B_{R-2t}} \mathbf{A}_w\left(\left(1-P_t\chi_R\right)\mathbf{I}\right)(x)\cdot t\nabla\left(\left(1+t^2\mathcal{L}_w\right)^{-1}\right)^*\overline{\chi_{R/4}\mathbf{g}}(x)\ dx\right| \\
\leq &\ C\sum_{j=0}^{\infty}\left|\int_{B_{2^jR}\setminus B_{2^{j-1}R}} t\nabla\left(\left(1+t^2\mathcal{L}_w\right)^{-1}\right)^*\overline{\chi_{R/4}\mathbf{g}}(x)\ wdx\right| \\
\leq &\ C\sum_{j=0}^{\infty} w\left(B_{2^jR}\right)^{\frac{1}{2}}\left(\int_{B_{2^jR}\setminus B_{2^{j-1}R}}\left|t\nabla\left(\left(1+t^2\mathcal{L}_w\right)^{-1}\right)^*\overline{\chi_{R/4}\mathbf{g}}(x)\right|^2\ wdx\right)^{\frac{1}{2}}.
\end{aligned}$$

Lemma 2.10 and the doubling property of w imply

$$\begin{aligned}
&\left|\left\langle U_t\left(\chi_R \mathbf{I}\right), \chi_{R/4}\mathbf{g}\right\rangle_w\right| \\
\leq &\ Cw\left(B_R\right)^{\frac{1}{2}}\|\mathbf{g}\|_{L^2(w)}\sum_{j=0}^{\infty}\left(C_w\right)^{\frac{j}{2}}\exp\left(-c\frac{2^jR}{t}\right) \\
= &\ Cw\left(B_R\right)^{\frac{1}{2}}\|\mathbf{g}\|_{L^2(w)}\sum_{j=0}^{\infty}\exp\left(-\left[\frac{c}{t}2^jR-\frac{\ln C_w}{2}j\right]\right),
\end{aligned}$$

where C_w is the doubling constant of w. The right hand side tends to zero as $R\to\infty$.

On the other hand, by Lemma 4.3,

$$\begin{aligned}
&\left|\left\langle U_t\left(\chi_R \mathbf{I}\right), \left(1-\chi_{R/4}\right)\mathbf{g}\right\rangle_w\right| \\
&=\left|\int_{\mathbb{R}^n\setminus R/4}\left(\int_{B_R} U_t(x,y)\,\mathbf{I}\ dy\right)\cdot\ \overline{\mathbf{g}}(x)\ wdx\right| \\
&\leq\left(\int_{\mathbb{R}^n\setminus R/4}\left|\mathbf{g}(x)\right|^2\ wdx\right)^{1/2}\left(\int_{\mathbb{R}^n\setminus R/4}\left|\int_{B_R}U_t(x,y)\,\mathbf{I}\ dy\right|^2\ wdx\right)^{1/2} \\
&\leq\|\mathbf{g}\|_{L^2(w)}\left(\int_{\mathbb{R}^n\setminus R/4}\left(\int_{B_R}\left|U_t(x,y)\right|^2 e^{\frac{c}{8}\frac{|x-y|}{t}}\ dy\right)\left(\int_{B_R}e^{-\frac{c}{8}\frac{|x-y|}{t}}\ dy\right)\ wdx\right)^{1/2} \\
&\leq\|\mathbf{g}\|_{L^2(w)}\left(C\frac{e^{-\frac{c}{32}\frac{R}{t}}}{t^n}\int_{B_R}w\left(B_t(y)\right)\ dy\right)^{1/2} \\
&\leq C\|\mathbf{g}\|_{L^2(w)}\frac{e^{-\frac{c}{64}\frac{R}{t}}}{t^{n/2}}\left(\int_{B_t}\int_{B_R}w(x+y)\ dy\ dx\right)^{1/2} \\
&\leq C\|\mathbf{g}\|_{L^2(w)}e^{-\frac{c}{64}\frac{R}{t}}w\left(B_{2R}\right).
\end{aligned}$$

Since $e^{-\frac{c}{64}\frac{R}{t}}w\left(B_{2R}\right)$ is bounded independently of R because of the doubling property of w, this terms also vanishes as $R\to\infty$. This concludes the proof of the lemma. □

Corollary 4.6. *Let $K_t\mathbf{f}(x) = U_t^*U_t\mathbf{f}(x)$, where U_t^* denotes the adjoint operator of U_t in $L^2(w)$. Then K_t is given by a kernel $K_t(x,y)$ with respect to Lebesgue* *measure, satisfying the following pointwise upper bounds:*

$$|K_t(x,y)| \leq C\frac{e^{-\frac{c}{9}\frac{|x-y|}{t}}}{w(x)}\frac{\min\{w(B_t(x)),w(B_t(y))\}}{t^{2n}}.$$

Proof. Write

$$K_t\mathbf{f} = \left(K_t^1 f_1,\ldots,K_t^n f_n\right) \quad \text{and} \quad U_t = \left(U_t^1,\ldots,U_t^n\right),$$

where $\mathbf{f} = (f_1,\ldots,f_n)$; it follows that

$$K_t^\ell(x,y) = \frac{1}{w(x)}\sum_{j=1}^n \int_{\mathbb{R}^n} \overline{U_t^j}(z,x)\,U_t^\ell(z,y)\ wdz.$$

From Lemma 4.3 we then have

$$\begin{aligned} e^{\frac{c}{8}\frac{|x-y|}{t}}|K_t(x,y)| &\leq \frac{1}{w(x)}\sum_{j,\ell=1}^n\int_{\mathbb{R}^n} e^{\frac{c}{8}\frac{|x-z|}{t}}\left|\overline{U_t^j}(z,x)\right| e^{\frac{c}{8}\frac{|y-z|}{t}}\left|U_t^\ell(z,y)\right|\ wdz \\ &\leq \frac{C}{w(x)}\frac{\sqrt{w(B_t(x))\,w(B_t(y))}}{t^{2n}}. \end{aligned}$$

The doubling property of the weight implies that $e^{-\alpha\frac{|x-y|}{t}}w(B_t(x)) \leq C_{\alpha,D}w(B_t(y))$ for any $\alpha > 0$, where D is the doubling constant of w. Then

$$|K_t(x,y)| \leq C\frac{e^{-\frac{c}{9}\frac{|x-y|}{t}}}{w(x)}\frac{\min\{w(B_t(x)),w(B_t(y))\}}{t^{2n}},$$

as wanted. □

Proof of Lemma 4.4. We first note that $K_t = U_t^*U_t$ satisfies a conservation property in the sense of Lemma 4.5. This follows from Lemma 4.5 by duality and the boundedness (4.10) of U_t in $L^2(w)$. Therefore, we can write

$$\begin{aligned} K_tQ_s\mathbf{f}(x) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K_t(x,y)\,\psi_s(y-z)\ \mathbf{f}(z)\ dz\ dy \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K_t(x,y)\left(\psi_s(y-z)-\psi_s(x-z)\right)\ dy\ \mathbf{f}(z)\ dz. \end{aligned}$$

Since

$$|\psi_s(y-z)-\psi_s(x-z)| \lesssim \frac{t}{s}s^{-n}\frac{|y-x|}{t}\left(\chi_{B_s(x)}(z)+\chi_{B_s(y)}(z)\right),$$

by Corollary 4.6 we have that

$$\begin{aligned} |K_tQ_s\mathbf{f}(x)| &\lesssim \frac{t}{s}\frac{w(B_t(x))}{w(x)\,t^{2n}}\Theta_s\mathbf{f}(x)\int_{\mathbb{R}^n} e^{-\frac{c}{10}\frac{|x-y|}{t}}\ dy \\ &\quad +\frac{t}{s}\frac{w(B_t(x))}{w(x)\,t^{2n}}\int_{\mathbb{R}^n} e^{-\frac{c}{10}\frac{|x-y|}{t}}\Theta_s\mathbf{f}(y)\ dy \\ &\lesssim \frac{t}{s}\frac{w(B_t(x))}{w(x)\,t^{n}}\Theta_s\mathbf{f}(x)+\frac{t}{s}\frac{w(B_t(x))}{w(x)\,t^{2n}}\int_{\mathbb{R}^n} e^{-\frac{c}{10}\frac{|x-y|}{t}}\Theta_s\mathbf{f}(y)\ dy. \qquad (4.19) \end{aligned}$$

Here recall that $\Theta_s f(x)$ denotes the averaging operator $\Theta_s f(x) = ⨍_{B_s(x)}|f(y)|\ dy$.

We claim that the pair $(u(x), v(x)) = \left(\frac{w(B_t(x))^2}{t^{2n}w(x)}, w(x)\right) \in A_2^t$, as given in Definition 2.6. Indeed, this is immediate from the doubling and the A_2 properties of w :

$$\begin{aligned}
&\left(\fint_{B_r} u(x)\ dx\right)\left(\fint_{B_r} v(x)^{1-2'}\ dx\right)^{2-1} \\
= &\left(\fint_{B_r}\left(\frac{w(B_t)^2}{t^{2n}w(x)}\right) dx\right)\left(\fint_{B_r} w(x)^{-1}\ dx\right) \lesssim \left(\frac{w(B_r)}{|B_r|}\frac{w^{-1}(B_r)}{|B_r|}\right)^2 \lesssim 1.
\end{aligned}$$

Furthermore, since $w^{-1} \in A_2$, it satisfies a reverse Hölder condition:

$$\fint_{B_r} w(x)^{-1}\ dx \le C\left(\fint_{B_r} w(x)^{-\frac{1}{1+\delta}}\ dx\right)^{1+\delta}, \qquad \text{for some } \delta > 0.$$

Therefore, the pair (u, v) satisfies the hypotheses of Theorem 2.7 with $p = 2$, and so the operator Θ_t is bounded from $L^2(w)$ into $L^2\left(\frac{w(B_t)^2}{t^{2n}w}\right) = L^2(u)$.

From the inequality $\Theta_s f(x) \le 2^n\Theta_t\Theta_{2s}f(x) \le 2^n\Theta_t Mf(x)$, where M is the Hardy-Littlewood maximal operator and $0 < t \le s$, we have that the $L^2(w)$-norm of the first term in (4.19) satisfies

$$\begin{aligned}
\left\|\frac{w(B_t(\cdot))}{t^n w}\Theta_s\mathbf{f}\right\|_{L^2(w)} &= \|\Theta_s\mathbf{f}\|_{L^2(u)} \\
&\le 2^n\|\Theta_t M\mathbf{f}\|_{L^2(u)} \lesssim \|M\mathbf{f}\|_{L^2(w)} \lesssim \|\mathbf{f}\|_{L^2(w)}.
\end{aligned} \tag{4.20}$$

On the other hand, since

$$\begin{aligned}
&t^{-n}\int_{\mathbb{R}^n} e^{-\frac{c}{10}\frac{|x-y|}{t}}\Theta_s\mathbf{f}(y)\ dy \\
\lesssim\ & t^{-n}\sum_{j=0}^{\infty} e^{-\frac{c}{10}2^j}\int_{|x-y|\approx 2^j t}\Theta_s\mathbf{f}(y)\ dy \lesssim \sum_{j=0}^{\infty} e^{-\frac{c}{20}2^j}\Theta_{2^j t}\Theta_s\mathbf{f}(x) \\
\lesssim\ & \sum_{j=0}^{\infty} e^{-\frac{c}{20}2^j}\Theta_t\Theta_{2^{j+1}t}\Theta_s\mathbf{f}(x) \lesssim \sum_{j=0}^{\infty} e^{-\frac{c}{20}2^j}\Theta_t M^2\mathbf{f}(x),
\end{aligned}$$

we have

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left|\frac{w(B_t(x))}{w(x)t^{2n}}\int_{\mathbb{R}^n} e^{-\frac{c}{10}\frac{|x-y|}{t}}\Theta_s\mathbf{f}(y)\ dy\right|^2 w dx \\
\lesssim\ & \int_{\mathbb{R}^n}\left|\sum_{j=0}^{\infty} e^{-\frac{c}{20}2^j}\Theta_t M^2\mathbf{f}(x)\right|^2 \frac{w(B_t(x))^2}{w(x)t^{2n}}dx \\
\lesssim\ & \sum_{j=0}^{\infty} e^{-\frac{c}{10}2^j}\int_{\mathbb{R}^n}\left|\Theta_t M^2\mathbf{f}(x)\right|^2 u dx \lesssim \sum_{j=0}^{\infty} e^{-\frac{c}{20}2^j}\left\|M^2\mathbf{f}\right\|_{L^2(w)} \lesssim \|\mathbf{f}\|_{L^2(w)}.
\end{aligned}$$

Applying this to the second term on (4.19), and (4.20) to the first term on the right of (4.19), we obtain

$$\|K_t Q_s\mathbf{f}\|_{L^2(w)} = \|U_t^* U_t Q_s\mathbf{f}\|_{L^2(w)} \lesssim \frac{t}{s}\|\mathbf{f}\|_{L^2(w)}.$$

This implies $\|U_tQ_s\|_{\mathcal{B}(L^2(w))} \lesssim \frac{t}{s}$ since

$$\begin{aligned}\|U_tQ_s\mathbf{f}\|_{L^2(w)}^2 &= \langle U_tQ_s\mathbf{f}, U_tQ_s\mathbf{f}\rangle_{L^2(w)} = \langle U_t^*U_tQ_s\mathbf{f}, Q_s\mathbf{f}\rangle_{L^2(w)} \\ &\lesssim \frac{t}{s}\|\mathbf{f}\|_{L^2(w)}\|Q_s\mathbf{f}\|_{L^2(w)} \lesssim \frac{t}{s}\|\mathbf{f}\|_{L^2(w)}^2 .\end{aligned}$$

Finally, since $P_tQ_s\mathbf{f} = Q_sP_t\mathbf{f}$, and P_t is bounded on $L^2(w)$,

$$\|U_tP_tQ_s\mathbf{f}\|_{L^2(w)}^2 \lesssim \frac{t}{s}\|P_t\mathbf{f}\|_{L^2(w)}^2 \lesssim \frac{t}{s}\|\mathbf{f}\|_{L^2(w)}^2 .$$

□

5. The T(b) argument

To finish the proof of Theorem 1.1 it remains to prove the weighted Carleson measure condition (4.8). We will do so by adapting the original proof from [1] as we did in [7] to get a weighted version of the T(b) theorem for square roots (cf. [3]).

To prove (4.8) we have to estimate

$$\|\gamma_t\|_{C,w}^2 = \sup_Q \fint_Q \int_0^{\ell(Q)} |\gamma_t(x)|^2 \frac{dt}{t}\, dw,$$

where $\gamma_t(x) = \theta_t\mathbf{I}(x) = -t\left(1+t^2\mathcal{L}_w\right)^{-1}\frac{1}{w}\operatorname{div}\mathbf{A}_w\mathbf{I}(x)$. We will obtain the desired inequality as an immediate application of the two lemmas below. Before we can state these results we need some definitions. For any cube Q with center x_Q we set $\Phi_Q = x - x_Q$. Then

$$\gamma_t = \theta_t\nabla\Phi_Q = -t\left(1+t^2\mathcal{L}_w\right)^{-1}\frac{1}{w}\operatorname{div}\mathbf{A}_w\nabla\Phi_Q = -t\left(1+t^2\mathcal{L}_w\right)^{-1}\mathcal{L}_w\Phi_Q.$$

Now for any $\varepsilon \in (0,1)$ and unit vector $\nu \in \mathbb{C}^n$ we define the scalar function

$$f_{Q,\nu}^{\varepsilon} = \left(1+(\varepsilon\ell(Q))^2\mathcal{L}_w\right)^{-1}(\Phi_Q\cdot\nu^*).$$

The dyadic grid in $\mathbb{R}^n$ is the collection of semi-open cubes

$$\mathfrak{D} = \{Q = 2^n([0,1)^n + k),\ k\in\mathbb{Z}^n,\ n\in\mathbb{Z}\},$$

and we define the dyadic averaging operators S_t as

$$S_t\mathbf{f}(x) = \fint_{Q'_{\mathfrak{D}}(x)} \mathbf{f}(y)\ dy, \tag{5.1}$$

where $Q' = Q'_{\mathfrak{D}}(x)$ is the unique cube in $\mathfrak{D}$ such that $x \in Q'$ and $\frac{1}{2}\ell(Q') < t \le \ell(Q')$.

Lemma 5.1. *There exists $\varepsilon > 0$ depending on n, λ, Λ and w and a set U of unit vectors ν in $\mathbb{C}^n$ whose cardinality depend on ε and n such that*

$$\sup_Q \fint_Q \int_0^{\ell(Q)} |\gamma_t(x)|^2 \frac{dt}{t}\, dw \le C\sum_{\nu\in U}\sup_Q \fint_Q \int_0^{\ell(Q)} \left|\gamma_t(x)\cdot\left(S_t\nabla f_{Q,\nu}^{\varepsilon}\right)(x)\right|^2 \frac{dt}{t}\, dw,$$

where C depends on n, λ, Λ and w. The supremum is taken over all cubes $Q \subset \mathbb{R}^n$.

Lemma 5.2. *For C depending on n, λ, Λ, w and $\varepsilon > 0$, we have*

$$\fint_Q \int_0^{\ell(Q)} \left|\gamma_t(x)\cdot\left(S_t\nabla f_{Q,\nu}^{\varepsilon}\right)(x)\right|^2 \frac{dt}{t}\, dw \le C.$$

It is clear that the conjunction of these two lemmas immediately yields (4.8), and hence our main theorem is proven. These results follow in a similar way as in the original proof for the Lebesgue measure case ($w \equiv 1$) [1], with some significant differences.

Proof Lemma 5.2. We begin with two inequalities that are immediate consequences of Corollary 2.13:

$$\int_{5Q} \left| f^{\varepsilon}_{Q,\nu} - (\Phi_Q \cdot v^*) \right|^2 \, wdx \leq C_1 \varepsilon^2 \ell(Q)^2 w(Q), \tag{5.2}$$

and

$$\int_{5Q} \left| \nabla \left(f^{\varepsilon}_{Q,\nu} - (\Phi_Q \cdot \nu^*) \right) \right|^2 \, wdx \leq C_1 w(Q), \tag{5.3}$$

where C does not depend on ε, Q, or v.

Now let χ be a smooth cutoff function supported in $4Q$, $\chi \equiv 1$ on $2Q$, and such that $\|\chi\|_\infty + \ell(Q) \|\nabla \chi\|_\infty \leq C = C(n)$. Since for $x \in Q$ we have that $S_t \nabla f(x) = S_t \nabla \chi f(x)$, we have that

$$\begin{aligned} &\int_Q \int_0^{\ell(Q)} |\gamma_t(x) \cdot (S_t \nabla f)(x)|^2 \frac{dt}{t} \, wdx \\ \leq\ &\int_Q \int_0^{\ell(Q)} \left| \gamma_t(x) \cdot \left(\left(S_t - P_t^2 \right) \nabla \chi f \right)(x) \right|^2 \frac{dt}{t} \, wdx \\ &+ \int_Q \int_0^{\ell(Q)} \left| \gamma_t(x) \cdot \left(P_t^2 \nabla \chi f \right)(x) \right|^2 \frac{dt}{t} \, wdx. \end{aligned} \tag{5.4}$$

To estimate the first term, we claim that

$$\int_Q \int_0^{\ell(Q)} \left| \gamma_t(x) \cdot \left(\left(S_t - P_t^2 \right) \nabla \chi f \right)(x) \right|^2 \frac{dt}{t} \, wdx \leq C \|\nabla \chi f\|^2_{L^2(w)}. \tag{5.5}$$

The proof of this square function estimate is the same as the proof of (4.7). It is straightforward to show that inequality (4.15) holds for $W_t \mathbf{f} = \gamma_t \cdot \left(S_t - P_t^2 \right) \mathbf{f}$. Therefore, we can adapt the proofs of Lemmas 4.2 and 4.3 to show that W_t is given by a kernel $W_t(x,y)$ uniformly in $L^1(dx)$ for all y, and (using the off-diagonal estimates from Lemma 2.10) that $W_t(x,y)$ satisfies exponential bounds similar to those in Lemma 4.3. We can then prove a conservation property and use Proposition 2.3 to get (5.5).

Now, the second term on the right of (5.4) is bounded by

$$\begin{aligned} &\int_0^{\ell(Q)} \int_Q |\theta_t \nabla \chi f|^2 \, wdx \frac{dt}{t} + \int_0^{\ell(Q)} \int_Q \left| \gamma_t(x) \cdot \left(P_t^2 \nabla \chi f \right) - \theta_t P_t^2 \nabla \chi f \right|^2 wdx \frac{dt}{t} \\ &+ \int_0^{\ell(Q)} \int_Q \left| \theta_t \left(I - P_t^2 \right) \nabla \chi f \right|^2 wdx \frac{dt}{t}. \end{aligned}$$

Because of Lemma 4.1 the second term on the right is bounded by $C \|\nabla \chi f\|^2_{L^2(w)}$. The last term on the right is also bounded by $C \|\nabla \chi f\|^2_{L^2(w)}$ because of (4.5). It follows that

$$\int_Q \int_0^{\ell(Q)} \left| \gamma_t(x) \cdot \left(P_t^2 \nabla \chi f \right) \right|^2 \frac{dt}{t} \, wdx \leq \int_0^{\ell(Q)} \int_Q |\theta_t \nabla \chi f|^2 \, wdx \frac{dt}{t} + C \|\nabla \chi f\|^2_{L^2(w)}. \tag{5.6}$$

Hence, to prove Lemma 5.2 it suffices to show that the right hand side of (5.6) is bounded by a constant times $w(Q)$ when $f = f^{\varepsilon}_{Q,\nu}$.

First, by (5.2) and (5.3) it follows that

$$\begin{aligned}
\int_{\mathbb{R}^n} \left|\nabla \chi f^{\varepsilon}_{Q,\nu}\right|^2 \, wdx &\leq \frac{C(n)}{\ell(Q)^2} \int_{4Q} \left|f^{\varepsilon}_{Q,\nu}\right|^2 \, wdx + \int_{4Q} \left|\nabla f^{\varepsilon}_{Q,\nu}\right|^2 \, wdx \\
&\leq \frac{C(n)}{\ell(Q)^2} \int_{4Q} \left|f^{\varepsilon}_{Q,\nu} - (\Phi_Q \cdot v^*)\right|^2 \, wdx \\
&\quad + \int_{4Q} \left|\nabla \left(f^{\varepsilon}_{Q,\nu} - (\Phi_Q \cdot v^*)\right)\right|^2 \, wdx \\
&\quad + \frac{C(n)}{\ell(Q)^2} \int_{4Q} \left|\Phi_Q \cdot v^*\right|^2 \, wdx + \int_{4Q} \left|\nabla \Phi_Q \cdot v^*\right|^2 \, wdx \\
&\leq Cw(Q) + \frac{C(n)}{\ell(Q)^2} \int_{4Q} |x - x_Q|^2 \, wdx + w(4Q) \\
&\leq Cw(Q), \qquad (5.7)
\end{aligned}$$

where in the last line we used the doubling property of w.

Next, we write

$$\begin{aligned}
\theta_t \nabla \chi f &= t\left(1 + t^2 \mathcal{L}_w\right)^{-1} \mathcal{L}_w \chi f \\
&= t\left(1 + t^2 \mathcal{L}_w\right)^{-1} \chi \mathcal{L}_w f - t\left(1 + t^2 \mathcal{L}_w\right)^{-1} \frac{1}{w} (\nabla \chi) \cdot \mathbf{A}_w \nabla f \\
&\quad - t\left(1 + t^2 \mathcal{L}_w\right)^{-1} \frac{1}{w} \operatorname{div} \mathbf{A}_w f (\nabla \chi).
\end{aligned}$$

Therefore, the right hand side of (5.6) (with $f = f^{\varepsilon}_{Q,\nu}$) is bounded by

$$\begin{aligned}
&\int_0^{\ell(Q)} \int_Q |\theta_t \nabla \chi f|^2 \, wdx \frac{dt}{t} \\
&= \int_0^{\ell(Q)} \int_Q \left| t\left(1 + t^2 \mathcal{L}_w\right)^{-1} \chi \mathcal{L}_w f^{\varepsilon}_{Q,\nu} \right|^2 wdx \frac{dt}{t} \\
&\quad + \int_0^{\ell(Q)} \int_Q \left| t\left(1 + t^2 \mathcal{L}_w\right)^{-1} \frac{1}{w} \operatorname{div} \mathbf{A}_w f^{\varepsilon}_{Q,\nu} (\nabla \chi) \right|^2 wdx \frac{dt}{t} \\
&\quad + \int_0^{\ell(Q)} \int_Q \left| t\left(1 + t^2 \mathcal{L}_w\right)^{-1} \frac{1}{w} (\nabla \chi) \cdot \mathbf{A}_w \nabla f^{\varepsilon}_{Q,\nu} \right|^2 wdx \frac{dt}{t} \\
&= I + II + III. \qquad (5.8)
\end{aligned}$$

First, since $\mathcal{L}_w f^{\varepsilon}_{Q,\nu} = (\varepsilon \ell(Q))^{-2} \left((\Phi_Q \cdot \nu^*) - f^{\varepsilon}_{Q,\nu}\right)$, from the resolvent bound (2.5) and from (5.2) we have that

$$\begin{aligned}
I &\leq \int_0^{\ell(Q)} \left\| t\left(1 + t^2 \mathcal{L}_w\right)^{-1} (\varepsilon \ell(Q))^{-2} \chi \left((\Phi_Q \cdot \nu^*) - f^{\varepsilon}_{Q,\nu}\right) \right\|^2_{L^2(w)} \frac{dt}{t} \\
&\leq (\varepsilon \ell(Q))^{-4} \int_0^{\ell(Q)} \left\| t \chi \left((\Phi_Q \cdot \nu^*) - f^{\varepsilon}_{Q,\nu}\right) \right\|^2_{L^2(w)} \frac{dt}{t} \\
&\leq (\varepsilon \ell(Q))^{-4} \int_0^{\ell(Q)} C_1 \varepsilon^2 \ell(Q)^2 w(Q) \; tdt \leq C \varepsilon^{-2} w(Q). \qquad (5.9)
\end{aligned}$$

Since support $(\nabla\chi) \subset 4Q\backslash 2Q$, by the off-diagonal estimate (2.11) and (5.2) we have

$$\begin{aligned}
&\int_Q \left| t\left(1+t^2\mathcal{L}_w\right)^{-1} \frac{1}{w} \operatorname{div} \mathbf{A}_w f^{\varepsilon}_{Q,\nu}(\nabla\chi)\right|^2 wdx \\
\leq\ & Ce^{-c\frac{\ell(Q)}{t}} \int_{4Q} \left|f^{\varepsilon}_{Q,\nu}(\nabla\chi)\right|^2 wdx \leq Ce^{-c\frac{\ell(Q)}{t}} \frac{C(n)}{\ell(Q)^2} \int_{4Q} \left|f^{\varepsilon}_{Q,\nu}\right|^2 wdx \\
\leq\ & Ce^{-c\frac{\ell(Q)}{t}} \frac{C(n)}{\ell(Q)^2} \int_{4Q} \left|f^{\varepsilon}_{Q,\nu} - (\Phi_Q \cdot v^*)\right|^2 wdx \\
& + Ce^{-c\frac{\ell(Q)}{t}} \frac{C(n)}{\ell(Q)^2} \int_{4Q} \left|\Phi_Q \cdot v^*\right|^2 wdx \\
\leq\ & Ce^{-c\frac{\ell(Q)}{t}} w(Q).
\end{aligned}$$

Thus

$$II \leq Cw(Q) \int_0^{\ell(Q)} e^{-c\frac{\ell(Q)}{t}} \frac{dt}{t} \leq \frac{C}{c} w(Q). \tag{5.10}$$

Finally, by the resolvent bound (2.5) and by (5.3),

(5.11)

$$\begin{aligned}
III &\leq \int_0^{\ell(Q)} \int_Q \left| t\left(1+t^2\mathcal{L}_w\right)^{-1} \frac{1}{w} (\nabla\chi) \cdot \mathbf{A}_w \nabla f^{\varepsilon}_{Q,\nu}\right|^2 wdx \frac{dt}{t} \\
&\leq \int_0^{\ell(Q)} \int_{\mathbb{R}^n} \left| t(\nabla\chi) \cdot \nabla f^{\varepsilon}_{Q,\nu}\right|^2 wdx \frac{dt}{t} \leq \frac{C}{\ell(Q)^2} \int_0^{\ell(Q)} t \int_{4Q} \left|\nabla f^{\varepsilon}_{Q,\nu}\right|^2 wdxdt \\
&\leq C \int_{4Q} \left|\nabla\left(f^{\varepsilon}_{Q,\nu} - (\Phi_Q \cdot v^*)\right)\right|^2 wdx + C \int_{4Q} \left|\nabla\Phi_Q \cdot v^*\right|^2 wdx \\
&\leq Cw(Q).
\end{aligned}$$

The lemma follows from (5.7) and by applying (5.9), (5.10), and (5.11) to (5.8). □

To finish the proof of Theorem 1.1 we must establish Lemma 5.1. We will attain this as a consequence of the following proposition.

Proposition 5.3. *There exists $\varepsilon > 0$ depending only on n, λ, Λ, and $[w]_{A_2}$, and $\eta = \eta(\varepsilon) > 0$ such that for each unit vector $\nu \in \mathbb{C}^n$ and cube Q there exists a collection $\mathcal{S}'_\nu = \{Q'\}$ of nonoverlapping subcubes of Q with the properties:*

1. *The union of the cubes in $\mathcal{S}'_\nu$ has measure not exceeding $(1-\eta)|Q|$.*
2. *If $Q'' \in \mathcal{S}''_\nu$, the collection of dyadic subcubes of Q not contained in any $Q' \in \mathcal{S}'_\nu$, then*

$$\fint_{Q''} \operatorname{Re}\left(\nabla f^{\varepsilon}_{Q,\nu}(y) \cdot \nu\right)\ dy \geq \frac{3}{4} \tag{5.12}$$

and

$$\fint_{Q''} \left|\nabla f^{\varepsilon}_{Q,\nu}(y)\right|\ dy \leq (4\varepsilon)^{-1}. \tag{5.13}$$

We will also use the following geometric result.

Lemma 5.4 (Lemma 5.10 in [1]). *Let ν be a unit vector in a Hilbert space H, p and q vectors in H and $0<\varepsilon\leq 1$ be such that:*

(1) $|p-(p\cdot\nu^*)\nu|\leq\varepsilon|p\cdot\nu^*|$,
(2) $\operatorname{Re}(q\cdot\nu)\geq\frac{3}{4}$,
(3) $|q|\leq(4\varepsilon)^{-1}$.

Then $|p|\leq 4|p\cdot q|$.

Proof of Lemma 5.1. Let $\varepsilon>0$ be given by Proposition 5.3. Then $\mathbb{C}^n=\bigcup_{\nu\in O(n,\varepsilon)}\Gamma_\nu$, where Γ_ν is the cone

$$\Gamma_\nu=\{p\in\mathbb{C}^n:|p-(p\cdot\nu^*)\nu|\leq\varepsilon|p\cdot\nu^*|\},$$

and $U(n,\varepsilon)$ is a finite set of unit vectors $\nu\in\mathbb{C}^n$, with cardinality depending only on n and ε. Since

$$|\gamma_t(x)|^2\leq\left|\sum_{\nu\in U(n,\varepsilon)}\chi_{\Gamma_\nu}(\gamma_t(x))\gamma_t(x)\right|^2\lesssim\sum_{\nu\in U(n,\varepsilon)}|\gamma_{t,\nu}(x)|^2,$$

to prove Lemma 5.1 it is enough to prove that for all ν in $U(n,\varepsilon)$,

$$\sup_Q\fint_Q\int_0^{\ell(Q)}|\gamma_{t,\nu}(x)|^2\frac{dt}{t}\,dw\leq C\sup_Q\fint_Q\int_0^{\ell(Q)}\left|\gamma_{t,\nu}(x)\cdot\left(S_t\nabla f^{\varepsilon}_{\tilde{Q},\nu}\right)(x)\right|^2\frac{dt}{t}\,dw,$$

where the supremum is taken over all cubes Q.

Fix a cube Q and $\nu\in U(n,\varepsilon)$, and let $Q''\in\mathcal{S}''_\nu$ as defined in Proposition 5.3. Let

$$q=S_t\nabla f^{\varepsilon}_{Q'',\nu}=\frac{1}{|Q''|}\int_{Q''}\nabla f^{\varepsilon}_{Q,\nu}(y)\,\,dy\in\mathbb{C}^n,$$

where S_t is the averaging operator (5.1). By (5.12) and (5.13) we have that ν and q satisfy (2) and (3) in Lemma 5.4. Moreover, by the definitions of Γ_ν and $\gamma_{t,\nu}$, we have that $p=\gamma_{t,\nu}$, q, and ν satisfy (1) in Lemma 5.4. Therefore,

$$|\gamma_{t,\nu}(x)|\leq 4\left|\gamma_{t,\nu}(x)\cdot S_t\nabla f^{\varepsilon}_{Q,\nu}\right|. \tag{5.14}$$

Next, note that the Carleson box $Q\times(0,\ell(Q)]$ may be partitioned into the Carleson boxes $Q'\times(0,\ell(Q')]$ for $Q'\in\mathcal{S}'_\nu$ and the Whitney rectangles $Q''\times\left(\frac{1}{2}\ell(Q''),\ell(Q'')\right]$ for $Q''\in S''_\nu$. Hence,

$$\begin{aligned}(5.15)\quad&\int_Q\int_0^{\ell(Q)}|\gamma_{t,\nu}(x)|^2\frac{dt}{t}\,wdx\\ &=\sum_{Q'\in S'_w}\int_{Q'}\int_0^{\ell(Q')}|\gamma_{t,\nu}(x)|^2\frac{dt}{t}\,wdx+\sum_{Q''\in S''_w}\int_{Q''}\int_{\frac{1}{2}\ell(Q'')}^{\ell(Q'')}|\gamma_{t,\nu}(x)|^2\frac{dt}{t}\,wdx.\end{aligned}$$

If we define

$$M=\sup_Q\fint_Q\int_0^{\ell(Q)}|\gamma_{t,\nu}(x)|^2\frac{dt}{t}\,dw,$$

where the supremum is taken over all cubes Q, it follows that the first term on the right of (5.15) is bounded by

$$\sum_{Q'\in S'_w}\int_{Q'}\int_0^{\ell(Q')}|\gamma_{t,\nu}(x)|^2\frac{dt}{t}\,wdx\leq M\sum_{Q'\in S'_w}w(Q').$$

Since $w \in A_2 \subset A_\infty$, given $0 < \eta < 1$ and $E \subset Q$ for any cube Q, with $|E|/|Q| \leq 1-\eta$, there exists $0<\tilde{\eta}<1$ such that $w(E)/w(Q) \leq 1-\tilde{\eta}$. Then, by Proposition 5.3,

$$\left|\bigcup_{Q'\in\mathcal{S}'_\nu} Q'\right| \leq (1-\eta)|Q|, \quad \Longrightarrow \quad w\left(\bigcup_{Q'\in\mathcal{S}'_\nu} Q'\right) \leq (1-\tilde{\eta})\, w(Q).$$

Thus,

$$\sum_{Q'\in S'_w} \int_{Q'} \int_0^{\ell(Q')} |\gamma_{t,\nu}(x)|^2 \frac{dt}{t}\, w dx \leq M(1-\tilde{\eta})\, w(Q). \tag{5.16}$$

By (5.14), the second term on the right of (5.15) is bounded by

$$\begin{aligned}
&\sum_{Q''\in S''_w} \int_{Q''} \int_{\frac{1}{2}\ell(Q'')}^{\ell(Q'')} |\gamma_{t,\nu}(x)|^2 \frac{dt}{t}\, wdx \\
\leq\ & 16 \sum_{Q''\in S''_w} \int_{Q''} \int_{\frac{1}{2}\ell(Q'')}^{\ell(Q'')} \left|\gamma_{t,\nu}(x)\cdot S_t\nabla f^{\varepsilon}_{Q,\nu}\right|^2 \frac{dt}{t}\, wdx \\
\leq\ & 16 \int_Q \int_0^{\ell(Q)} \left|\gamma_{t,\nu}(x)\cdot S_t\nabla f^{\varepsilon}_{Q,\nu}\right|^2 \frac{dt}{t}\, wdx. \qquad (5.17)
\end{aligned}$$

Hence, combining (5.15), (5.16) and (5.17), we get

$$⨍_Q \int_0^{\ell(Q)} |\gamma_{t,\nu}(x)|^2 \frac{dt}{t}\, dw \leq M(1-\tilde{\eta}) + 16 ⨍_Q \int_0^{\ell(Q)} \left|\gamma_{t,\nu}(x)\cdot S_t\nabla f^{\varepsilon}_{Q,\nu}\right|^2 \frac{dt}{t}\, dw.$$

Taking supremum over all cubes Q, recalling the definition of M, and rearranging, we have

$$\begin{aligned}
&\sup_Q ⨍_Q \int_0^{\ell(Q)} |\gamma_{t,\nu}(x)|^2 \frac{dt}{t}\, dw \\
\leq\ & \frac{16}{\eta} \sup_Q ⨍_Q \int_0^{\ell(Q)} \left|\gamma_{t,\nu}(x)\cdot S_t\nabla f^{\varepsilon}_{Q,\nu}\right|^2 \frac{dt}{t}\, dw.
\end{aligned}$$

□

Finally, we prove Proposition 5.3, and finish this way the proof of Theorem 1.1. We will first need an additional lemma.

Lemma 5.5. *If $w\in A_p$, $1<p<\infty$, then there exist constants C and $0<\tau<1$ depending on dimension n and $[w]_{A_p}$ such that for all $h \in W^{1,p}_{\mathrm{loc}}(w)$,*

$$\left|⨍_Q \nabla h\, dx\right| \leq C\ell(Q)^{-\tau} \left(⨍_Q |h|^p\, dw\right)^{\frac{\tau}{p}} \left(⨍_Q |\nabla h|^p\, dw\right)^{\frac{1-\tau}{p}}. \tag{5.18}$$

Proof. Using standard properties of A_∞ weights, the proof of this lemma is an extension of the unweighted version for $p=2$; see Lemma 5.15 in [1]. Let $M = \left(⨍_Q |h|^p\, dw\right)^{\frac{1}{p}}$ and $M' = \left(⨍_Q |\nabla h|^p\, dw\right)^{\frac{1}{p}}$. We assume first that $M < \ell(Q) M'$. Then given $t\in(0,1)$ take a cutoff function $\varphi \in \mathcal{C}_0^\infty(Q)$ such that $0\leq\varphi\leq 1$, $\varphi \equiv 1$

on $Q_{1-t} := (1-t)\,Q$, and $\|\nabla\varphi\|_\infty \le C(n)\,(t\ell(Q))^{-1}$. Then multiplying and dividing by $w^{1/p}$, and applying Hölder's inequality,

$$\begin{aligned}\left|\fint_Q \nabla h\, dx\right| &= |Q|^{-1}\left|\int_Q (1-\varphi)\nabla h\, dx - \int_Q h\nabla\varphi\, dx\right| \\ &\le \left(\left(\int_Q |\nabla h|^p\ \, wdx\right)^{\frac{1}{p}} + C(n)\,(t\ell(Q))^{-1}\left(\int_Q h^2\ wdx\right)^{\frac{1}{2}}\right) \\ &\quad\cdot |Q|^{-1}\left(w^{-\frac{1}{p-1}}\,(Q\backslash Q_{1-t})\right)^{\frac{p-1}{p}} \\ &= \left(M' + C(n)\,(t\ell(Q))^{-1}M\right)|Q|^{-1}\left(w^{-\frac{1}{p-1}}\,(Q\backslash Q_{1-t})\right)^{\frac{p-1}{p}} w(Q)^{\frac{1}{p}}. \qquad (5.19)\end{aligned}$$

Since $w \in A_p \Longleftrightarrow w^{-\frac{1}{p-1}} \in A_{p'} \subset A_\infty$, there exists $\alpha > 0$ and $\tau > 0$ such that

$$\frac{w^{-\frac{1}{p-1}}\,(E)}{w^{-\frac{1}{p-1}}\,(Q)} \le \alpha\left(\frac{|E|}{|Q|}\right)^{\frac{p}{p-1}\tau}, \qquad \text{for all cubes } Q,\ E \subset Q.$$

Then, since $|Q\backslash Q_{1-t}| \le C(n)\,t\,|Q|$, it follows that

$$\left(w^{-\frac{1}{p-1}}\,(Q\backslash Q_{1-t})\right)^{\frac{p-1}{p}} \le C(n,\alpha)\,t^\tau\left(w^{-\frac{1}{p-1}}\,(Q)\right)^{\frac{p-1}{p}}. \qquad (5.20)$$

Note that $|Q|^{-1}\left(w^{-\frac{1}{p-1}}\,(Q)\right)^{\frac{p-1}{p}} w(Q)^{\frac{1}{p}} \le [w]_{A_p}^{\frac{1}{p}}$. Then applying (5.20) to the right of (5.19) we obtain

$$\left|\fint_Q \nabla h\, dx\right| \le C\left(n,[w]_{A_p}\right)\left(M'\,(t\ell(Q))^\tau + M\,(t\ell(Q))^{\tau-1}\right)\ell(Q)^{-\tau}.$$

Taking $t = M/\left(\ell(Q)\,(M')\right) < 1$ yields (5.18). This proves the lemma in the case $\ell(Q)\,M < M'$. On the other hand, if $M' \le \ell(Q)^{-1}M$,

$$\begin{aligned}\left|\fint_Q \nabla h\, dx\right| &\le \left(\frac{1}{|Q|}\int_Q |\nabla h|^p\ \, wdx\right)^{\frac{1}{p}}\left(w^{-\frac{1}{p-1}}\,(Q)\right)^{\frac{p-1}{p}} \\ &\le [w]_{A_p}^{\frac{1}{p}}\left(\frac{1}{w(Q)}\int_Q |\nabla h|^2\ \, wdx\right)^{\frac{1}{p}} \\ &= [w]_{A_p}^{\frac{1}{p}} M' \le [w]_{A_p}^{\frac{1}{p}}\,\ell(Q)^{-\tau} M^\tau\,(M')^{1-\tau}.\end{aligned}$$

□

Proof of Proposition 5.3. From the identity $\nabla(\Phi_Q\cdot v^*)(x)\cdot v = v^*\cdot v = |v|^2 = 1$, we have that

$$\nabla\left(f^\varepsilon_{Q,\nu} - (\Phi_Q\cdot\nu^*)\right)\cdot v = \nabla f^\varepsilon_{Q,\nu}\cdot v - 1.$$

Therefore, by Lemma 5.5 (with $p = 2$) and inequalities (5.2) and (5.3) we have that

$$\begin{aligned} &\left|\int_Q \nabla f^{\varepsilon}_{Q,\nu} \cdot v - 1 \, dx\right| \\ &\leq C(n,\tau)\,|Q|\,\ell(Q)^{-\tau}\left(\frac{1}{w(Q)}\int_Q \left|f^{\varepsilon}_{Q,\nu} - (\Phi_Q \cdot \nu^*)\right|^2 \, wdx\right)^{\frac{\tau}{2}} \\ &\quad \cdot \left(\frac{1}{w(Q)}\int_Q \left|\nabla\left(f^{\varepsilon}_{Q,\nu} - (\Phi_Q \cdot \nu^*)\right)\right|^2 \, wdx\right)^{\frac{1-\tau}{2}} \\ &\leq C(n,w)\,|Q|\,\varepsilon^{\tau}. \end{aligned} \tag{5.21}$$

Then, for ε small enough,

$$\fint_Q \operatorname{Re}\left(\nabla f^{\varepsilon}_{Q,\nu} \cdot v\right)\, dx \geq \frac{7}{8}.$$

On the other hand, by (5.3),

$$\begin{aligned} \fint_Q \left|\nabla f^{\varepsilon}_{Q,\nu}\right| \, dx &\leq \fint_Q \left|\nabla\left(f^{\varepsilon}_{Q,\nu} - (\Phi_Q \cdot \nu^*)\right)\right| \, dx + C \\ &\leq \frac{\left(w^{-1}(Q)\right)^{\frac{1}{2}}}{|Q|}\left(\int_Q \left|\nabla\left(f^{\varepsilon}_{Q,\nu} - (\Phi_Q \cdot \nu^*)\right)\right|^2 \, wdx\right)^{\frac{1}{2}} + C \\ &\leq C\left([w]_{A_2}^{\frac{1}{2}} + 1\right). \end{aligned} \tag{5.22}$$

Now perform a stopping-time decomposition to select a collection $\mathcal{S}'_\nu$ of dyadic subcubes Q' of Q which are maximal with respect to either

$$\fint_{Q'} \operatorname{Re}\left(\nabla f^{\varepsilon}_{Q,\nu}(y) \cdot \nu\right)\, dy < \frac{3}{4} \tag{5.23}$$

or

$$\fint_{Q'} \left|\nabla f^{\varepsilon}_{Q,\nu}(y)\right| \, dy > (4\varepsilon)^{-1}. \tag{5.24}$$

(If necessary, we may take ε smaller than chosen above; we will fix its value below.) That is, we dyadically subdivide Q and stop the first time either one of the inequalities hold. By construction, we have that (2) holds.

It remains to establish (1). Set $E = \bigcup_{\mathcal{S}'_\nu} Q'$. We have to prove that $|E| \leq (1-\eta)\,|Q|$. Let E_1 consist of those cubes in $\mathcal{S}'_\nu$ which satisfy (5.23), and let E_2 be the union of the cubes in $\mathcal{S}'_\nu$ which satisfy (5.24). Then $|E| \leq |E_1| + |E_2|$.

Since the cubes in $\mathcal{S}'_\nu$ are nonoverlapping, it follows from (5.22) and (5.24) that

$$|E_2| \leq (4\varepsilon)\int_Q \left|\nabla f^{\varepsilon}_{Q,\nu}(y)\right| \, dy \leq (4\varepsilon)\, C\left([w]_{A_2}^{\frac{1}{2}} + 1\right)|Q|. \tag{5.25}$$

Setting $b(x) = 1 - \operatorname{Re}\left(\nabla f^{\varepsilon}_{Q,\nu}(y) \cdot \nu\right)$, if (5.23) holds we have $\frac{1}{|Q'|}\int_{Q'} b(y)\, dy > \frac{1}{4}$. Hence

$$|E_1| \leq 4\sum \int_{Q'} b(y)\, dy = 4\int_Q b(y)\, dy - 4\int_{Q\setminus E_1} b(y)\, dy,$$

where the sum is taken over all the cubes comprising E_1. By (5.21) the first term is bounded by $C(n,w)\,|Q|\,\varepsilon^{\tau}$, while the absolute value of the second term is bounded by

$$\begin{aligned} & 4\left|Q\backslash E_1\right|+4\int_{\left|Q\backslash E_1\right|}\left|\nabla f_{Q,\nu}^{\varepsilon}\left(y\right)\right|\ dy \\ \leq\ & 4\left|Q\backslash E_1\right|+4w^{-1}\left(Q\backslash E_1\right)^{\frac{1}{2}}\left(\int_Q\left|\nabla\left(f_{Q,\nu}^{\varepsilon}\left(y\right)-\left(\Phi_Q\cdot\nu^*\right)\right)\right|^2\ wdy+w\left(Q\right)\right)^{\frac{1}{2}} \\ \leq\ & 4\left|Q\backslash E_1\right|+4Cw^{-1}\left(Q\backslash E_1\right)^{\frac{1}{2}}w\left(Q\right)^{\frac{1}{2}}. \end{aligned}$$

By the A_∞ property of w^{-1}, we have that for some constants $\alpha,\sigma>0$,

$$\frac{w^{-1}\left(Q\backslash E_1\right)}{w^{-1}\left(Q\right)}\leq\alpha^2\left(\frac{\left|Q\backslash E_1\right|}{\left|Q\right|}\right)^{2\sigma}.$$

Then

$$\begin{aligned} & 4\left|Q\backslash E_1\right|+4\int_{\left|Q\backslash E_1\right|}\left|\nabla f_{Q,\nu}^{\varepsilon}\left(y\right)\right|\ dy \\ \leq\ & 4\left|Q\backslash E_1\right|+4C\alpha\left[w\right]_{A_2}\left|Q\backslash E_1\right|^{\sigma}\left|Q\right|^{1-\sigma} \\ \leq\ & 4\left|Q\backslash E_1\right|+\left(4C\alpha\left[w\right]_{A_2}\right)^{\rho/\sigma}\sigma\varepsilon^{-\tau(1-\sigma)/\sigma}\left|Q\backslash E_1\right|+\left(1-\sigma\right)\varepsilon^{\tau}\left|Q\right|. \end{aligned}$$

Since $\left|Q\backslash E_1\right|=\left|Q\right|-\left|E_1\right|$, we get

$$\left|E_1\right|\leq\frac{4+\left(C\left(n,w\right)+1\right)\varepsilon^{\tau}+\sigma\left(4C\alpha\left[w\right]_{A_2}\right)^{\rho/\sigma}\varepsilon^{-\tau(1-\sigma)/\sigma}}{5+\sigma\left(4C\alpha\left[w\right]_{A_2}\right)^{\rho/\sigma}\varepsilon^{-\tau(1-\sigma)/\sigma}}\left|Q\right|. \tag{5.26}$$

Taking ε small enough, from (5.25) and (5.26) we conclude that for some $0<\eta<1$, $\left|E\right|\leq\left(1-\eta\right)\left|Q\right|$, as wanted. □

References

[1] Pascal Auscher, Steve Hofmann, Michael Lacey, Alan McIntosh, and Ph. Tchamitchian, *The solution of the Kato square root problem for second order elliptic operators on* $\mathbb{R}^n$, Ann. of Math. (2) **156** (2002), no. 2, 633–654, DOI 10.2307/3597201. MR1933726 (2004c:47096c)

[2] Pascal Auscher and José María Martell, *Weighted norm inequalities, off-diagonal estimates and elliptic operators. II. Off-diagonal estimates on spaces of homogeneous type*, J. Evol. Equ. **7** (2007), no. 2, 265–316, DOI 10.1007/s00028-007-0288-9. MR2316480 (2008m:47059)

[3] Pascal Auscher and Philippe Tchamitchian, *Square root problem for divergence operators and related topics* (English, with English and French summaries), Astérisque **249** (1998), viii+172. MR1651262 (2000c:47092)

[4] Filippo Chiarenza and Michelangelo Franciosi, *Quasiconformal mappings and degenerate elliptic and parabolic equations*, Matematiche (Catania) **42** (1987), no. 1-2, 163–170 (1989). MR1030914 (91b:35019)

[5] Filippo Chiarenza and Raul Serapioni, *A remark on a Harnack inequality for degenerate parabolic equations*, Rend. Sem. Mat. Univ. Padova **73** (1985), 179–190. MR799906 (87a:35110)

[6] David Cruz-Uribe and Cristian Rios, *Gaussian bounds for degenerate parabolic equations*, J. Funct. Anal. **255** (2008), no. 2, 283–312, DOI 10.1016/j.jfa.2008.01.017. MR2419963 (2009k:35162)

[7] David Cruz-Uribe and Cristian Rios, *The solution of the Kato problem for degenerate elliptic operators with Gaussian bounds*, Trans. Amer. Math. Soc. **364** (2012), no. 7, 3449–3478, DOI 10.1090/S0002-9947-2012-05380-3. MR2901220

[8] Nelson Dunford and B. J. Pettis, *Linear operations on summable functions*, Trans. Amer. Math. Soc. **47** (1940), 323–392. MR0002020 (1,338b)

[9] Eugene B. Fabes, Carlos E. Kenig, and Raul P. Serapioni, *The local regularity of solutions of degenerate elliptic equations*, Comm. Partial Differential Equations **7** (1982), no. 1, 77–116, DOI 10.1080/03605308208820218. MR643158 (84i:35070)

[10] Michael Frazier, Björn Jawerth, and Guido Weiss, *Littlewood-Paley theory and the study of function spaces*, CBMS Regional Conference Series in Mathematics, vol. 79, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1991. MR1107300 (92m:42021)

[11] F. W. Gehring, *The L^p-integrability of the partial derivatives of quasiconformal mapping*, Bull. Amer. Math. Soc. **79** (1973), 465–466. MR0320308 (47 #8847)

[12] Markus Haase, *The functional calculus for sectorial operators*, Operator Theory: Advances and Applications, vol. 169, Birkhäuser Verlag, Basel, 2006. MR2244037 (2007j:47030)

[13] Tosio Kato, *Fractional powers of dissipative operators*, J. Math. Soc. Japan **13** (1961), 246–274. MR0138005 (25 #1453)

[14] C. Kenig, *Featured review: The solution of the Kato square root problem for second order elliptic operators on* $\mathbb{R}^n$, Mathematical Reviews MR1933726 (2004c:47096c) (2004).

[15] J.-L. Lions, *Espaces d'interpolation et domaines de puissances fractionnaires d'opérateurs* (French), J. Math. Soc. Japan **14** (1962), 233–241. MR0152878 (27 #2850)

[16] Alan McIntosh, *Square roots of operators and applications to hyperbolic PDE*, (Canberra, 1983), Proc. Centre Math. Anal. Austral. Nat. Univ., vol. 5, Austral. Nat. Univ., Canberra, 1984, pp. 124–136. MR757577 (85h:47016)

[17] Alan McIntosh and Atsushi Yagi, *Operators of type ω without a bounded H_∞ functional calculus*, Miniconference on Operators in Analysis (Sydney, 1989), Proc. Centre Math. Anal. Austral. Nat. Univ., vol. 24, Austral. Nat. Univ., Canberra, 1990, pp. 159–172. MR1060121 (91e:47013)

[18] Nicholas Miller, *Weighted Sobolev spaces and pseudodifferential operators with smooth symbols*, Trans. Amer. Math. Soc. **269** (1982), no. 1, 91–109, DOI 10.2307/1998595. MR637030 (83f:47036)

[19] El Maati Ouhabaz, *Analysis of heat equations on domains*, London Mathematical Society Monographs Series, vol. 31, Princeton University Press, Princeton, NJ, 2005. MR2124040 (2005m:35001)

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