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From r-dual sets to uniform contractions

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Abstract. Let $\mathbb{M}^d$ denote the d-dimensional Euclidean, hyperbolic, or spherical space. The r-dual set of a given set in $\mathbb{M}^d$ is the intersection of balls of radii r centered at the points of the a given set. In this paper we prove that for any set of given volume in $\mathbb{M}^d$ the volume of the r-dual set becomes maximal if the set is a ball. As an application we prove the following. The Kneser–Poulsen Conjecture states that if the centers of a family of N congruent balls in Euclidean d-space is contracted, then the volume of the intersection does not decrease. A uniform contraction is a contraction where all the pairwise distances in the first set of centers are larger than all the pairwise distances in the second set of centers, that is, when the pairwise distances of the two sets are separated by some positive real number. We prove a special case of the Kneser–Poulsen conjecture namely, we prove the conjecture for uniform contractions (with sufficiently large N) in $\mathbb{M}^d$.

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1. Introduction

Let $\mathbb{M}^d, d > 1$ denote the d-dimensional Euclidean, hyperbolic, or spherical space, i.e., one of the simply connected complete Riemannian manifolds of constant sectional curvature. Since simply connected complete space forms, the sectional curvature of which have the same sign are similar, we may assume without loss of generality that the sectional curvature κ of $\mathbb{M}^d$ is $0, -1$, or 1. Let $\mathbf{R}_+$ denote the set of positive real numbers for $\kappa \leq 0$ and the half-open interval $(0, \frac{\pi}{2}]$ for $\kappa = 1$. Let $\mathrm{dist}_{\mathbb{M}^d}(\mathbf{x}, \mathbf{y})$ stand for the geodesic distance between the points $\mathbf{x} \in \mathbb{M}^d$ and $\mathbf{y} \in \mathbb{M}^d$. Furthermore, let $\mathbf{B}_{\mathbb{M}^d}[\mathbf{x}, r]$ denote the closed d-dimensional ball with center $\mathbf{x} \in \mathbb{M}^d$ and radius $r \in \mathbf{R}_+$ in $\mathbb{M}^d$, i.e.,

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let $\mathbf{B}_{\mathbb{M}^d}[\mathbf{x}, r] := \{\mathbf{y} \in \mathbb{M}^d \mid \mathrm{dist}_{\mathbb{M}^d}(\mathbf{x}, \mathbf{y}) \le r\}$. Now, we are ready to introduce the central notion of this paper.

Definition 1. For a set $X \subseteq \mathbb{M}^d, d > 1$ and $r \in \mathbf{R}_+$ let the r-dual set X^r of X be defined by $X^r := \bigcap_{\mathbf{x} \in X} \mathbf{B}_{\mathbb{M}^d}[\mathbf{x}, r]$. If the interior $\mathrm{int}(X^r) \ne \emptyset$, then we call X^r the r-dual body of X.

We note that either $X^r = \emptyset$, or X^r is a point in $\mathbb{M}^d$, or $\mathrm{int}(X^r) \ne \emptyset$. Perhaps not surprisingly, r-dual sets of $\mathbb{E}^d$ have already been investigated in a number of papers, however, under various names such as "überkonvexe Menge" [13], "r-convex domain" [8], "spindle convex set" [2,11], "ball convex set" [12], and "hyperconvex set" [9]. r-dual sets satisfy some basic identities such as

$$((X^r)^r))^r = X^r \text{ and } (X \cup Y)^r = X^r \cap Y^r,$$

which hold for any $X \subseteq \mathbb{M}^d$ and $Y \subseteq \mathbb{M}^d$. Clearly, also monotonicity holds namely, $X \subseteq Y \subseteq \mathbb{M}^d$ implies $Y^r \subseteq X^r$. Thus, there is a good deal of similarity between r-dual sets and polar sets (resp., spherical polar sets) in $\mathbb{E}^d$ (resp., $\mathbb{S}^d$). In this paper we explore further this similarity by investigating a volumetric relation between X^r and X in $\mathbb{M}^d$. For this reason let $V_{\mathbb{M}^d}(\cdot)$ denote the Lebesgue measure in $\mathbb{M}^d$, to which we are going to refer as volume in $\mathbb{M}^d$. Now, recall the recent theorem of Gao et al. [10] stating that for any convex body of given volume in $\mathbb{S}^d$ the volume of the spherical polar body becomes maximal if the convex body is a ball. We prove the following extension of their theorem.

Theorem 1. *Let $A \subseteq \mathbb{M}^d, d > 1$ be a compact set of volume $V_{\mathbb{M}^d}(A) > 0$ and $r \in \mathbf{R}_+$. If $B \subseteq \mathbb{M}^d$ is a ball with $V_{\mathbb{M}^d}(A) = V_{\mathbb{M}^d}(B)$, then $V_{\mathbb{M}^d}(A^r) \le V_{\mathbb{M}^d}(B^r)$.*

Note that the Gao–Hug–Schneider theorem is a special case of Theorem 1 namely, when $\mathbb{M}^d = \mathbb{S}^d$ and $r = \frac{\pi}{2}$. As this theorem of [10] is often called a spherical counterpart of the Blaschke–Santaló inequality, one may refer to Theorem 1 as a Blaschke–Santaló-type inequality for r-duality in $\mathbb{M}^d$.

From our point view, the importance of Theorem 1 lies in the following application. To state it in a proper way we recall the following notion from [5].

Definition 2. We say that the (labeled) point set $\{\mathbf{q}_1, \dots, \mathbf{q}_N\} \subset \mathbb{M}^d$ is a uniform contraction of the (labeled) point set $\{\mathbf{p}_1, \dots, \mathbf{p}_N\} \subset \mathbb{M}^d$ with separating value $\lambda > 0$ in $\mathbb{M}^d, d > 1$ if

$$\mathrm{dist}_{\mathbb{M}^d}(\mathbf{q}_i, \mathbf{q}_j) \le \lambda \le \mathrm{dist}_{\mathbb{M}^d}(\mathbf{p}_i, \mathbf{p}_j)$$

holds for all $1 \le i < j \le N$.

Now, recall the following recent theorem of the author and Naszódi [5] : Let $d \in \mathbb{Z}$ and $\delta, \lambda \in \mathbb{R}$ be given such that $d > 1$ and $0 < \lambda \le \sqrt{2}\delta$. If $Q := \{\mathbf{q}_1, \dots, \mathbf{q}_N\} \subset \mathbb{E}^d$ is a uniform contraction of $P := \{\mathbf{p}_1, \dots, \mathbf{p}_N\} \subset \mathbb{E}^d$ with separating value λ in $\mathbb{E}^d$ and $N \ge (1 + \sqrt{2})^d$, then $V_{\mathbb{E}^d}(P^\delta) \le V_{\mathbb{E}^d}(Q^\delta)$.

As it is explained in [5], this proves the Kneser–Poulsen conjecture for uniform contractions. For the sake of completeness we mention here that according to the Kneser–Poulsen conjecture if a finite set of balls in $\mathbb{E}^d$ is rearranged so that the distance between each pair of centers does not increase, then the volume of the intersection does not decrease. This is proved for $d = 2$ in [1] and it remains open for $d > 2$. For more details on the Kneser–Poulsen conjecture we refer the interested reader to Chapter 3 in [3]. In this paper, we give a rather short and elementary proof of the above mentioned theorem of the author and Naszódi (replacing the Brunn–Minkowski inequality in [5] by Theorem 1) and perhaps, more importantly we extend it to hyperbolic as well as spherical spaces as follows.

Theorem 2. (i) *Let $d \in \mathbb{Z}$ and $\delta, \lambda \in \mathbb{R}$ be given such that $d > 1$ and $0 < \lambda \le \sqrt{2}\delta$. If $Q := \{\mathbf{q}_1, \dots, \mathbf{q}_N\} \subset \mathbb{E}^d$ is a uniform contraction of $P := \{\mathbf{p}_1, \dots, \mathbf{p}_N\} \subset \mathbb{E}^d$ with separating value λ in $\mathbb{E}^d$ and $N \ge (1+\sqrt{2})^d$, then $V_{\mathbb{E}^d}(P^\delta) < V_{\mathbb{E}^d}(Q^\delta)$.*

(ii) *Let $d \in \mathbb{Z}$ and $\delta, \lambda \in \mathbb{R}$ be given such that $d > 1, 0 < \delta < \frac{\pi}{2}$, and $0 < \lambda < \min\left\{\frac{2\sqrt{2}}{\pi}\delta, \pi - 2\delta\right\}$. If $Q := \{\mathbf{q}_1, \dots, \mathbf{q}_N\} \subset \mathbb{S}^d$ is a uniform contraction of $P := \{\mathbf{p}_1, \dots, \mathbf{p}_N\} \subset \mathbb{S}^d$ with separating value λ in $\mathbb{S}^d$ and $N \ge 2ed\pi^{d-1}\left(\frac{1}{2} + \frac{\pi}{2\sqrt{2}}\right)^d$, then $V_{\mathbb{S}^d}(P^\delta) < V_{\mathbb{S}^d}(Q^\delta)$.*

(iii) *Let $d, k \in \mathbb{Z}$ and $\delta, \lambda \in \mathbb{R}$ be given such $d > 1, k > 0$ and $0 < \frac{\sinh k}{\sqrt{2}k}\lambda \le \delta < k$. If $Q := \{\mathbf{q}_1, \dots, \mathbf{q}_N\} \subset \mathbb{H}^d$ is a uniform contraction of $P := \{\mathbf{p}_1, \dots, \mathbf{p}_N\} \subset \mathbb{H}^d$ with separating value λ in $\mathbb{H}^d$ and $N \ge (\frac{\sinh 2k}{2k})^{d-1} (\frac{\sqrt{2}\sinh k}{k} + 1)^d$, then $V_{\mathbb{H}^d}(P^\delta) < V_{\mathbb{H}^d}(Q^\delta)$.*

In the rest of the paper we prove the theorems stated.

2. Proof of Theorem 1

We adapt the two-point symmetrization method of the proof of the Gao-Hug-Schneider theorem from [10]. For this we need to recall the definition of two-point symmetrization, which is also known under the names "two-point rearrangement", "compression", or "polarization". (For more details on two-point symmetrization we refer the interested reader to the relevant section in [10] and the references mentioned there.)

Definition 3. Let H be a hyperplane in $\mathbb{M}^d$ with an orientation, which determines H^+ and H^- the two closed halfspaces bounded by H in $\mathbb{M}^d, d > 1$. Let σ_H denote the reflection about H in $\mathbb{M}^d$. If $K \subseteq \mathbb{M}^d$, then the two-point symmetrization τ_H with respect to H transforms K into the set

$$\tau_H K := (K \cap \sigma_H K) \cup \left((K \cup \sigma_H K) \cap H^+\right).$$

If $K_H := K \cap \sigma_H K$ stands for the H-symmetric core of K, then we call

$$\tau_H K = K_H \cup \big((K \cap H^+) \setminus K_H\big) \cup \sigma_H \left((K \cap H^-) \setminus K_H\right) \tag{1}$$

the canonical decomposition of $\tau_H K$.

Remark 3. The canonical decomposition of $\tau_H K$ is a disjoint decomposition of $\tau_H K$, which easily implies that two-point symmetrization preserves volume.

Definition 4. Let $K \subset \mathbb{M}^d, d > 1$ and $r \in \mathbf{R}_+$. Then the r-convex hull $\mathrm{conv}_r K$ of K is defined by

$$\mathrm{conv}_r K := \bigcap \{\mathbf{B}_{\mathbb{M}^d}[\mathbf{x}, r] \mid K \subseteq \mathbf{B}_{\mathbb{M}^d}[\mathbf{x}, r]\}.$$

Moreover, let the r-convex hull of $\mathbb{M}^d$ be $\mathbb{M}^d$. Furthermore, we say that $K \subseteq \mathbb{M}^d$ is an r-convex set if $K = \mathrm{conv}_r K$.

Remark 4. We note that clearly, $\mathrm{conv}_r K = \emptyset$ if and only if $K^r = \emptyset$.

Lemma 5. *If* $K \subseteq \mathbb{M}^d, d > 1$ *and* $r \in \mathbf{R}_+$, *then*

$$K^r = (\mathrm{conv}_r K)^r. \tag{2}$$

Proof. Based on Remark 4, the claim holds for $\mathrm{conv}_r K = \emptyset$. Thus, in what follows, we assume that $\mathrm{conv}_r K \neq \emptyset$, that is, $K^r \neq \emptyset$. Then $K \subseteq \mathrm{conv}_r K$ and therefore $(\mathrm{conv}_r K)^r \subseteq K^r$. On the other hand, we show that $K^r \subseteq (\mathrm{conv}_r K)^r$. So let $\mathbf{y} \in K^r$. Then clearly, $K \subseteq \mathbf{B}_{\mathbb{M}^d}[\mathbf{y}, r]$ and so, $\mathrm{conv}_r K \subseteq \mathbf{B}_{\mathbb{M}^d}[\mathbf{y}, r]$ implying that $\mathbf{y} \in (\mathrm{conv}_r K)^r$. Thus, (2) follows. □

The core part of our proof of Theorem 1 is

Lemma 6. *If* $K \subseteq \mathbb{M}^d, d > 1$ *and* $r \in \mathbf{R}_+$, *then*

$$\tau_H(K^r) \subseteq (\mathrm{conv}_r(\tau_H K))^r.$$

Proof. Lemma 5 implies that $(\mathrm{conv}_r(\tau_H K))^r = (\tau_H K)^r$ and so, it is sufficient to prove that $\tau_H(K^r) \subseteq (\tau_H K)^r$. For this we need to show that if $\mathbf{x} \in \tau_H(K^r)$, then $\mathbf{x} \in (\tau_H K)^r$, i.e.,

$$\tau_H K \subseteq \mathbf{B}_{\mathbb{M}^d}[\mathbf{x}, r]. \tag{3}$$

Remark 3 implies that

$$\tau_H(K^r) = (K^r)_H \cup \big((K^r \cap H^+) \setminus (K^r)_H\big) \cup \sigma_H \left((K^r \cap H^-) \setminus (K^r)_H\right)$$

is a disjoint decomposition of $\tau_H(K^r)$ with $(K^r)_H = K^r \cap \sigma_H(K^r)$. Thus, either $\mathbf{x} \in (K^r)_H$ (Case 1), or $\mathbf{x} \in (K^r \cap H^+) \setminus (K^r)_H$ (Case 2), or $\mathbf{x} \in \sigma_H((K^r \cap H^-) \setminus (K^r)_H)$ (Case 3). In all three cases we use (1) for the proof of (3).

Case 1: As $(K^r)_H = K^r \cap \sigma_H(K^r)$ we have $\mathbf{x}, \sigma_H\mathbf{x} \in (K^r)_H$. As $\mathbf{x} \in (K^r)_H \subseteq K^r$ we have $K_H \cup ((K \cap H^+) \setminus K_H) \subseteq K \subseteq \mathbf{B}_{\mathbb{M}^d}[\mathbf{x}, r]$. On the other hand, as $\sigma_H\mathbf{x} \in (K^r)_H \subseteq K^r$ we have $(K \cap H^-) \setminus K_H \subseteq K \subseteq \mathbf{B}_{\mathbb{M}^d}[\sigma_H\mathbf{x}, r]$ and so, $\sigma_H((K \cap H^-) \setminus K_H) \subseteq \mathbf{B}_{\mathbb{M}^d}[\mathbf{x}, r]$, finishing the proof of (3).

Case 2: As $\mathbf{x} \in (K^r \cap H^+) \setminus (K^r)_H \subseteq K^r$ we have $K_H \cup ((K \cap H^+) \setminus K_H) \subseteq K \subseteq \mathbf{B}_{\mathbb{M}^d}[\mathbf{x}, r]$. So, we are left to show that

$$\sigma_H \left((K \cap H^-) \setminus K_H\right) \subseteq \mathbf{B}_{\mathbb{M}^d}[\mathbf{x}, r]. \tag{4}$$

On the one hand, $\mathbf{x} \in (K^r \cap H^+) \setminus (K^r)_H \subseteq K^r$ implies that $(K \cap H^-) \setminus K_H \subseteq K \subseteq \mathbf{B}_{\mathbb{M}^d}[\mathbf{x}, r]$. On the other hand, for any $\mathbf{y} \in (K \cap H^-) \setminus K_H$ we have $\sigma_H \mathbf{y} \in \sigma_H ((K \cap H^-) \setminus K_H)$. As $\mathbf{x}, \sigma_H \mathbf{y} \in H^+$ and $\mathbf{y} \in H^-$ we have $\operatorname{dist}_{\mathbb{M}^d}(\sigma_H \mathbf{y}, \mathbf{x}) \leq \operatorname{dist}_{\mathbb{M}^d}(\mathbf{y}, \mathbf{x}) \leq r$. Thus, (4) follows.

Case 3: It follows from the assumption that $\sigma_H \mathbf{x} \in (K^r \cap H^-) \setminus (K^r)_H \subseteq K^r$ and we have $(K \cap H^-) \setminus K_H \subseteq K \subseteq \mathbf{B}_{\mathbb{M}^d}[\sigma_H \mathbf{x}, r]$ implying that $\sigma_H ((K \cap H^-) \setminus K_H) \subseteq \mathbf{B}_{\mathbb{M}^d}[\mathbf{x}, r]$. So, we are left to show that

$$K_H \cup \left((K \cap H^+) \setminus K_H\right) \subseteq \mathbf{B}_{\mathbb{M}^d}[\mathbf{x}, r]. \tag{5}$$

As $\sigma_H \mathbf{x} \in (K^r \cap H^-) \setminus (K^r)_H \subseteq K^r$ we have $K_H \cup ((K \cap H^+) \setminus K_H) \subseteq K \subseteq \mathbf{B}_{\mathbb{M}^d}[\sigma_H \mathbf{x}, r]$. Moreover, as $\sigma_H \mathbf{x} \in H^-$ and $\mathbf{x} \in H^+$, for all $\mathbf{y} \in (K \cap H^+) \setminus K_H \subseteq H^+$ (resp., $\mathbf{y} \in K_H \cap H^+ \subseteq H^+$) we have $\operatorname{dist}_{\mathbb{M}^d}(\mathbf{x}, \mathbf{y}) \leq \operatorname{dist}_{\mathbb{M}^d}(\sigma_H \mathbf{x}, \mathbf{y}) \leq r$ implying that $(K_H \cap H^+) \cup ((K \cap H^+) \setminus K_H) \subseteq \mathbf{B}_{\mathbb{M}^d}[\mathbf{x}, r]$. Finally, for any $\mathbf{y} \in K_H \cap H^-$ we have $\sigma_H \mathbf{y} \in K_H \cap H^+ \subseteq K_H$ with $\operatorname{dist}_{\mathbb{M}^d}(\mathbf{x}, \mathbf{y}) = \operatorname{dist}_{\mathbb{M}^d}(\sigma_H \mathbf{x}, \sigma_H \mathbf{y}) \leq r$ implying that $K_H \cap H^- \subseteq \mathbf{B}_{\mathbb{M}^d}[\mathbf{x}, r]$. This completes the proof of (5). □

Now, we are ready to prove Theorem 1. To avoid any trivial case we may assume that $V_{\mathbb{M}^d}(A^r) > 0$ for $A \subseteq \mathbb{M}^d$ with $a := V_{\mathbb{M}^d}(A) > 0$. In fact, our goal is to maximize the volume $V_{\mathbb{M}^d}(A^r)$ for compact sets $A \subseteq \mathbb{M}^d$ of given volume $V_{\mathbb{M}^d}(A) = a > 0$ and for given $d > 1$ and $r \in \mathbf{R}_+$. As according to Lemma 5 we have $A^r = (\operatorname{conv}_r A)^r$ with $A \subseteq \operatorname{conv}_r A$, it follows from the monotonicity of $V_{\mathbb{M}^d}((\cdot)^r)$ in a straightforward way that for the proof of Theorem 1 it is sufficient to maximize the volume $V_{\mathbb{M}^d}(A^r)$ for r-convex sets $A \subseteq \mathbb{M}^d$ of given volume $V_{\mathbb{M}^d}(A) = a$ with given d and r. Next, consider the extremal family $\mathcal{E}_{a,r,d}$ of r-convex sets $A \subseteq \mathbb{M}^d$ with $V_{\mathbb{M}^d}(A) = a$ and maximal $V_{\mathbb{M}^d}(A^r)$ for given a, d and r. As the Blaschke selection theorem [14] for nonempty, compact, convex subsets of $\mathbb{M}^d$ (using the convenient Hausdorff metric) extends to nonempty, r-convex subsets of $\mathbb{M}^d$ in a rather straightforward way, one obtains by standard arguments that $\mathcal{E}_{a,r,d} \neq \emptyset$.

Lemma 7. *The extremal family $\mathcal{E}_{a,r,d}$ is closed under two-point symmetrization.*

Proof. Let $A \in \mathcal{E}_{a,r,d}$ be an arbitrary extremal set and consider $\tau_H A$ for an arbitrary hyperplane H in $\mathbb{M}^d$. Lemmas 5 and 6 imply that $\tau_H(A^r) \subseteq (\operatorname{conv}_r(\tau_H A))^r = (\tau_H A)^r$ and therefore

$$V_{\mathbb{M}^d}(A^r) = V_{\mathbb{M}^d}(\tau_H(A^r)) \leq V_{\mathbb{M}^d}((\operatorname{conv}_r(\tau_H A))^r) = V_{\mathbb{M}^d}((\tau_H A)^r). \tag{6}$$

Here $\tau_H A \subseteq \operatorname{conv}_r(\tau_H A)$ implying that

$$a = V_{\mathbb{M}^d}(A) = V_{\mathbb{M}^d}(\tau_H A) \leq V_{\mathbb{M}^d}(\operatorname{conv}_r(\tau_H A)). \tag{7}$$

We are left to show that $\tau_H A \in \mathcal{E}_{a,r,d}$. Based on (6) and (7) we need to prove only that $\tau_H A$ is r-convex, i.e., $\tau_H A = \mathrm{conv}_r(\tau_H A)$. We prove this in an indirect way, i.e., assume that $\tau_H A \neq \mathrm{conv}_r(\tau_H A)$. As $\tau_H A \subseteq \mathrm{conv}_r(\tau_H A)$, this means that $\tau_H A \subset \mathrm{conv}_r(\tau_H A)$. Then there exists an r-convex set $A' \subset \mathrm{conv}_r(\tau_H A)$ with $V_{\mathbb{M}^d}(A') = a$. Thus, $(\mathrm{conv}_r(\tau_H A))^r \subset (A')^r$ implying that $V_{\mathbb{M}^d}\left((\mathrm{conv}_r(\tau_H A))^r\right) < V_{\mathbb{M}^d}\left((A')^r\right)$, a contradiction via (6). □

We finish the proof of Theorem 1 by adapting an argument from [10]. Namely, we are going to show that $B \in \mathcal{E}_{a,r,d}$, where $B \subseteq \mathbb{M}^d$ is a ball with $a = V_{\mathbb{M}^d}(A) = V_{\mathbb{M}^d}(B)$. By a standard argument there exists an r-convex set $C \in \mathcal{E}_{a,r,d}$ for which $V_{\mathbb{M}^d}(B \cap C)$ is maximal. Suppose that $B \neq C$. As $a = V_{\mathbb{M}^d}(B) = V_{\mathbb{M}^d}(C)$, there are congruent balls $C_1 \subseteq C \setminus B$ and $C_2 \subseteq B \setminus C$. Let H be the hyperplane in $\mathbb{M}^d$ with an orientation, which determines H^+ and H^- the two closed halfspaces bounded by H in $\mathbb{M}^d, d > 1$ such that $\sigma_H C_1 = C_2$ with $C_1 \subset H^-$. Clearly, $V_{\mathbb{M}^d}(B \cap \tau_H C) > V_{\mathbb{M}^d}(B \cap C)$ moreover, Lemma 7 implies that $\tau_H C \in \mathcal{E}_{a,r,d}$, a contradiction. Thus, $B = C \in \mathcal{E}_{a,r,d}$, finishing the proof of Theorem 1.

3. Proof of Theorem 2

Following [5], our proof is based on estimates of the following functionals.

Definition 5. Let

$$\begin{aligned} f_{\mathbb{M}^d}(N,\lambda,\delta) := \min\{V_{\mathbb{M}^d}(Q^\delta) \mid Q & \\ := \{\mathbf{q}_1,\ldots,\mathbf{q}_N\} \subset \mathbb{M}^d,\ \mathrm{dist}_{\mathbb{M}^d}(\mathbf{q}_i,\mathbf{q}_j) & \\ \leq \lambda \text{ for all } 1 \leq i < j \leq N\} & \end{aligned} \tag{8}$$

and

$$\begin{aligned} g_{\mathbb{M}^d}(N,\lambda,\delta) := \max\{V_{\mathbb{M}^d}(P^\delta) \mid P & \\ := \{\mathbf{p}_1,\ldots,\mathbf{p}_N\} \subset \mathbb{M}^d,\ \lambda & \\ \leq \mathrm{dist}_{\mathbb{M}^d}(\mathbf{p}_i,\mathbf{p}_j) \text{ for all } 1 \leq i < j \leq N\}. & \end{aligned} \tag{9}$$

(We note that in this paper the maximum of the empty set is zero.) We need also

Definition 6. The circumradius $\mathrm{cr}X$ of the set $X \subseteq \mathbb{M}^d, d > 1$ is defined by

$$\mathrm{cr}X := \inf\{r \mid X \subseteq \mathbf{B}_{\mathbb{M}^d}[\mathbf{x}, r]\}.$$

For the proof that follows we need the following straightforward extension of the rather obvious but very useful Euclidean identity (9) of [5]: for any $X = \{\mathbf{x}_1,\ldots,\mathbf{x}_n\} \subset \mathbb{M}^d, n > 1, d > 1, r \in \mathbf{R}_+, r^* \in \mathbf{R}_+$ with $r + r^* \in \mathbf{R}_+$ one has

$$X^r = \left(\bigcup_{i=1}^{n} \mathbf{B}_{\mathbb{M}^d}[\mathbf{x}_i, r^*]\right)^{r+r^*}. \tag{10}$$

3.1. Proof of (i) in Theorem 2

First, we give a lower bound for (8). Jung's theorem [7] implies in a straightforward way that $\mathrm{cr}Q \leq \sqrt{\frac{2d}{d+1}}\frac{\lambda}{2} < \frac{1}{\sqrt{2}}\lambda$ and so, $\mathbf{B}_{\mathbb{E}^d}\left[\mathbf{x}, \delta - \frac{1}{\sqrt{2}}\lambda\right] \subset Q^\delta$ for some $\mathbf{x} \in \mathbb{E}^d$. (We note that by assumption $\delta - \frac{1}{\sqrt{2}}\lambda \geq 0$.) As a result we get that

$$f_{\mathbb{E}^d}(N, \lambda, \delta) > V_{\mathbb{E}^d}\left(\mathbf{B}_{\mathbb{E}^d}\left[\mathbf{x}, \delta - \frac{1}{\sqrt{2}}\lambda\right]\right). \tag{11}$$

Second, we give an upper bound for (9). (10) implies that

$$P^\delta = \left(\bigcup_{i=1}^{N} \mathbf{B}_{\mathbb{E}^d}\left[\mathbf{p}_i, \frac{\lambda}{2}\right]\right)^{\delta+\frac{\lambda}{2}}, \tag{12}$$

where the balls $\mathbf{B}_{\mathbb{E}^d}[\mathbf{p}_1, \frac{\lambda}{2}], \dots, \mathbf{B}_{\mathbb{E}^d}[\mathbf{p}_N, \frac{\lambda}{2}]$ are pairwise non-overlapping in $\mathbb{E}^d$. Thus,

$$V_{\mathbb{E}^d}\left(\bigcup_{i=1}^{N} \mathbf{B}_{\mathbb{E}^d}\left[\mathbf{p}_i, \frac{\lambda}{2}\right]\right) = N V_{\mathbb{E}^d}\left(\mathbf{B}_{\mathbb{E}^d}\left[\mathbf{p}_1, \frac{\lambda}{2}\right]\right). \tag{13}$$

Let $\mu > 0$ be chosen so that $N V_{\mathbb{E}^d}\left(\mathbf{B}_{\mathbb{E}^d}\left[\mathbf{p}_1, \frac{\lambda}{2}\right]\right) = V_{\mathbb{E}^d}\left(\mathbf{B}_{\mathbb{E}^d}[\mathbf{p}_1, \mu]\right)$. Clearly,

$$\mu = \frac{1}{2} N^{\frac{1}{d}} \lambda. \tag{14}$$

Now Theorem 1, (12), (13), and (14) imply in a straightforward way that

$$\begin{aligned} V_{\mathbb{E}^d}\left(P^\delta\right) &= V_{\mathbb{E}^d}\left(\left(\bigcup_{i=1}^{N} \mathbf{B}_{\mathbb{E}^d}\left[\mathbf{p}_i, \frac{\lambda}{2}\right]\right)^{\delta+\frac{\lambda}{2}}\right) \\ &\leq V_{\mathbb{E}^d}\left(\left(\mathbf{B}_{\mathbb{E}^d}\left[\mathbf{p}_1, \frac{1}{2} N^{\frac{1}{d}} \lambda\right]\right)^{\delta+\frac{\lambda}{2}}\right). \end{aligned} \tag{15}$$

Clearly, $\left(\mathbf{B}_{\mathbb{E}^d}\left[\mathbf{p}_1, \frac{1}{2} N^{\frac{1}{d}} \lambda\right]\right)^{\delta+\frac{\lambda}{2}} = \mathbf{B}_{\mathbb{E}^d}\left[\mathbf{p}_1, \delta - \frac{N^{\frac{1}{d}}-1}{2}\lambda\right]$ with the convention that if $\delta - \frac{N^{\frac{1}{d}}-1}{2}\lambda < 0$, then $\mathbf{B}_{\mathbb{E}^d}\left[\mathbf{p}_1, \delta - \frac{N^{\frac{1}{d}}-1}{2}\lambda\right] = \emptyset$. Hence (15) yields

$$g_{\mathbb{E}^d}(N, \lambda, \delta) \leq V_{\mathbb{E}^d}\left(\mathbf{B}_{\mathbb{E}^d}\left[\mathbf{p}_1, \delta - \frac{N^{\frac{1}{d}} - 1}{2}\lambda\right]\right) \tag{16}$$

(with $V_{\mathbb{E}^d}(\emptyset)=0$). Finally, as $N\geq(1+\sqrt{2})^d$ we have $\frac{N^{\frac{1}{d}}-1}{2}\lambda\geq\frac{1}{\sqrt{2}}\lambda$ and so, (11) and (16) yield $g_{\mathbb{E}^d}(N,\lambda,\delta)<f_{\mathbb{E}^d}(N,\lambda,\delta)$, finishing the proof of (i) in Theorem 2.

3.2. Proof of (ii) in Theorem 2

First, we lower bound (8). Let $R:=\mathrm{cr}Q$. Then Jung's theorem [7] yields $\sin R\leq\sqrt{\frac{2d}{d+1}}\sin\frac{\lambda}{2}$. By assumption $0<\lambda<\frac{\pi}{2}$ and so,

$$0\leq\frac{2}{\pi}R<\sin R\leq\sqrt{\frac{2d}{d+1}}\sin\frac{\lambda}{2}<\sqrt{\frac{2d}{d+1}}\frac{\lambda}{2}<\frac{1}{\sqrt{2}}\lambda$$

implying that $0\leq R<\frac{\pi}{2\sqrt{2}}\lambda$. Thus, $\mathbf{B}_{\mathbb{S}^d}\left[\mathbf{x},\delta-\frac{\pi}{2\sqrt{2}}\lambda\right]\subset Q^{\delta}$ for some $\mathbf{x}\in\mathbb{S}^d$. (We note that by assumption $\delta-\frac{\pi}{2\sqrt{2}}\lambda>0$.) As a result we get that

$$f_{\mathbb{S}^d}(N,\lambda,\delta)>V_{\mathbb{S}^d}\left(\mathbf{B}_{\mathbb{S}^d}\left[\mathbf{x},\delta-\frac{\pi}{2\sqrt{2}}\lambda\right]\right). \tag{17}$$

Second, we upper bound (9). By assumption $0<\delta+\frac{\lambda}{2}<\frac{\pi}{2}$ and therefore (10) implies that

$$P^{\delta}=\left(\bigcup_{i=1}^{N}\mathbf{B}_{\mathbb{S}^d}\left[\mathbf{p}_i,\frac{\lambda}{2}\right]\right)^{\delta+\frac{\lambda}{2}}, \tag{18}$$

where the balls $\mathbf{B}_{\mathbb{S}^d}[\mathbf{p}_1,\frac{\lambda}{2}],\ldots,\mathbf{B}_{\mathbb{S}^d}[\mathbf{p}_N,\frac{\lambda}{2}]$ are pairwise non-overlapping in $\mathbb{S}^d$. Thus,

$$V_{\mathbb{S}^d}\left(\bigcup_{i=1}^{N}\mathbf{B}_{\mathbb{S}^d}\left[\mathbf{p}_i,\frac{\lambda}{2}\right]\right)=NV_{\mathbb{S}^d}\left(\mathbf{B}_{\mathbb{S}^d}\left[\mathbf{p}_1,\frac{\lambda}{2}\right]\right). \tag{19}$$

Let $\mu>0$ be chosen so that

$$NV_{\mathbb{S}^d}\left(\mathbf{B}_{\mathbb{S}^d}\left[\mathbf{p}_1,\frac{\lambda}{2}\right]\right)=V_{\mathbb{S}^d}\left(\mathbf{B}_{\mathbb{S}^d}\left[\mathbf{p}_1,\mu\right]\right). \tag{20}$$

Proposition 8. *If* $0<\mu<\frac{\pi}{2}$, *then* $\left(\frac{1}{2ed\pi^{d-1}}\right)^{\frac{1}{d}}N^{\frac{1}{d}}\lambda<\mu$.

Proof. One can rewrite (20) using the integral representation of the volume of balls in $\mathbb{S}^d$ [6] as follows:

$$Nd\omega_d\int_{\frac{\pi}{2}-\frac{\lambda}{2}}^{\frac{\pi}{2}}(\cos t)^{d-1}dt=d\omega_d\int_{\frac{\pi}{2}-\mu}^{\frac{\pi}{2}}(\cos t)^{d-1}dt,$$

where $\omega_d := V_{\mathbb{E}^d}(\mathbf{B}_{\mathbb{E}^d}[\mathbf{x}, 1]), \mathbf{x} \in \mathbb{E}^d$. Then Lemma 4.7 of [4] yields the following chain of inequalities in a rather straightforward way:

$$\begin{aligned}\frac{N}{2ed\pi^{d-1}}\lambda^d &< \frac{N}{ed}\frac{\lambda}{2}\left(\sin\frac{\lambda}{2}\right)^{d-1} \\ &\leq N\int_{\frac{\pi}{2}-\frac{\lambda}{2}}^{\frac{\pi}{2}}(\cos t)^{d-1}dt \\ &= \int_{\frac{\pi}{2}-\mu}^{\frac{\pi}{2}}(\cos t)^{d-1}dt \leq \mu(\sin\mu)^{d-1} \leq \mu^d.\end{aligned}$$

From this the claim follows. □

Now Theorem 1, (18), (19), and (20) imply in a straightforward way that

$$\begin{aligned}V_{\mathbb{S}^d}\left(P^\delta\right) &= V_{\mathbb{S}^d}\left(\left(\bigcup_{i=1}^{N}\mathbf{B}_{\mathbb{S}^d}\left[\mathbf{p}_i, \frac{\lambda}{2}\right]\right)^{\delta+\frac{\lambda}{2}}\right) \\ &\leq V_{\mathbb{S}^d}\left(\left(\mathbf{B}_{\mathbb{S}^d}\left[\mathbf{p}_1, \mu\right]\right)^{\delta+\frac{\lambda}{2}}\right).\end{aligned} \tag{21}$$

Clearly, $\left(\mathbf{B}_{\mathbb{S}^d}\left[\mathbf{p}_1, \mu\right]\right)^{\delta+\frac{\lambda}{2}} = \mathbf{B}_{\mathbb{S}^d}\left[\mathbf{p}_1, \delta+\frac{\lambda}{2}-\mu\right]$ (with the usual convention that if $\delta+\frac{\lambda}{2}-\mu < 0$, then $\mathbf{B}_{\mathbb{S}^d}\left[\mathbf{p}_1, \delta+\frac{\lambda}{2}-\mu\right] = \emptyset$). By assumption $0 < \delta+\frac{\lambda}{2} < \frac{\pi}{2}$ and so, if $\delta+\frac{\lambda}{2}-\mu \geq 0$, then necessarily $0 < \mu < \frac{\pi}{2}$. Thus, Proposition 8 and (21) yield

$$g_{\mathbb{S}^d}(N, \lambda, \delta) \leq V_{\mathbb{S}^d}\left(\mathbf{B}_{\mathbb{S}^d}\left[\mathbf{p}_1, \delta - \left(\left(\frac{1}{2ed\pi^{d-1}}\right)^{\frac{1}{d}}N^{\frac{1}{d}} - \frac{1}{2}\right)\lambda\right]\right) \tag{22}$$

(with $V_{\mathbb{S}^d}(\emptyset) = 0$). As $N \geq 2ed\pi^{d-1}\left(\frac{1}{2}+\frac{\pi}{2\sqrt{2}}\right)^d$ we have $\left(\left(\frac{1}{2ed\pi^{d-1}}\right)^{\frac{1}{d}}N^{\frac{1}{d}} - \frac{1}{2}\right)\lambda \geq \frac{\pi}{2\sqrt{2}}\lambda$ and so, (17) and (22) yield $g_{\mathbb{S}^d}(N, \lambda, \delta) < f_{\mathbb{S}^d}(N, \lambda, \delta)$, finishing the proof of (ii) in Theorem 2.

3.3. Proof of (iii) in Theorem 2

Let us lower bound (8) in a way similar to the previous cases. Let $R := \mathrm{cr}Q$. Then Jung's theorem [7] yields $\sinh R \leq \sqrt{\frac{2d}{d+1}}\sinh\frac{\lambda}{2}$. By assumption we have $0 < \frac{1}{2}\lambda < \frac{\sinh k}{\sqrt{2}k}\lambda \leq \delta < k$ and so,

$$0 \leq R \leq \sinh R \leq \sqrt{\frac{2d}{d+1}}\sinh\frac{\lambda}{2} < \sqrt{2}\frac{\sinh k}{k}\frac{\lambda}{2}, \tag{23}$$

where for the last inequality we have used $0 < x < \sinh x < \frac{\sinh k}{k}x$, which holds for all $0 < x < k$. From (24) it follows that $0 \leq R < \frac{\sinh k}{\sqrt{2}k}\lambda$. Thus,

$\mathbf{B}_{\mathbb{H}^d}\left[\mathbf{x}, \delta - \frac{\sinh k}{\sqrt{2}k}\lambda\right] \subset Q^{\delta}$ for some $\mathbf{x} \in \mathbb{H}^d$. (We note that by assumption $\delta - \frac{\sinh k}{\sqrt{2}k}\lambda \geq 0$.) As a result we get that

$$f_{\mathbb{H}^d}(N, \lambda, \delta) > V_{\mathbb{H}^d}\left(\mathbf{B}_{\mathbb{H}^d}\left[\mathbf{x}, \delta - \frac{\sinh k}{\sqrt{2}k}\lambda\right]\right). \tag{24}$$

Next, we upper bound (9). (10) implies that

$$P^{\delta} = \left(\bigcup_{i=1}^{N} \mathbf{B}_{\mathbb{H}^d}\left[\mathbf{p}_i, \frac{\lambda}{2}\right]\right)^{\delta + \frac{\lambda}{2}}, \tag{25}$$

where the balls $\mathbf{B}_{\mathbb{H}^d}[\mathbf{p}_1, \frac{\lambda}{2}], \ldots, \mathbf{B}_{\mathbb{H}^d}[\mathbf{p}_N, \frac{\lambda}{2}]$ are pairwise non-overlapping in $\mathbb{H}^d$. Thus,

$$V_{\mathbb{H}^d}\left(\bigcup_{i=1}^{N} \mathbf{B}_{\mathbb{H}^d}\left[\mathbf{p}_i, \frac{\lambda}{2}\right]\right) = N V_{\mathbb{H}^d}\left(\mathbf{B}_{\mathbb{H}^d}\left[\mathbf{p}_1, \frac{\lambda}{2}\right]\right). \tag{26}$$

Let $\mu > 0$ be chosen so that

$$N V_{\mathbb{H}^d}\left(\mathbf{B}_{\mathbb{H}^d}\left[\mathbf{p}_1, \frac{\lambda}{2}\right]\right) = V_{\mathbb{H}^d}\left(\mathbf{B}_{\mathbb{H}^d}\left[\mathbf{p}_1, \mu\right]\right). \tag{27}$$

Now Theorem 1, (25), (26), and (27) imply in a straightforward way that

$$\begin{aligned} V_{\mathbb{H}^d}\left(P^{\delta}\right) &= V_{\mathbb{H}^d}\left(\left(\bigcup_{i=1}^{N} \mathbf{B}_{\mathbb{H}^d}\left[\mathbf{p}_i, \frac{\lambda}{2}\right]\right)^{\delta + \frac{\lambda}{2}}\right) \\ &\leq V_{\mathbb{H}^d}\left(\left(\mathbf{B}_{\mathbb{H}^d}\left[\mathbf{p}_1, \mu\right]\right)^{\delta + \frac{\lambda}{2}}\right) \\ &= V_{\mathbb{H}^d}\left(\mathbf{B}_{\mathbb{H}^d}\left[\mathbf{p}_1, \delta + \frac{\lambda}{2} - \mu\right]\right) \end{aligned} \tag{28}$$

with the usual convention that if $\delta + \frac{\lambda}{2} - \mu < 0$, then $\mathbf{B}_{\mathbb{H}^d}\left[\mathbf{p}_1, \delta + \frac{\lambda}{2} - \mu\right] = \emptyset$.

Proposition 9. *If* $0 < \mu \leq \delta + \frac{\lambda}{2}$, *then* $\left(\frac{2k}{\sinh 2k}\right)^{\frac{d-1}{d}} N^{\frac{1}{d}} \frac{\lambda}{2} < \mu$.

Proof. One can rewrite (27) using the integral representation of the volume of balls in $\mathbb{H}^d$ [6] as follows:

$$N d\omega_d \int_0^{\frac{\lambda}{2}} (\sinh t)^{d-1} dt = d\omega_d \int_0^{\mu} (\sinh t)^{d-1} dt.$$

As $0 < \mu \leq \delta + \frac{\lambda}{2}$, by assumption also the inequalities $0 < \mu \leq \delta + \frac{\lambda}{2} < 2\delta < 2k$ hold. Hence the following inequalities follow in a rather straightforward way:

$$\begin{aligned}\frac{N}{d}\left(\frac{\lambda}{2}\right)^d &= N\int_0^{\frac{\lambda}{2}} t^{d-1}dt < N\int_0^{\frac{\lambda}{2}} (\sinh t)^{d-1}dt\\ &= \int_0^{\mu} (\sinh t)^{d-1}dt < \int_0^{\mu}\left(\frac{\sinh 2k}{2k}t\right)^{d-1} dt\\ &= \left(\frac{\sinh 2k}{2k}\right)^{d-1}\frac{\mu^d}{d},\end{aligned}$$

where for the last inequality we have used $0 < x < \sinh x < \frac{\sinh 2k}{2k}x$, which holds for all $0 < x < 2k$. From this the claim follows. □

Thus, Proposition 9 and (28) yield

$$g_{\mathbb{H}^d}(N,\lambda,\delta) \leq V_{\mathbb{H}^d}\left(\mathbf{B}_{\mathbb{H}^d}\left[\mathbf{p}_1, \delta - \left(\left(\frac{2k}{\sinh 2k}\right)^{\frac{d-1}{d}} N^{\frac{1}{d}} - 1\right)\frac{\lambda}{2}\right]\right) \tag{29}$$

(with $V_{\mathbb{H}^d}(\emptyset) = 0$). As $N \geq \left(\frac{\sinh 2k}{2k}\right)^{d-1}\left(\frac{\sqrt{2}\sinh k}{k} + 1\right)^d$ therefore

$$\left(\left(\frac{2k}{\sinh 2k}\right)^{\frac{d-1}{d}} N^{\frac{1}{d}} - 1\right)\frac{\lambda}{2} \geq \frac{\sinh k}{\sqrt{2}k}\lambda$$

and so, (24) and (29) yield $g_{\mathbb{H}^d}(N,\lambda,\delta) < f_{\mathbb{H}^d}(N,\lambda,\delta)$, finishing the proof of (iii) in Theorem 2.

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