Construction of relative discrete series representations for p-adic $\mathbf{GL}_n$

by

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Abstract

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Let F be a nonarchimedean local field of characteristic zero and odd residual characteristic. Let G be the F-points of a connected reductive group defined over F and let θ be an F-rational involution of G. Define H to be the closed subgroup of θ-fixed points in G. The quotient variety $H\backslash G$ is a p-adic symmetric space. A fundamental problem in the harmonic analysis on $H\backslash G$ is to understand the irreducible subrepresentations of the right-regular representation of G acting on the space $L^2(H\backslash G)$ of complex-valued square integrable functions on $H\backslash G$. The irreducible subrepresentations of $L^2(H\backslash G)$ are called relative discrete series representations.

In this thesis, we give an explicit construction of relative discrete series representations for three p-adic symmetric spaces, all of which are quotients of the general linear group. We consider $\mathbf{GL}_n(F) \times \mathbf{GL}_n(F)\backslash\mathbf{GL}_{2n}(F)$, $\mathbf{GL}_n(F)\backslash\mathbf{GL}_n(E)$, and $\mathbf{U}_{E/F}(F)\backslash\mathbf{GL}_{2n}(E)$, where E is a quadratic Galois extension of F and $\mathbf{U}_{E/F}$ is a quasi-split unitary group. All of the representations that we construct are parabolically induced from θ-stable parabolic subgroups admitting a certain type of Levi subgroup. In particular, we give a sufficient condition for the relative discrete series representations that we construct to be non-relatively supercuspidal. Finally, in an appendix, we describe the relative discrete series for the space $\mathbf{GL}_{n-1}(F) \times \mathbf{GL}_1(F)\backslash\mathbf{GL}_n(F)$.

To my teachers

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Introduction

The work in this thesis fits into the general program of harmonic analysis on reductive p-adic symmetric spaces. Let F be a p-adic field with residual characteristic different from two. Let $\mathbf{G}$ be a connected reductive group defined over F and $G = \mathbf{G}(F)$ the group of F-points. Let θ be an F-automorphism of $\mathbf{G}$ of order two, and let $\mathbf{H}$ be the closed subgroup of θ-fixed points in $\mathbf{G}$ with F-points $H = \mathbf{H}(F)$. The quotient F-variety $X = H\backslash G$ is a p-adic symmetric space and, in particular, a homogeneous space for G. One of the main goals of harmonic analysis on X is to understand the space $L^2(X)$ of square integrable complex-valued functions on X. In particular, the group G acts on $L^2(X)$ by right-translation of functions and it is desirable to characterize the irreducible subrepresentations of $L^2(X)$, such representations are said to lie in the *discrete spectrum* of X. We denote the discrete spectrum of X by $L^2_{\mathrm{disc}}(X)$, it is the subspace of $L^2(X)$ generated by all irreducible subrepresentations of $L^2(X)$. An irreducible subrepresentation of $L^2(X)$ is also called a *relative discrete series (RDS)* representation. The main content of this thesis is a construction of representations that occur in the discrete spectrum of certain symmetric quotients of the general linear group.

Many classical representation theoretic results have been generalized to the symmetric space setting (see, for instance, [14, 16, 17, 23, 44, 45, 47, 48, 61]). One can often reconstruct the harmonic analysis on the reductive group G' by studying the so-called *group case*. In the group case, we take $G = G' \times G'$ to be the direct product of G' with itself and define the involution θ by $(g_1, g_2) \mapsto (g_2, g_1)$. In this setting, $H = \Delta G'$ is the diagonal subgroup $\{(g, g) : g \in G'\}$ and $H\backslash G$ is isomorphic to G' as an F-variety. In analogy with the harmonic analysis on groups, one may define *relative* (symmetric space) analogs of supercuspidal and square integrable representations in terms of generalized matrix coefficients. The present work provides an application of a symmetric space version of Casselman's Criterion for square-integrability due to Kato and Takano [45].

A representation (π, V) of G on a complex vector space V is smooth if the stabilizer of any vector is an open subgroup of G. All of the representations considered in this thesis will be assumed to be smooth. The representation (π, V) of G is H-distinguished if there exists a nonzero H-invariant linear form on V. It is exactly the H-distinguished smooth representations of G that contribute to the harmonic analysis on the symmetric space $H\backslash G$.

Sakellaridis and Venkatesh [66] have laid out a general program of harmonic analysis on p-adic spherical varieties, of which p-adic symmetric spaces are a special case. Precisely, a *spherical variety* for G is a normal F-variety $\mathbf{X}$, that is a homogeneous space for $\mathbf{G}$, on which a Borel subgroup of $\mathbf{G}$ acts with a Zariski-dense orbit. In a broad sense, the work [66] lays the formalism and foundations of a *relative Langlands program*. Sakellaridis and Venkatesh strive to provide the correct setting to link the study of global automorphic period integrals and local harmonic analysis. In particular, for certain p-adic spherical varieties X, they give an explicit Plancherel formula describing $L^2(X)$ up to a description of the discrete spectrum [66, Theorem 6.2.1]. In the case that X is strongly factorizable (*cf.* [66]), which includes all symmetric varieties, the discrete spectrum is described in terms of toric families of relative discrete series representations [66, Conjecture 9.4.6]. The decomposition of the norm on $L^2_{disc}(X)$ provided by this result is not necessarily a direct integral, the images of certain intertwining operators packaged with the toric families of RDS may be non-orthogonal. On the other hand, Sakellaridis and Venkatesh believe that their conjectures on $\mathbf{X}$-distinguished Arthur parameters may give a canonical choice of mutually orthogonal toric families of RDS that span $L^2_{disc}(X)$. However, Sakellaridis and Venkatesh do *not* give an explicit description of the relative discrete series representations required to build such algebraic families. For this reason, it is desirable to explicitly construct representations that occur in the discrete spectrum of p-adic spherical varieties. The present author takes p-adic symmetric spaces as a natural starting point. By the results of Kato and Takano [45], an H-distinguished discrete series representation of G will lie in the discrete spectrum of $H\backslash G$. We are thus interested in constructing H-distinguished representations of G that are relatively discrete but do not lie in the discrete spectrum of G. Recently, for Galois symmetric spaces, Zhang [77] has verified refined conjectures of Prasad, related to the conjectures of Sakellaridis and Venkatesh, for depth-zero supercuspidal L-packets. As part of future work, the author hopes to address the truth of the conjectures [66, Conjecture 1.3.1] and [66, Conjecture 16.2.2] for the representations constructed in Theorems 4.2.1 and 5.2.22. It is expected that [66, Conjecture 16.2.2] is related to our construction of RDS via parabolic induction from representations of θ-elliptic Levi subgroups.

The aim of the present author's doctoral research has been to construct, at least in part, the discrete spectrum of some specific p-adic symmetric spaces. In fact, we give a construction of representations in the discrete spectrum of certain quotients of p-adic general linear groups. In Chapter 4, we consider $\mathbf{GL}_n(F) \times \mathbf{GL}_n(F)\backslash \mathbf{GL}_{2n}(F)$ and $\mathbf{GL}_n(F)\backslash \mathbf{GL}_n(E)$, where E/F is a quadratic Galois extension. While in Chapter 5, we study the quotient of $\mathbf{GL}_{2n}(E)$ by a quasi-split unitary group $\mathbf{U}_{E/F}(F)$, associated to the extension E/F. The representations that we obtain are parabolically induced from distinguished discrete series representations of certain Levi subgroups of G, and in particular, do not occur in $L^2_{disc}(G)$. Our results parallel a forthcoming construction of Murnaghan [57] (in a more general setting) of non-supercuspidal relatively supercuspidal representations of G. Murnaghan's result involves the exploitation of certain symmetry properties, relative to the involution, of the K-types appearing in the representations induced from supercuspidals which allows her to prove that certain Jacquet modules are not distinguished. In contrast, in our construction, we induce from (non-supercuspidal) discrete series representations and we must conduct a detailed study of distinction of Jacquet modules. The author expects that the construction of non-discrete relative discrete series representations outlined for the cases considered here will generalize to many other symmetric spaces. At present, no such generalization has been obtained; however, it is work-in-progress to extend our results to any symmetric quotient of $\mathbf{GL}_n$.

After giving a few basic definitions, we provide a statement of the main results of this thesis. A Levi subgroup L of G is θ-elliptic if it is not contained in any proper θ-split parabolic subgroup, where a parabolic subgroup P is θ-split if $\theta(P)$ is the parabolic subgroup opposite to P. An element $g \in G$ is said to be θ-split if $\theta(g) = g^{-1}$ and a θ-stable subset Y of G is θ-split if every element $y \in Y$ is θ-split. An F-torus S is (θ, F)-split if it is both F-split and θ-split. A representation τ of a Levi subgroup L of G is regular if for every non-trivial element $w \in N_G(L)/L$ we have that the twist ${}^w\tau = \tau(w^{-1}(\cdot)w)$ is not equivalent to τ.

Let A_0 be a θ-stable maximal F-split torus of G containing a fixed maximal (θ, F)-split torus S_0. Let $L_0 = C_G((A_0^\theta)^\circ)$ be a minimal θ-elliptic Levi subgroup of G containing A_0. A Levi subgroup L is a semi-standard-θ-elliptic Levi if L contains L_0. We make a particular choice Δ^{ell} of simple roots for the root system $\Phi(G, A_0)$. The F-split component of the centre of L_0 is determined by a proper nonempty subset $\Delta^{ell}_{\min}$ of Δ^{ell}. A Levi subgroup L is a standard-θ-elliptic Levi subgroup if L is standard with respect to Δ^{ell} and contains L_0. The main result of Chapter 4 is the following.

Theorem (Theorem 4.2.1)**.** *Let $n \geq 2$ (respectively, $n \geq 4$) be an integer. Let G be equal to $\mathbf{GL}_{2n}(F)$ (respectively, $\mathbf{GL}_n(E)$) and let H be equal to $\mathbf{GL}_n(F) \times \mathbf{GL}_n(F)$ (respectively, $\mathbf{GL}_n(F)$). Let $\Omega^{ell} \subset \Delta^{ell}$ be a proper subset such that Ω^{ell} contains $\Delta^{ell}_{\min}$. The proper Δ^{ell}-standard parabolic subgroup $Q = Q_{\Omega^{ell}}$ is θ-stable with standard-θ-elliptic Levi subgroup $L = L_{\Omega^{ell}}$. Let τ be a regular L^θ-distinguished discrete series representation of L. The parabolically induced representation $\pi = \iota_Q^G \tau$ is an H-distinguished relative discrete series representation of G. Moreover, π is in the complement of the discrete series of G.*

If τ is not regular, then we cannot currently determine if π is (or is not) relatively discrete due to a lack of understanding of the invariant forms of Lagier [48] and Kato–Takano [44] on the Jacquet modules of π along proper θ-split parabolic subgroups. In this case, it is still possible that (a subquotient of) π is relatively discrete. We do not prove that Theorem 4.2.1 provides an exhaustion of the non-discrete RDS.

We are able to prove a result, analogous to Theorem 4.2.1, for the symmetric space $\mathbf{U}_{E/F}(F)\backslash\mathbf{GL}_{2n}(E)$, where $\mathbf{U}_{E/F}$ is a quasi-split unitary group associated to E/F. In this case, we only produce representations induced from the standard upper-triangular parabolic subgroup $P_{(n,n)}$. Let σ denote the non-trivial element of the Galois group of E over F. Note that in Theorem 5.2.22, L is, in fact, θ-elliptic and τ is regular. The main theorem of Chapter 5 is the following.

Theorem (Theorem 5.2.22)**.** *Let $n \geq 2$ be an integer. Let $Q = P_{(n,n)}$ be the upper-triangular parabolic subgroup of $\mathbf{GL}_{2n}(E)$, with standard Levi factorization $L = M_{(n,n)}$, $U = N_{(n,n)}$. Let $\pi = \iota_Q^G \tau$, where $\tau = \tau' \otimes {}^\sigma\tau'$, and τ' is a discrete series representation of $\mathbf{GL}_n(E)$ such that τ' is not Galois invariant, i.e., $\tau' \ncong {}^\sigma\tau'$. The representation π is a relative discrete series representation for $\mathbf{U}_{E/F}(F)\backslash\mathbf{GL}_{2n}(E)$ that does not occur in the discrete series of $\mathbf{GL}_{2n}(E)$.*

We now give an outline of the contents of this thesis. In Chapter 1, we review the structure of reductive p-adic groups and symmetric spaces. In Chapter 2, the relevant representation theoretic background is taken up; in particular, we give a precise formulation of the Relative Casselman's Criterion 2.2.18, as established by Kato and Takano [45]. Also in Chapter 2, we recall results on distinguished discrete series representations relevant to the construction given in Theorem 4.2.1. Chapter 3 consists of general results on distinction of induced representations, preliminaries on exponents and some technical results that will be useful in the sequel. Chapter 4 is the technical heart of the thesis. In this chapter, we study the two symmetric pairs $\mathbf{GL}_n(F) \times \mathbf{GL}_n(F)\backslash\mathbf{GL}_{2n}(F)$ and $\mathbf{GL}_n(F)\backslash\mathbf{GL}_n(E)$, where E/F is a quadratic Galois extension. In Chapter 5, we extend the construction given in Theorem 4.2.1 to the symmetric pair, $\mathbf{U}_{E/F}(F)\backslash\mathbf{GL}_{2n}(E)$, where $\mathbf{U}_{E/F}$ is a quasi-split unitary group. In both Chapters 4 and 5, we note existence results (Proposition 4.2.8, respectively 5.2.23) which, with mild conditions on n, guarantee that our construction provides infinitely many equivalence classes of RDS. In Chapter 6, we discuss an approach to determining when an H-distinguished representation is not relatively supercuspidal. In Appendix A, by adapting work of Venketasubramanian [74], we give a description of (nearly all) the representations in the discrete spectrum of $\mathbf{GL}_{n-1}(F) \times \mathbf{GL}_1(F)\backslash\mathbf{GL}_n(F)$.

Basic notation

We end the introduction by setting some basic notation. For a field F, let $\mathbf{M}_n(F)$ denote the set of all n by n matrices with entries in F. We use $\mathrm{diag}(a_1, a_2, \ldots, a_n)$ to denote an $n \times n$ diagonal matrix with entries $a_1, \ldots, a_n$. Let $\mathbf{GL}_n$ denote the general linear group of n by n invertible matrices. As is customary, we denote the block-upper triangular parabolic subgroup of $\mathbf{GL}_n$, corresponding to a partition $(\underline{m}) = (m_1, \ldots, m_k)$ of n, by $\mathbf{P}_{(\underline{m})}$, with block-diagonal Levi subgroup $\mathbf{M}_{(\underline{m})} \cong \prod_{i=1}^{k} \mathbf{GL}_{m_i}$ and unipotent radical $\mathbf{N}_{(\underline{m})}$.

Given a finite extension K of F and an algebraic variety $\mathbf{Y}$ defined over K, we denote by $R_{K/F}\mathbf{Y}$ the F-variety obtained by Weil's restriction of scalars [72]. The important fact for us is that the set of F-points of $R_{K/F}\mathbf{Y}$ may be identified with the K-points of $\mathbf{Y}$.

Given a real number r we let $\lfloor r \rfloor$ denote the greatest integer that is less than or equal to r. We use $\widehat{(\cdot)}$ to denote that a symbol is omitted. For instance, $\mathrm{diag}(\widehat{a_1}, a_2, \ldots, a_n)$ may be used to denote the diagonal matrix $\mathrm{diag}(a_2, \ldots, a_n)$. Most commonly, $\widehat{(\cdot)}$ will be used to denote that a particular entry of an $n \times n$ diagonal matrix is omitted, depending on if n even or odd.

We use e to denote the identity element of a group. Given a group G and $g \in G$, we write ${}^g x = gxg^{-1}$ for any $x \in G$. More generally, for any subset Y of G, we'll write ${}^g Y = \{{}^g y : y \in Y\}$. Let $C_G(Y)$ denote the centralizer of Y in G and let $N_G(Y)$ be the normalizer of Y in G.

Chapter 1

Structure of reductive p-adic groups and symmetric spaces

In this chapter, we set out the notation and terminology for our study of reductive p-adic groups and the associated symmetric spaces. Of particular note, we set conventions for determining a standard choice of Levi decomposition for parabolic subgroups (see §1.4). We also recall various structural results on p-adic symmetric spaces following [31, 30]. In particular, we are concerned with tori, root systems and parabolic subgroups *relative* to involutions. The authoritative work on reductive groups over local fields remains [10]; in addition, we find [72] and Springer's article [71] in [9] very useful.

1.1 Local fields

By a *p-adic field*, we mean a nonarchimedean local field of characteristic zero. In particular, a p-adic field is a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers, for some prime p [68]. Let F be a p-adic field with odd residual characteristic and let $\mathcal{O}_F$ be the ring of integers of F with prime ideal $\mathfrak{p}_F$. Fix a uniformizer ϖ_F of F and note that $\mathfrak{p}_F = \varpi_F \mathcal{O}_F$. Let q be the cardinality of the residue field $k_F = \mathcal{O}_F / \mathfrak{p}_F$ of F. Let $|\cdot|_F$ denote the normalized absolute value on F such that $|\varpi_F|_F = q^{-1}$. Recall that the ring of integers is equal to $\mathcal{O}_F = \{a \in F : |a|_F \leq 1\}$, the prime ideal is equal to $\mathfrak{p}_F = \{a \in F : |a|_F < 1\}$, and the units in $\mathcal{O}_F$ are equal to the set of elements $\mathcal{O}_F^\times = \mathcal{O}_F \setminus \mathfrak{p}_F = \{u \in F : |u|_F = 1\}$ with absolute value 1.

Let E be a quadratic Galois extension of F. We obtain $\mathcal{O}_E, \mathfrak{p}_E, \varpi_E, k_E$, and $|\cdot|_E$, as above, and we normalize the absolute value on E such that $|\cdot|_E = |N_{E/F}(\cdot)|_F^{1/2}$, where $N_{E/F} : E \to F$ is the norm map. Fix a generator ε of the extension E/F such that $E = F(\varepsilon)$. Let $\mathrm{Gal}(E/F)$ be the Galois group of E over F. Let $\sigma \in \mathrm{Gal}(E/F)$ be a generator. Since the extension E/F is quadratic, $\sigma^2 = 1$ and we have that

$$\sigma(a + b\varepsilon) = a - b\varepsilon,$$

for any $a, b \in F$. The norm map is given explicitly by

$$N_{E/F}(a + b\varepsilon) = (a + b\varepsilon)\sigma(a + b\varepsilon) = a^2 - b^2\varepsilon^2,$$

where $a, b \in F$.

1.2 Reductive groups and involutions

Let $\mathbf{G}$ be a connected reductive group defined over F and let $G = \mathbf{G}(F)$ denote the group of F-points. The group G inherits a topology from the field F such that G is locally compact totally disconnected and Hausdorff; in particular, G is locally profinite. We let $Z_G = \mathbf{Z_G}(F)$ denote the centre of G while $A_G = \mathbf{A_G}(F)$ denotes the F-split component of the centre of G, that is, the largest F-split torus contained in Z_G (*cf.* §1.3). As is the custom, we will often abuse notation and identify an algebraic group defined over F with its group of F-points. When the distinction is to be made, we will use boldface to denote the algebraic group and regular typeface to denote the group of F-points.

Let θ be an F-involution of $\mathbf{G}$, that is, an order-two automorphism of $\mathbf{G}$ defined over F. Define $\mathbf{H}$ to be the closed subgroup of θ-fixed points $\mathbf{G}^\theta$ of $\mathbf{G}$. The identity component $\mathbf{H}^\circ$ of the subgroup $\mathbf{H}$ is also a connected reductive group. The quotient variety $H\backslash G$ is a *p-adic symmetric space*.

Definition 1.2.1. We say that an involution θ_1 of G is G-conjugate (or G-equivalent) to another involution θ_2 if there exists $g \in G$ such that $\theta_1 = \operatorname{Int} g^{-1} \circ \theta_2 \circ \operatorname{Int} g$, where $\operatorname{Int} g$ denotes the inner F-automorphism of $\mathbf{G}$ given by $\operatorname{Int} g(x) = gxg^{-1}$, for all $x \in \mathbf{G}$.

We write $g \cdot \theta$ to denote the involution $\operatorname{Int} g^{-1} \circ \theta \circ \operatorname{Int} g$.

1.3 Tori, root systems and Weyl groups

The material that we recall in this section, and the next, is standard and we refer to [36, 71, 72] for the details. An *F-torus* is a connected diagonalizable algebraic group defined over F. An F-torus $\mathbf{A}$ is *F-split* if $\mathbf{A}$ is F-isomorphic to a product of a finite number of copies of the multiplicative group $\mathbb{G}_m$. For any F-torus $\mathbf{A}$, denote by $X^*(\mathbf{A})$ the free abelian group of F-rational characters of A. Let $X_*(\mathbf{A})$ denote the F-rational cocharacters of $\mathbf{A}$. Precisely, $X^*(\mathbf{A}) = \operatorname{Hom}_F(\mathbf{A}, \mathbb{G}_m)$ and $X_*(\mathbf{A}) = \operatorname{Hom}_F(\mathbb{G}_m, \mathbf{A})$. In subsequent chapters, we will often abuse notation and write $X^*(A)$, respectively $X_*(A)$.

Let $\mathbf{A}$ be a maximal F-split torus contained in $\mathbf{G}$, with F-points $A = \mathbf{A}(F)$. By considering the adjoint action of $\mathbf{A}$ on the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$, we obtain a direct sum decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha,$$

where $\mathfrak{t}$ is the Lie algebra of the centralizer $\mathbf{T} = C_{\mathbf{G}}(\mathbf{A})$ of $\mathbf{A}$ in $\mathbf{G}$, $\alpha \in X^*(\mathbf{A})$ are F-rational characters of $\mathbf{A}$, and $\mathfrak{g}_\alpha$ is the $\operatorname{Ad}(\mathbf{A})$-eigenspace for the non-trivial eigencharacter α. The finite subset of $X^*(\mathbf{A})$ consisting of all α such that $\mathfrak{g}_\alpha \neq \{0\}$ is denoted $\Phi(\mathbf{G}, \mathbf{A})$. If $\alpha \in \Phi(G, A)$, we call α a *root* of A in G. One refers to $\Phi(\mathbf{G}, \mathbf{A})$ as the *root system* of $\mathbf{G}$ (with respect to $\mathbf{A}$).

Remark 1.3.1. By abuse of notation, we'll write $\Phi(G, A)$ to denote $\Phi(\mathbf{G}, \mathbf{A})$ and refer to $\Phi(G, A)$ as the root system of G (with respect to A).

In this subsection, we'll write $\Phi = \Phi(G, A)$.

Remark 1.3.2. In the literature, since F is not algebraically closed, Φ is often called the *relative root system* [71]; however, we avoid this terminology to prevent confusion with our use of the word "relative" in the symmetric space setting.

Definition 1.3.3. A subset Δ of Φ is called a *base* if

1. Δ is a basis of the $\mathbb{Q}$-vector space spanned by Φ in $X^*(\mathbf{A}) \otimes_{\mathbb{Z}} \mathbb{Q}$, and

2. each root $\beta \in \Phi$ can be written as an integral linear combination $\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$, $c_\alpha \in \mathbb{Z}$, of the roots in Δ such that the coefficients c_α are all nonnegative or all nonpositive.

Let Δ be a base for Φ. We also refer to Δ as a choice of *simple roots* in Φ and $\alpha \in \Delta$ is called a *simple root*. Note that the representation of $\beta \in \Phi$ as an integral linear combination of the simple roots Δ is unique. In addition, the simple roots Δ determine a choice Φ^+ of *positive roots* in Φ. The positive roots Φ^+ are exactly those roots that can be written as nonnegative linear combinations of the simple roots Δ, i.e.,

$$\Phi^+ = \left(\operatorname{span}_{\mathbb{Z}_{\geq 0}} \Delta\right) \cap \Phi.$$

The *negative roots* Φ^- are exactly those roots that can be written as nonpositive linear combinations of the simple roots Δ; in particular $\Phi^- = -\Phi^+$.

The *Weyl group* of $\mathbf{G}$ (with respect to $\mathbf{A}$) is the finite reflection group $W = N_{\mathbf{G}}(\mathbf{A})/C_{\mathbf{G}}(\mathbf{A})$. The Weyl group acts on $\mathbf{A}$ by conjugation. Indeed, given $w \in W$ there exists a representative $n \in N_{\mathbf{G}}(\mathbf{A})(F)$ of w [71, Section 3.5], and we define ${}^w a = nan^{-1}$, for any $a \in \mathbf{A}$. The element ${}^w a$ of $\mathbf{A}$ is well-defined (independent of the choice of representative for w). Given $a \in \mathbf{A}$ and $w \in W$ we'll write ${}^w a = waw^{-1}$ without mention of the choice of representative of w in $N_{\mathbf{G}}(\mathbf{A})(F)$. Since W acts on $\mathbf{A}$, there is an action of W on $X^*(\mathbf{A})$ given by

$$(w\chi)(a) = \chi(w^{-1}aw), \qquad (a \in \mathbf{A})$$

for any $\chi \in X^*(\mathbf{A})$ and $w \in W$. The root system Φ of G with respect to A is stable under the action of W on $X^*(\mathbf{A})$. The Weyl group acts transitively on the set of bases for Φ; in particular, for any $w \in W$ the set

$$w\Delta = \{w\alpha : \alpha \in \Delta\}$$

is a base for Φ. The base $w\Delta$ determines the set of positive roots $\{w\beta : \beta \in \Phi^+\}$, where Φ^+ is the set of positive roots with respect to Δ.

1.4 Standard parabolic subgroups and Levi factors

Again, in this section, we refer to [36, 71, 72] for details. Recall that a *parabolic subgroup* $\mathbf{P}$ of $\mathbf{G}$ is a (Zariski) closed subgroup of $\mathbf{G}$ such that the quotient variety $\mathbf{G}/\mathbf{P}$ is projective. A parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ is an *F-parabolic subgroup* if $\mathbf{P}$ is defined over F.

Let $\mathbf{P}$ be an F-parabolic subgroup of $\mathbf{G}$. The unipotent radical $\mathbf{N}$ of $\mathbf{P}$ is the unique maximal (Zariski) closed Zariski-connected unipotent subgroup of $\mathbf{P}$; moreover, $\mathbf{N}$ is defined over F. The unipotent radical $\mathbf{N}$ is normal in $\mathbf{P}$ and the quotient $\mathbf{P}/\mathbf{N}$ is reductive. A closed subgroup $\mathbf{M}$ of $\mathbf{P}$ such that the product map $\mathbf{M} \times \mathbf{N} \to \mathbf{P}$ is bijective is called a *Levi subgroup*. A Levi subgroup $\mathbf{M}$ of $\mathbf{P}$ (defined over F) exists and $\mathbf{P}$ is the semi-direct product $\mathbf{P} = \mathbf{M} \ltimes \mathbf{N}$; in particular $\mathbf{M} \cong \mathbf{P}/\mathbf{N}$ is reductive. A realization of $\mathbf{P}$ as the product $\mathbf{P} = \mathbf{M}\mathbf{N}$ is often referred to as a *Levi decomposition* or *Levi factorization* of $\mathbf{P}$. Moreover, at the level of F-points, we have that $P = \mathbf{P}(F)$ is equal to the semi-direct product of the F-points of the Levi $M = \mathbf{M}(F)$ and unipotent radical $N = \mathbf{N}(F)$, i.e., $P = M \ltimes N$. Given a Levi subgroup $\mathbf{M}$ of $\mathbf{P}$ (defined over F), any other Levi subgroup is of the form $n\mathbf{M}n^{-1}$ for a unique element $n \in N$; in particular, the Levi subgroups of $\mathbf{P}$ are conjugate over F. We also refer to a Levi subgroup (defined over F) of an F-parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ as a Levi subgroup of $\mathbf{G}$.

Let $A = \mathbf{A}(F)$ be (the F-points of) a maximal F-split torus $\mathbf{A}$ of $\mathbf{G}$ and let Δ be a base for the root system $\Phi = \Phi(G, A)$. The positive roots Φ^+ are uniquely determined by Δ; moreover, Φ^+ uniquely determines a minimal F-parabolic subgroup $\mathbf{P}_0$ of $\mathbf{G}$. The *standard Levi* subgroup $\mathbf{M}_0 = C_{\mathbf{G}}(\mathbf{A})$ of $\mathbf{P}_0$ is the centralizer in $\mathbf{G}$ of the torus $\mathbf{A}$. The torus $\mathbf{A}$ is a maximal F-split torus of $\mathbf{M}_0$ and $\mathbf{P}_0$. For any root $\alpha \in \Phi$, there is a unique unipotent F-subgroup $\mathbf{N}_\alpha$ of $\mathbf{G}$ normalized by $\mathbf{A}$, such that the Lie algebra of $\mathbf{N}_\alpha$ is equal to $\mathfrak{g}_\alpha$. We will refer the subgroups $\mathbf{N}_\alpha$, where $\alpha \in \Phi$, as *root groups*. Note that the dimension of $\mathfrak{g}_\alpha$ over F (respectively, $\mathbf{N}_\alpha$) need not be equal to one. The unipotent radical $\mathbf{N}_0$ of $\mathbf{P}_0$ is generated by the root groups $\mathbf{N}_\alpha$ corresponding to the positive roots $\alpha \in \Phi^+$. The *opposite parabolic* subgroup $\mathbf{P}_0^{op}$ to $\mathbf{P}_0$ (with respect to $\mathbf{A}$) is obtained by interchanging the sets of positive and negative roots in Φ. In particular, the unipotent radical $\mathbf{N}_0^{op}$ of $\mathbf{P}_0^{op}$ is generated by the root groups $\mathbf{N}_\beta$, where $\beta \in \Phi^-$. We have that $\mathbf{M}_0 = \mathbf{P}_0 \cap \mathbf{P}_0^{op}$ is a Levi subgroup of both $\mathbf{P}_0$ and $\mathbf{P}_0^{op}$.

The F-parabolic subgroups of $\mathbf{G}$ containing $\mathbf{P}_0$ are referred to as Δ-*standard parabolic* subgroups. Often the Δ is omitted from this terminology when understood by context. Given a subset Θ of Δ, one may canonically associate a Δ-standard parabolic subgroup $\mathbf{P}_\Theta$ of $\mathbf{G}$ and a standard choice of Levi subgroup. First, let Φ_Θ be the subsystem of Φ generated by the simple roots Θ, that is,

$$\Phi_\Theta = (\operatorname{span}_{\mathbb{Z}} \Theta) \cap \Phi$$

and note that Θ is a base for Φ_Θ. Let Φ_Θ^+ be the set of positive roots determined by the simple roots Θ in Φ_Θ. Of course, Φ_Θ^+ is contained in Φ^+. The unipotent radical $\mathbf{N}_\Theta$ of $\mathbf{P}_\Theta$ is generated by the root groups $\mathbf{N}_\alpha$, where $\alpha \in \Phi^+ \setminus \Phi_\Theta^+$. The parabolic subgroup $\mathbf{P}_\Theta$ admits a Levi factorization $\mathbf{P}_\Theta = \mathbf{M}_\Theta \mathbf{N}_\Theta$, where $\mathbf{M}_\Theta$ is the centralizer in $\mathbf{G}$ of the F-split torus

$$\mathbf{A}_\Theta = \left(\bigcap_{\alpha \in \Theta} \ker \alpha \right)^\circ .$$

Here $(\cdot)^\circ$ indicates the Zariski-connected component of the identity. In fact, $\mathbf{A}_\Theta$ is the F-split component of the centre of $\mathbf{M}_\Theta$ and Φ_Θ is the root system $\Phi(M_\Theta, A)$ of $A = \mathbf{A}(F)$ in $M_\Theta = \mathbf{M}_\Theta(F)$. The parabolic $\mathbf{P}_\Theta^{op}$ opposite to $\mathbf{P}_\Theta$ (with respect to $\mathbf{A}_\Theta$) has unipotent radical $\mathbf{N}_\Theta^{op}$, generated by $\mathbf{N}_\beta$, where $\beta \in \Phi^- \setminus \Phi_\Theta^-$, and standard Levi subgroup $\mathbf{M}_\Theta$.

Any F-parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ is G-conjugate to a unique Δ-standard parabolic subgroup $\mathbf{P}_\Theta$, for some $\Theta \subset \Delta$. In particular, if $\mathbf{P} = g\mathbf{P}_\Theta g^{-1}$, then the opposite parabolic to $\mathbf{P}$ (with respect to ${}^g\mathbf{A}_\Theta$) is equal to $\mathbf{P}^{op} = g\mathbf{P}_\Theta^{op} g^{-1}$. Observe that any Levi subgroup of $\mathbf{G}$ is the centralizer in $\mathbf{G}$ of an F-split torus. Conversely, the centralizer in $\mathbf{G}$ of any F-split torus occurs as the Levi factor of some F-parabolic subgroup [71].

Remark 1.4.1. In subsequent chapters, we will refer to the F-points of an F-parabolic subgroup of $\mathbf{G}$ simply as a parabolic subgroup of G. By a Levi subgroup M (respectively, the unipotent radical N) of $P = \mathbf{P}(F)$, we mean the F-points of a Levi F-subgroup $\mathbf{M}$ (respectively, the F-points of the unipotent radical $\mathbf{N}$) of $\mathbf{P}$. When considering standard parabolic subgroups P_Θ, associated to $\Theta \subset \Delta$, we will always work with the Levi factorization $P_\Theta = M_\Theta N_\Theta$, where $M_\Theta = C_{\mathbf{G}}(\mathbf{A}_\Theta)(F)$ is (the group of F-points of) the *standard Levi subgroup* of P_Θ and N_Θ is (the group of F-points of) the unipotent radical of P_Θ.

1.4.1 Measures and modular functions

The group G of F-points of a connected reductive F-group $\mathbf{G}$ is unimodular (we give a sketch of the proof below, *cf.* Proposition 1.4.2). Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ and let $\mathbf{N}$ be the unipotent radical of $\mathbf{P}$. Fix a Levi decomposition $\mathbf{P} = \mathbf{MN}$, where $\mathbf{M} \cong \mathbf{P}/\mathbf{N}$ is a Levi subgroup of $\mathbf{P}$. Let μ_P be a left Haar measure on $P = \mathbf{P}(F)$. By the uniqueness of the Haar measure, for any continuous compactly supported function f on P and any $q \in P$ we have

$$\int_P f(pq)\ d\mu_P(p) = \Delta_P(q) \int_P f(p)\ d\mu_P(p),$$

for some positive real number $\Delta_P(q)$. The *modular function* $\Delta_P : P \to \mathbb{R}_{>0}$ is a continuous homomorphism. Define the *modular character* δ_P of P by

$$\delta_P(p) = \Delta_P(p)^{-1}, \tag{1.1}$$

for all $p \in P$. Since $N = \mathbf{N}(F)$ is unipotent, N is equal to the union of its compact open subgroups. It follows that any continuous $\mathbb{R}_{>0}$-valued character is trivial on N; in particular, δ_P is trivial on N. Let $\mathfrak{n}$ be the Lie algebra of $\mathbf{N}$. Then $\delta_P(p) = |\det \mathrm{Ad}_{\mathfrak{n}}(p)|$, for all $p \in P$, where $\mathrm{Ad}_{\mathfrak{n}}$ denotes the adjoint action of $\mathbf{P}$ on $\mathfrak{n}$ [15].

Proposition 1.4.2. *The group $G = \mathbf{G}(F)$ of F-points of a connected reductive F-group $\mathbf{G}$ is unimodular.*

Proof. It is enough to prove that $\delta_G : G \to \mathbb{R}_{>0}$ is the trivial character. The radical of $\mathbf{G}$ is the maximal central F-torus $(\mathbf{Z_G})^\circ$ in $\mathbf{G}$ [72, Proposition 7.3.1]. In particular, the group of F-points of the radical of $\mathbf{G}$ is contained in the kernel of δ_G. Thus it suffices to show that δ_G is trivial on the F-points of the (connected) semisimple quotient $\mathbf{G}/(\mathbf{Z_G})^\circ$. By [58, Proposition 3.17], when $\mathbf{G}$ is semisimple, $\mathbf{G}(F)$ has no infinite abelian quotient. In particular, the image of $\mathbf{G}/(\mathbf{Z_G})^\circ(F)$ under the continuous character δ_G is a finite connected subgroup of $\mathbb{R}_{>0}$ and is thus equal to $\{1\}$. ☐

For a proof of Proposition 1.4.2 in the case that $\mathbf{G}$ is semisimple, we refer the reader to [58, Corollary to Theorem 3.18].

1.5 Tori and root systems associated to involutions

In this section, we follow [44, 45] and recall the results of [31, 30]. First, we recall that an element $g \in G$ is said to be θ-split if $\theta(g) = g^{-1}$. A subgroup G' of G is θ-split if every element of G' is θ-split.

Definition 1.5.1. An F-torus $\mathbf{S}$ contained in $\mathbf{G}$ is (θ, F)*-split* if $\mathbf{S}$ is both F-split and θ-split.

Let $\mathbf{S}_0$ be a maximal (θ, F)-split torus of G. By [31, Lemma 4.5(iii)], there exists a θ-stable maximal F-split torus $\mathbf{A}_0$ of G that contains $\mathbf{S}_0$. Let $\Phi_0 = \Phi(G, A_0)$ be the root system of G with respect to A_0. Recall that we abuse notation and write $\Phi(G, A_0)$ for $\Phi(\mathbf{G}, \mathbf{A}_0)$ (*cf.* Remark 1.3.1). Since $\mathbf{A}_0$ is θ-stable, there is an action of θ on the F-rational characters $X^*(\mathbf{A}_0)$ of $\mathbf{A}_0$. Explicitly, given $\chi \in X^*(\mathbf{A}_0)$, we have

$$(\theta\chi)(a) = \chi(\theta(a)),$$

for all $a \in \mathbf{A}_0$. Moreover, Φ_0 is stable under the action of θ on $X^*(\mathbf{A}_0)$. Let Φ_0^θ denote the subset of θ-fixed roots in Φ_0.

Definition 1.5.2. A base Δ_0 of Φ_0 is called a *θ-base* if for every positive root $\alpha \in \Phi_0^+$ with respect to Δ_0 that is not fixed by θ, we have that $\theta(\alpha) \in \Phi_0^-$.

As noted in [30, §1], and shown in [32], a θ-base of Φ_0 exists. As above (*cf.* §1.3), the choice of base Δ_0 for Φ_0 determines a set of positive roots Φ_0^+ in Φ_0. The negative roots in Φ_0, relative to this choice of Δ_0, are given by $\Phi_0^- = -\Phi_0^+$. Let Δ_0 be a θ-base for Φ_0 and denote the θ-fixed simple roots by Δ_0^θ.

Let $\mathbf{M}$ be any Levi subgroup of $\mathbf{G}$ (defined over F). Let $\mathbf{A_M}$ denote the F-split component of the centre of $\mathbf{M}$. The torus $\mathbf{A_M}$ is the largest F-split torus contained in the centre $\mathbf{Z_M}$ of $\mathbf{M}$.

Definition 1.5.3. The (θ, F)*-split component* of $\mathbf{M}$ is the largest (θ, F)-split torus $\mathbf{S_M}$ contained in the centre of $\mathbf{M}$.

In fact, we have that $\mathbf{S_M}$ is the connected component (of the identity) of the subgroup of θ-split elements in $\mathbf{A_M}$, that is,

$$\mathbf{S_M} = \left(\{x \in \mathbf{A_M} : \theta(x) = x^{-1}\}\right)^\circ . \tag{1.2}$$

Indeed, the F-split torus $\mathbf{S_M}$ given in (1.2) is (θ, F)-split; moreover, $\mathbf{S_M}$ is the largest (θ, F)-split torus contained in the centre of $\mathbf{M}$.

Remark 1.5.4. In subsequent chapters, we will write S_M for the group of F-points of $\mathbf{S_M}$ and, by abuse of notation, we refer to S_M as the (θ, F)-split component of M. Similarly, we will abuse notation and refer to the F-points $A_M = \mathbf{A_M}(F)$ as the F-split component of M.

1.5.1 The restricted root system

Let $p : X^*(\mathbf{A}_0) \to X^*(\mathbf{S}_0)$ be the surjective homomorphism defined by restricting the F-rational characters on $\mathbf{A}_0$ to the subtorus $\mathbf{S}_0$. The *restricted root system* of $H\backslash G$ (relative to our choice of $(\mathbf{A}_0, \mathbf{S}_0, \Delta_0)$) is defined to be

$$\overline{\Phi}_0 = p(\Phi_0) \setminus \{0\} = p(\Phi_0 \setminus \Phi_0^\theta). \tag{1.3}$$

As shown in [30], the set $\overline{\Phi}_0$ coincides with the set $\Phi(G, S_0)$ of roots with respect to S_0 and this is a (not necessarily reduced) root system by [31, Proposition 5.9]. A choice of simple roots for $\overline{\Phi}_0$ is given by

$$\overline{\Delta}_0 = p(\Delta_0) \setminus \{0\} = p(\Delta_0 \setminus \Delta_0^\theta). \tag{1.4}$$

Given a subset $\overline{\Theta} \subset \overline{\Delta}_0$, define the subset

$$[\overline{\Theta}] = p^{-1}(\overline{\Theta}) \cup \Delta_0^\theta \tag{1.5}$$

of Δ_0. Subsets of Δ_0 of the form $[\overline{\Theta}]$, for $\overline{\Theta} \subset \overline{\Delta}_0$, are said to be *$\theta$-split*. The maximal θ-split subsets of Δ_0 are of the form

$$[\overline{\Delta}_0 \setminus \{\bar{\alpha}\}], \tag{1.6}$$

where $\bar{\alpha} \in \overline{\Delta}_0$.

Remark 1.5.5. We're interested in the restricted root system as a way to parametrize θ-split parabolic subgroups of G (*cf.* §1.6). The analogous restricted root system in the spherical variety setting plays a crucial role in the work of Sakellaridis and Venkatesh [66], particularly in their study of the "boundary degenerations" and "geometry at infinity" of spherical varieties.

1.6 Parabolic subgroups relative to involutions

There are two types of parabolic subgroups important for the work in this thesis, those that are θ-stable and those that are mapped by θ to their opposites. We first consider the latter.

1.6.1 θ-split parabolic subgroups

Definition 1.6.1. A parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ is called θ-split if $\theta(\mathbf{P})$ is opposite to $\mathbf{P}$. In this case, $\mathbf{M} = \mathbf{P} \cap \theta(\mathbf{P})$ is a θ-stable Levi subgroup of both $\mathbf{P}$ and $\theta(\mathbf{P}) = \mathbf{P}^{op}$.

Lemma 1.6.2. *Let $\mathbf{P}$ be a θ-split parabolic subgroup with θ-stable Levi $\mathbf{M} = \mathbf{P} \cap \theta(\mathbf{P})$. The Levi subgroup $\mathbf{M}$ is equal to the centralizer in $\mathbf{G}$ of its (θ, F)-split component $\mathbf{S_M}$.*

Proof. The containment $\mathbf{M} \subset C_{\mathbf{G}}(\mathbf{S_M})$ is automatic since $\mathbf{S_M}$ is central in $\mathbf{M}$. Let $\mathbf{S}$ be a maximal (θ, F)-split torus of $\mathbf{P}$ that contains $\mathbf{S_M}$. By [31, Lemma 4.5(iii)], there exists a θ-stable maximal F-split torus $\mathbf{A}$ of $\mathbf{P}$ that contains $\mathbf{S}$. By [31, Lemma 4.6], there exists an F-rational cocharacter $\chi^\vee \in X_*(\mathbf{S})$ of $\mathbf{S}$ such that $\mathbf{M} = C_{\mathbf{G}}(\operatorname{Im} \chi^\vee)$. In particular, we have that $\operatorname{Im} \chi^\vee$ is contained in $\mathbf{S_M}$ and it follows that $C_{\mathbf{G}}(\mathbf{S_M}) \subset C_{\mathbf{G}}(\operatorname{Im} \chi^\vee) = \mathbf{M}$. □

Let $\mathbf{S}_0$ be a maximal (θ, F)-split torus of $\mathbf{G}$, contained in a maximal θ-stable F-split torus $\mathbf{A}_0$, and let Δ_0 be a θ-base for $\Phi_0 = \Phi(G, A_0)$ (*cf.* Definition 1.5.2). Given a θ-split subset $\Theta \subset \Delta_0$, the Δ_0-standard parabolic subgroup $\mathbf{P}_\Theta = \mathbf{M}_\Theta \mathbf{N}_\Theta$ (*cf.* §1.4) is θ-split [45]. Let $\mathbf{S}_\Theta$ denote the (θ, F)-split component of $\mathbf{M}_\Theta$. We have that

$$\mathbf{S}_\Theta = \left(\{s \in \mathbf{A}_\Theta : \theta(s) = s^{-1}\}\right)^\circ = \left(\bigcap_{\bar{\alpha} \in p(\Theta)} \ker(\bar{\alpha} : \mathbf{S}_0 \to F^\times) \right)^\circ, \tag{1.7}$$

where $p : X^*(\mathbf{A}_0) \to X^*(\mathbf{S}_0)$ is the restriction map. For the second equality in (1.7), see [45, §1.5]. By [30, Theorem 2.9], the subset Δ_0^θ of θ-fixed roots in Δ_0 determines a minimal θ-split parabolic subgroup $\mathbf{P}_0 = \mathbf{P}_{\Delta_0^\theta}$. By [31, Proposition 4.7(iv)], the minimal θ-split parabolic subgroup $\mathbf{P}_0$ has standard θ-stable Levi $\mathbf{M}_0 = C_{\mathbf{G}}(\mathbf{S}_0)$. Let $\mathbf{P}_0 = \mathbf{M}_0 \mathbf{N}_0$ be the standard Levi factorization of the Δ_0-standard minimal θ-split parabolic subgroup $\mathbf{P}_0$ of $\mathbf{G}$.

Lemma 1.6.3. *If $\mathbf{M}$ is the centralizer in $\mathbf{G}$ of a non-central (θ, F)-split torus $\mathbf{S}$, then $\mathbf{M}$ is the Levi subgroup of a proper θ-split parabolic subgroup of $\mathbf{G}$.*

Proof. Let $\mathbf{S}_0$ be a maximal (θ, F)-split torus of $\mathbf{G}$ containing $\mathbf{S}$ and let $\mathbf{A}_0$ a θ-stable maximal F-split torus of $\mathbf{G}$ containing $\mathbf{S}_0$. The subgroup $\mathbf{M} = C_{\mathbf{G}}(\mathbf{S})$ is a θ-stable Levi subgroup of $\mathbf{G}$ since $\mathbf{S}$ is a θ-stable F-split torus. Since $\mathbf{S}$ is not central in $\mathbf{G}$, $\mathbf{M}$ is a proper Levi subgroup. Let $\mathbf{P}_0 = \mathbf{M}_0 \mathbf{N}_0$ be a minimal θ-split parabolic subgroup of $\mathbf{G}$ with θ-stable Levi subgroup $\mathbf{M}_0 = C_{\mathbf{G}}(\mathbf{S}_0)$. Since $\mathbf{S}$ is contained in $\mathbf{S}_0$, we have that $\mathbf{M}_0$ is contained in $\mathbf{M}$. Let $\mathbf{P} = \mathbf{M}\mathbf{N}_0$. Note that $\mathbf{P}$ is a closed subgroup containing $\mathbf{P}_0$; therefore, $\mathbf{P}$ is a proper parabolic subgroup of $\mathbf{G}$ with θ-stable Levi subgroup $\mathbf{M} \neq \mathbf{G}$. It remains to show that $\mathbf{P}$ is θ-split. Since $\mathbf{P}_0$ is θ-split, we have that $\theta(\mathbf{N}_0) = \mathbf{N}_0^{op}$ is opposite to $\mathbf{N}_0$ (with respect to $\mathbf{A}_0$). Since $\mathbf{M}$ is θ-stable, it follows that $\theta(\mathbf{P}) = \mathbf{M}\mathbf{N}_0^{op}$ is opposite to $\mathbf{P}$. □

Let $\Theta \subset \Delta_0$ be a θ-split subset. For any $0 < \epsilon \leq 1$, define

$$S_\Theta^-(\epsilon) = \{s \in S_\Theta : |\alpha(s)|_F \leq \epsilon, \text{ for all } \alpha \in \Delta_0 \setminus \Theta\}. \tag{1.8}$$

We write S_Θ^- for $S_\Theta^-(1)$ and refer to S_Θ^- as the *dominant part* of S_Θ. The dominant parts of the tori $S_\Theta = \mathbf{S}_\Theta(F)$ are involved in the Relative Casselman's Criterion 2.2.18, which we review in Chapter 2.

1.6.2 Conjugacy of θ-split parabolic subgroups

The minimal θ-split parabolic subgroups of $\mathbf{G}$ are not always conjugate over $H = \mathbf{H}(F)$ (*cf.* [31, Example 4.12]). Here we record [44, Lemma 2.5], which is based on the results of [31].

Lemma 1.6.4 ([44, Lemma 2.5])**.** *Let $\mathbf{S}_0 \subset \mathbf{A}_0$, Δ_0, and $\mathbf{P}_0 = \mathbf{M}_0\mathbf{N}_0$ be as in the previous subsection.*

1. *Any θ-split parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ is conjugate to a Δ_0-standard θ-split parabolic subgroup by an element $g \in (\mathbf{H}\mathbf{M}_0)(F)$.*

2. *If the group of F-points of the product $(\mathbf{H}\mathbf{M}_0)(F)$ is equal to HM_0, then any θ-split parabolic subgroup of $\mathbf{G}$ is H-conjugate to a Δ_0-standard θ-split parabolic subgroup.*

Let $\mathbf{P} = \mathbf{M}\mathbf{N}$ be a θ-split parabolic subgroup and choose $g \in (\mathbf{H}\mathbf{M}_0)(F)$ such that $\mathbf{P} = g\mathbf{P}_\Theta g^{-1}$ for some Δ_0-standard θ-split parabolic subgroup $\mathbf{P}_\Theta$. Since $g \in (\mathbf{H}\mathbf{M}_0)(F)$ we have that $g^{-1}\theta(g) \in \mathbf{M}_0(F)$. Indeed, $g = \underline{h} \cdot \underline{m} \in \mathbf{G}(F)$, where $\underline{h} \in \mathbf{H}(K)$ and $\underline{m} \in \mathbf{M}_0(K)$ for some finite extension K of F, and

$$g^{-1}\theta(g) = m^{-1}h^{-1}\theta(h)\theta(m) = m^{-1}\theta(m) \in \mathbf{M}_0(K) \cap \mathbf{G}(F) = \mathbf{M}_0(F) = M_0.$$

We have that $g^{-1}\theta(g) \in M_0$ and $\mathbf{M}_0 \subset \mathbf{M}_\Theta$; therefore, we may take $\mathbf{S}_\mathbf{M} = g\mathbf{S}_\Theta g^{-1}$. Indeed, if $s \in S_\Theta$, then

$$\theta(gsg^{-1}) = \theta(g)\theta(s)\theta(g)^{-1} = \theta(g)s^{-1}\theta(g)^{-1} = gg^{-1}\theta(g)s^{-1}\theta(g)^{-1}gg^{-1} = gs^{-1}g^{-1} = (gsg^{-1})^{-1},$$

since $g^{-1}\theta(g) \in M_0$ commutes with $\mathbf{S}_\Theta \subset \mathbf{S}_0$. For a given $\epsilon > 0$, one may extend the definition (1.8) of S_Θ^- to the non-Δ_0-standard torus $S_M = \mathbf{S}_\mathbf{M}(F)$. Indeed, we set

$$S_M^-(\epsilon) = gS_\Theta^-(\epsilon)g^{-1} \tag{1.9}$$

and we define $S_M^- = S_M^-(1)$. Again, we refer to S_M^- as the dominant part of S_M.

1.6.3 θ-elliptic Levi subgroups

We give the following definition, following the terminology of Murnaghan [57], in analogy with the notion of an elliptic Levi subgroup.

Definition 1.6.5. A θ-stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ is *θ-elliptic* if and only if $\mathbf{L}$ is not contained in any proper θ-split parabolic subgroup of $\mathbf{G}$.

We note the following simple lemma, which follows immediately from Definition 1.6.5.

Lemma 1.6.6. *If a θ-stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ contains a θ-elliptic Levi subgroup, then $\mathbf{L}$ is θ-elliptic.*

Next, we record a useful characterization of θ-elliptic Levi subgroups in terms of their (θ, F)-split components.

Lemma 1.6.7. *A θ-stable Levi subgroup $\mathbf{L}$ is θ-elliptic if and only if $\mathbf{S_L} = \mathbf{S_G}$.*

Proof. If $\mathbf{L} = \mathbf{G}$, then there is nothing to do. Without loss of generality, $\mathbf{L}$ is a proper subgroup of $\mathbf{G}$. Suppose that $\mathbf{L}$ is θ-elliptic. We have that $\mathbf{A_G}$ is contained in $\mathbf{A_L}$ and it follows that $\mathbf{S_G}$ is contained in $\mathbf{S_L}$. If $\mathbf{S_L}$ properly contains $\mathbf{S_G}$, then $\mathbf{L}$ is contained in the Levi subgroup $\mathbf{M} = C_{\mathbf{G}}(\mathbf{S_L})$. By Lemma 1.6.3, $\mathbf{M}$ is a Levi subgroup of a proper θ-split parabolic subgroup. It follows that $\mathbf{L} \subset \mathbf{M}$ is contained in a proper θ-split parabolic subgroup. This contradicts the fact that $\mathbf{L}$ is θ-elliptic, so we must have that $\mathbf{S_L} = \mathbf{S_G}$.

On the other hand, suppose that $\mathbf{S_L}$ is equal to $\mathbf{S_G}$. Argue by contradiction, and suppose that $\mathbf{L}$ is contained in a proper θ-split parabolic $\mathbf{P} = \mathbf{MN}$ with θ-stable Levi subgroup $\mathbf{M} = \mathbf{P} \cap \theta(\mathbf{P})$. We have that $\mathbf{L} = \theta(\mathbf{L})$ is contained in $\mathbf{M}$ and $\mathbf{S_M} \subset \mathbf{S_L}$. By Lemma 1.6.2, $\mathbf{M}$ is the centralizer of its (θ, F)-split component $\mathbf{S_M}$. Since $\mathbf{M}$ is a proper Levi subgroup of $\mathbf{G}$, we have that $\mathbf{S_M}$ properly contains $\mathbf{S_G}$. However, by assumption $\mathbf{S_L} = \mathbf{S_G}$, which implies that $\mathbf{M} = \mathbf{G}$, and this is impossible. We conclude that $\mathbf{L}$ must be θ-elliptic. $\square$

We conclude this chapter with the following proposition which qualitatively describes the parabolic subgroups of $\mathbf{G}$ that admit θ-elliptic Levi subgroups.

Proposition 1.6.8. *Let $\mathbf{Q}$ be a parabolic subgroup of $\mathbf{G}$. If $\mathbf{Q}$ admits a θ-elliptic Levi factor $\mathbf{L}$, then $\mathbf{Q}$ is θ-stable.*

Proof. By assumption, $\mathbf{Q}$ has Levi decomposition $\mathbf{Q} = \mathbf{LU}$ where $\mathbf{L}$ is θ-elliptic and, in particular, θ-stable. It is enough to show that the unipotent radical $\mathbf{U}$ of $\mathbf{Q}$ is θ-stable. Let $\mathbf{A}$ be a θ-stable maximal F-split torus of $\mathbf{Q}$ contained in $\mathbf{L}$. Such a torus exists by [63, Lemma 5.3]. In fact, $\mathbf{A}$ is a maximal F-split torus of $\mathbf{G}$. Let $\Phi = \Phi(G, A)$ be the root system of G with respect to A and let $\Phi_L = \Phi(L, A)$ be the root system of L with respect to A. Since $\mathbf{A}$ and $\mathbf{L}$ are θ-stable, the involution θ acts on Φ and Φ_L is a θ-stable subsystem of Φ. The parabolic subgroup $\mathbf{Q}$ is standard with respect to some choice Δ of simple roots in Φ. We can arrange our choice of Δ so that $\mathbf{L}$ is a standard Levi subgroup of $\mathbf{Q}$. Let Φ^+ denote the set of positive roots in Φ with respect to Δ. Then $\Phi_L^+ = \Phi_L \cap \Phi^+$ is a choice of positive roots in Φ_L. The unipotent radical $\mathbf{U}$ of $\mathbf{Q}$ is generated by the root groups $\mathbf{N}_\beta$, where $\beta \in \Phi^+ \setminus \Phi_L^+$.

To prove that $\mathbf{U}$ is θ-stable it is enough to prove that the set of roots $\Phi(U, A) = \Phi^+ \setminus \Phi_L^+$ is θ-stable. The map $\Phi \to \Phi(G, A_L)$ defined by restriction from $\mathbf{A}$ to $\mathbf{A_L}$ is surjective. In particular, the restriction map $\Phi(U, A)$ to $\Phi(U, A_L)$ is surjective; moreover, if $\beta \in \Phi(U, A_L)$ then any lift of β to Φ must lie in $\Phi(U, A)$. To see the second claim, one can choose a subset $\Theta \subset \Delta$ such that $\mathbf{Q} = \mathbf{P}_\Theta$ and then use that the restriction of $\Delta \setminus \Theta$ to $\mathbf{A_L}$ is a set of simple roots for $\Phi(U, A_L)$. Writing $\alpha \in \Phi^+$ as a nonnegative integral combination of the roots in Δ, one can then see that $\alpha|\mathbf{A_L}$ cannot lie in $\Phi(U^{op}, A_L)$. For this reason, if $\Phi(U, A_L)$ is θ-stable we must also have that $\Phi(U, A)$ is θ-stable. The θ-stable F-split torus $\mathbf{A_L}$

is the almost direct product of $(\mathbf{A}_{\mathbf{L}}^{\theta})^{\circ}$ and the (θ, F)-split component $\mathbf{S}_{\mathbf{L}}$ of $\mathbf{L}$ [33]. By assumption, $\mathbf{L}$ is θ-elliptic so, by Lemma 1.6.7, $\mathbf{S}_{\mathbf{L}}$ is equal to $\mathbf{S}_{\mathbf{G}}$. Let $a \in \mathbf{A}_{\mathbf{L}} = (\mathbf{A}_{\mathbf{L}}^{\theta})^{\circ}\mathbf{S}_{\mathbf{G}}$ be the (non-unique) product of $a_+ \in (\mathbf{A}_{\mathbf{L}}^{\theta})^{\circ}$ and $a_- \in \mathbf{S}_{\mathbf{G}}$. Since a_- is central in $\mathbf{G}$ and a_+ is θ-fixed, for any root $\alpha \in \Phi$, we have that

$$\alpha(a) = \alpha(a_+a_-) = \alpha(a_+) = \alpha(\theta(a_+)) = (\theta\alpha)(a_+) = (\theta\alpha)(a);$$

in particular, the restriction of α to $\mathbf{A}_{\mathbf{L}}$ is θ-fixed for any $\alpha \in \Phi$. It follows that $\Phi(U, A_L)$ is θ-stable. Thus, we have that $\Phi(U, A)$ is θ-stable, completing the proof. □

Remark 1.6.9. It would be desirable to obtain a combinatorial characterization of θ-elliptic Levi subgroups for any $\mathbf{G}$ and θ. No such characterization is currently known to the author.

Remark 1.6.10. For the remainder of this thesis, we routinely abuse notation regarding algebraic groups defined over F and the corresponding groups of F-rational points. When necessary, we will use boldface to denote the algebraic group $\mathbf{G}$ and regular typeface to denote the group $G = \mathbf{G}(F)$ of F-rational points. See also Remarks 1.4.1 and 1.5.4.

Chapter 2

Representation-theoretic background

In this chapter, we review the necessary background material on the representation theory of p-adic groups. The results that we recall are well known and our main reference is [15]. We will discuss the work of Kato and Takano [45] on the Relative Casselman's Criterion 2.2.18, which is integral to the proof of the main theorems of Chapter 4 and 5. Finally, to conclude this chapter, we recall known results on distinction of discrete series for the two symmetric spaces discussed in Chapter 4. We refer to the case that $G = \mathbf{GL}_n(F)$, when n is even, and $H = \mathbf{GL}_{n/2}(F) \times \mathbf{GL}_{n/2}(F)$, as the *linear case*. We will refer to the case that $G = R_{E/F}\mathbf{GL}_n(F)$, and $H = \mathbf{GL}_n(F)$ as the *Galois case*. See Chapter 4 for more detail on the terminology and notation in these two cases.

Note. We will indicate below (§2.2.1) when we restrict ourselves to the case of a symmetric subgroup $\mathbf{H} = \mathbf{G}^\theta$. For now, we will let H denote any closed subgroup of G.

Recall that we will refer to the F-points P of an F-parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ as a parabolic subgroup of G (*cf.* Remark 1.4.1). We make a similar abuse of notation regarding the F-split and (θ, F)-split components of Levi subgroups (*cf.* Remark 1.5.4).

2.1 Smooth and admissible representations

Let G be the F-points of a connected reductive group $\mathbf{G}$ defined over F. A *representation* (π, V) of G consists of a group homomorphism π from G to the group $\mathbf{GL}(V)$ of invertible linear operators on a vector space V. All of the representations that we consider will be on complex vector spaces. For a vector $v \in V$, define the stabilizer of v in G to be the subgroup $\mathrm{Stab}_G(v)$ of G consisting of all elements that fix v pointwise, i.e., $\mathrm{Stab}_G(v) = \{g \in G : \pi(g)v = v\}$. In general, G is non-compact and the representations of interest are on infinite dimensional vector spaces, often spaces of functions on G. The appropriate notion of continuity for representations of G is that of *smoothness*.

Definition 2.1.1. A representation (π, V) of G is *smooth* if for every $v \in V$ the stabilizer of v in G is an open subgroup. A smooth representation (π, V) of G is *admissible* if, for every compact open subgroup K of G, the subspace of K-invariant vectors V^K is finite dimensional.

A representation is *irreducible* if it admits no non-trivial proper G-invariant subspaces. It is a theorem of Harish-Chandra and Jacquet that a smooth irreducible representation is admissible [39].

Let (π, V) be a smooth representation of G. Given any automorphism ψ of G, we define the twist of π by ψ to be the smooth representation $({}^{\psi}\pi, V)$ such that ${}^{\psi}\pi = \pi \circ \psi^{-1}$. For an inner automorphism $\operatorname{Int} x$, $x \in G$, we write ${}^{x}\pi$ instead of ${}^{\operatorname{Int} x}\pi$.

We obtain a representation (π^*, V^*) on the algebraic dual $V^* = \operatorname{Hom}_{\mathbb{C}}(V, 1)$ of linear functionals on the space V. We write $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{C}$ for the canonical pairing on V and its dual space. The action of G on V^* is given by

$$\langle \pi^*(g)v^*, v \rangle = \langle v^*, \pi(g^{-1})v \rangle, \tag{2.1}$$

for any $g \in G$, $v^* \in V^*$ and $v \in V$. The representation (π^*, V^*) need not be smooth (see [13] for an example). We set $\widetilde{V}$ to be equal to the subspace of smooth vectors in V^* and let $\widetilde{\pi}$ be the restriction of π^* to $\widetilde{V}$. The representation $(\widetilde{\pi}, \widetilde{V})$ is called the *contragredient* or *smooth dual* of (π, V). We also denote the canonical pairing $\widetilde{V} \times V \to \mathbb{C}$ by $(\widetilde{v}, v) \mapsto \langle \widetilde{v}, v \rangle$. The following result appears as [15, Proposition 2.1.10].

Proposition 2.1.2. *Let (π, V) be a smooth representation of G. The following are equivalent:*

1. *π is admissible,*
2. *$\widetilde{\pi}$ is admissible,*
3. *the contragredient $\widetilde{\widetilde{\pi}}$ of $\widetilde{\pi}$ is isomorphic to π.*

The following result, which appears as [13, Lemma 2.10], and the subsequent corollary are useful.

Proposition 2.1.3. *The functor that sends a smooth representation π to its contragredient $\widetilde{\pi}$ is contravariant and exact.*

Corollary 2.1.4. *An admissible representation (π, V) is irreducible if and only if $(\widetilde{\pi}, \widetilde{V})$ is irreducible.*

We refer to one-dimensional smooth representations as *quasi-characters*, we will refer to a unitary quasi-character as a *(unitary) character.* Let ω be a quasi-character of the centre Z_G of G. We say that π is an *ω-representation* if $\pi(z)v = \omega(z)v$, for every $z \in Z_G$ and $v \in V$. If π is an ω-representation, then we say that π has *central character* ω. The following is a consequence of Schur's Lemma.

Corollary 2.1.5. *If (π, V) is an irreducible smooth representation of G, then there exists a quasi-character ω of Z_G such that π is an ω-representation, that is, π admits a central character.*

The next lemma describes the central character of the contragredient of an irreducible representation.

Lemma 2.1.6. *Let (π, V) be an admissible representation of G with central character ω. The contragredient representation $(\widetilde{\pi}, \widetilde{V})$ has central character ω^{-1}.*

Proof. Let $z \in Z_G$ and let $\widetilde{v} \in \widetilde{V}$. For any $v \in V$, we have

$$\langle \widetilde{\pi}(z)\widetilde{v}, v \rangle = \langle \widetilde{v}, \pi(z^{-1})v \rangle = \langle \widetilde{v}, \omega(z^{-1})v \rangle = \langle \omega(z^{-1})\widetilde{v}, v \rangle;$$

therefore, $\widetilde{\pi}(z)\widetilde{v} = \omega^{-1}(z)\widetilde{v}$ for all $z \in Z_G$ and $\widetilde{v} \in \widetilde{V}$. $\square$

2.1.1 Matrix coefficients

Let $C^\infty(G)$ denote the space of complex-valued locally constant functions on G. We refer to $C^\infty(G)$ as the space of *smooth functions* on G. Let (π, V) be a smooth representation of G. Given $v \in V$ and $\widetilde{v} \in \widetilde{V}$, define a function $\phi_{\widetilde{v},v}$ on G by $\phi_{\widetilde{v},v}(g) = \langle \widetilde{v}, \pi(g)v \rangle$. The functions $\phi_{\widetilde{v},v}$ are referred to as the *matrix coefficients* of π. Since π is smooth, the function $\phi_{\widetilde{v},v}$ is locally constant and we have that $\phi_{\widetilde{v},v} \in C^\infty(G)$. Indeed, there exists an open subgroup K_v of G such that $\pi(k)v = v$, for all $k \in K_v$, and so $\phi_{\widetilde{v},v}(k) = \phi_{\widetilde{v},v}(e)$, for all $k \in K_v$. Many important analytic properties of a representation π may be interpreted in terms of properties of its matrix coefficients; we will discuss this further below (*cf.* 2.1.4).

2.1.2 Parabolic induction and Jacquet restriction

Induced representations

Let H be a closed subgroup of G and let (π, V) be a smooth representation of H. Define $\operatorname{Ind}_H^G \pi$ to be the representation of G, obtained by (smooth) induction from π, on the space of functions $f : G \to V$ such that:

1. $f(hg) = \pi(h)f(g)$ for all $h \in H$ and $g \in G$, and
2. $f(gk) = f(g)$ for all $g \in G$ and $k \in K_f$, for some open subgroup K_f of G depending on f,

with action of G given by right-translation of functions, i.e., $\left(\operatorname{Ind}_H^G \pi(g) f\right)(x) = f(xg)$, for all $x, g \in G$. The second requirement guarantees that $\operatorname{Ind}_H^G \pi$ is a smooth representation. Often both the space of an induced representation and the associated homomorphism are denoted by $\operatorname{Ind}_H^G \pi$. Define $\text{c-}\operatorname{Ind}_H^G \pi$ to be the restriction of $\operatorname{Ind}_H^G \pi$ to the subspace of functions in $\operatorname{Ind}_H^G \pi$ of compact support. The representation $\text{c-}\operatorname{Ind}_H^G \pi$ is also smooth. The following facts appear in [15, Theorem 2.4.1, Proposition 2.4.4].

Proposition 2.1.7. *Let H be a closed subgroup of G and let (π, V) be a smooth representation of H.*

1. *The representations $\operatorname{Ind}_H^G \pi$ and $\text{c-}\operatorname{Ind}_H^G \pi$ are smooth.*
2. *If $H\backslash G$ is compact and π is admissible, then $\operatorname{Ind}_H^G \pi$ is equal to $\text{c-}\operatorname{Ind}_H^G \pi$ and is admissible.*
3. *Frobenius Reciprocity holds: if (π', V') is any smooth representation of G, then there is a natural bijection between $\operatorname{Hom}_G(\pi', \operatorname{Ind}_H^G \pi)$ and $\operatorname{Hom}_H(\operatorname{Res}_H^G \pi', \pi)$. Here $\operatorname{Res}_H^G \pi'$ is the representation of H obtained by restricting π' to H.*
4. *The functor that sends π to $\operatorname{Ind}_H^G \pi$ is additive and exact.*

Let $\delta_H = \Delta_H^{-1} : H \to \mathbb{R}^\times$ be the modular character of H (*cf.* §1.4.1) and recall that G is unimodular.

Proposition 2.1.8 ([15, Theorem 2.4.2])**.** *Let H be a closed subgroup of G and let (π, V) be a smooth representation of H. The contragredient of $\text{c-}\operatorname{Ind}_H^G \pi$ is isomorphic to $\operatorname{Ind}_H^G(\delta_H \otimes \widetilde{\pi})$.*

Parabolically induced representations

Let P be a parabolic subgroup of G with Levi subgroup M and unipotent radical N. Given a smooth representation (ρ, V_ρ) of M we may inflate ρ to a representation of P, also denoted ρ, by declaring that N acts trivially. We define the representation $\iota_P^G \rho$ of G, obtained by *normalized parabolic induction* from ρ along P, to be the induced representation $\operatorname{Ind}_P^G(\delta_P^{1/2} \otimes \rho)$. Precisely, $\iota_P^G \rho$ is the representation of G acting by right-translation on the space of the functions $f : G \to V_\rho$ such that

1. $f(pg) = \delta_P^{1/2}(p)\rho(p)f(g)$ for all $p \in P$ and $g \in G$, and

2. $f(gk) = f(g)$ for all $g \in G$ and $k \in K_f$, for some open subgroup K_f of G depending on f,

with action $\left(\iota_P^G(g)f\right)(x) = f(xg)$ for all $x, g \in G$. We record here some basic facts on parabolic induction; the standard references in the case $\mathbf{G} = \mathbf{GL}_n$ remain [7, 75].

Proposition 2.1.9. *Let $P = MN$ be a parabolic subgroup of G with Levi factor M and unipotent radical N. Let ρ be a smooth representation of M.*

1. *The map that sends ρ to $\iota_P^G\rho$ is an additive exact functor from the category of smooth representations of M to the category of smooth representations of G.*

2. *If ρ is unitary, then $\iota_P^G\rho$ is unitary.*

3. *If ρ is finitely generated, then $\iota_P^G\rho$ is finitely generated.*

4. *If ρ is admissible, then $\iota_P^G\rho$ is admissible.*

5. *The contragredient of $\iota_P^G\rho$ is isomorphic to $\iota_P^G\widetilde{\rho}$.*

Remark 2.1.10. Proposition 2.1.9(2) is a consequence of (actually, a reason for) the normalization of $\iota_P^G\rho$ by twisting the inducing representation by the quasi-character $\delta_P^{1/2}$.

Remark 2.1.11. On a few occasions, most notably the proof of Proposition 4.2.23 and in §2.3.2, we'll use the Bernstein–Zelevinsky notation [7, 75] for parabolically induced representations of general linear groups. Given a block upper-triangular parabolic subgroup $P = P_{(m_1,\ldots,m_k)}$ of $G = \mathbf{GL}_n(F)$ and smooth representations ρ_i of $\mathbf{GL}_{m_i}(F)$, for each $1 \leq i \leq k$, write $\rho_1 \times \rho_2 \times \ldots \times \rho_k$ for the parabolically induced representation $\iota_P^G(\rho_1 \otimes \ldots \otimes \rho_k)$ of G.

The Jacquet module

We now describe the functor that is left-adjoint to ι_P^G and recall some of its properties. Let (π, V) be a smooth representation of G. Let P be a parabolic subgroup of G with Levi decomposition $P = MN$, where M is the Levi component and N is the unipotent radical of P. Define $V(N)$ to be the following subspace of V,

$$V(N) = \mathrm{span}\{\pi(n)v - v : n \in N, v \in V\}.$$

Lemma 2.1.12. *The subspace $V(N)$ of V is P-invariant.*

Proof. Let $p \in P$ and observe that for any $n \in N$ and $v \in V$ we have that

$$\pi(p)(\pi(n)v - v) = \pi(p)\pi(n)v - \pi(p)v = \pi(pnp^{-1})\pi(p)v - \pi(p)v.$$

Moreover, since N is normal in P, we have $pnp^{-1} \in N$. It follows that P preserves the spanning set of $V(N)$. By linearity of the operators $\pi(p)$ on V, the subspace $V(N)$ is P-invariant. ☐

Lemma 2.1.13. *If $V' \subset V$ is an N-invariant subspace and N acts trivially on the quotient V/V', then $V(N)$ must be contained in V'.*

Proof. By assumption, $\pi(n)v + V' = \pi(n)(v + V') = v + V'$, for all $v \in V$ and $n \in N$. It follows that $\pi(n)v - v \in V'$ for all $v \in V$ and $n \in N$. The subspace V' contains a spanning set for $V(N)$ and hence must contain $V(N)$. □

Define V_N to be the quotient vector space $V/V(N)$ and let π_N denote the quotient representation of P on V_N normalized by $\delta_P^{-1/2}$. That is, for $p \in P$ and $v + V(N) \in V_N$ we have

$$\pi_N(p)(v + V(N)) = \delta_P^{-1/2}(p)\pi(p)v + V(N). \tag{2.2}$$

By Lemmas 2.1.12 and 2.1.13, and since δ_P is trivial on N, we see that (π_N, V_N) is a representation of P on which N acts trivially. We will regard (π_N, V_N) as a representation of the Levi factor $M \cong P/N$ of P.

Definition 2.1.14. The representation (π_N, V_N) of P is called the *(normalized) Jacquet module* of (π, V) (with respect to P). The functor that sends π to π_N is often referred to as the *Jacquet restriction functor*.

A form of Frobenius Reciprocity holds for parabolic induction and Jacquet restriction.

Proposition 2.1.15 (Frobenius Reciprocity)**.** *Let $P = MN$ be a parabolic subgroup of G with Levi subgroup M and unipotent radical P. Let (π, V) be a smooth representation of G and let (ρ, V_ρ) a smooth representation of M. There is a natural bijection between $\mathrm{Hom}_G(\pi, \iota_P^G\rho)$ and $\mathrm{Hom}_M(\pi_N, \rho)$ that is functorial in both π and ρ.*

Theorem 2.1.16 (Jacquet)**.** *Let $P = MN$ be a parabolic subgroup of G with Levi subgroup M and unipotent radical P. Let (π, V) be a smooth representation of G. The Jacquet module (π_N, V_N) is a smooth representation of M. Moreover, we have the following.*

1. *If π is finitely generated, then π_N is finitely generated.*
2. *If π is admissible, then π_N is admissible.*

The Geometric Lemma [7, Lemma 2.12] (*cf.* Lemma 2.1.17) describes the structure of representations in the image of the composite functor of parabolic induction followed by Jacquet restriction, and is a fundamental tool in the study of induced representations. Let $P = MN$ and $Q = LU$ be two parabolic subgroups of G with Levi factors M and L, and unipotent radicals N and U respectively. Following [65], let

$$S(M, L) = \{y \in G : M \cap {}^yL \text{ contains a maximal } F\text{-split torus of } G\}. \tag{2.3}$$

There is a canonical bijection between the double-coset space $P\backslash G/Q$ and the set $M\backslash S(M, L)/L$. Let $y \in S(M, L)$. The subgroup $M \cap {}^yQ$ is a parabolic subgroup of M and $P \cap {}^yL$ is a parabolic subgroup of ${}^yL$. The unipotent radical of $M \cap {}^yQ$ is $M \cap {}^yU$ and the unipotent radical of $P \cap {}^yL$ is $N \cap {}^yL$; moreover, $M \cap {}^yL$ is a Levi subgroup of both $M \cap {}^yQ$ and $P \cap {}^yL$. Given a representation ρ of L, we obtain a representation ${}^y\rho = \rho \circ \mathrm{Int}\, y^{-1}$ of ${}^yL$.

Lemma 2.1.17 (The Geometric Lemma)**.** *Let ρ be a smooth representation of L. There is a filtration of the space of the representation $(\iota_Q^G\rho)_N$ such that the associated graded object is isomorphic to the direct sum*

$$\bigoplus_{y \in M\backslash S(M,L)/L} \iota^M_{M\cap {}^yQ}\left(({}^y\rho)_{N\cap {}^yL}\right). \tag{2.4}$$

Remark 2.1.18. We will write $\mathcal{F}_N^y(\rho)$ to denote the smooth representation $\iota_{M\cap {}^yQ}^M(({}^y\rho)_{N\cap {}^yL})$ of M.

Let Δ be a choice of simple roots for the root system Φ of G with respect to a fixed maximal F-split torus A. Let Φ^+ be the set of positive roots determined by Δ. Let W be the Weyl group of G with respect to A (*cf.* §1.3).

Lemma 2.1.19 ([15, Proposition 1.3.1]). *Let Θ and Ω be subsets of Δ. The set*

$$[W_\Theta\backslash W/W_\Omega] = \{w \in W : w\Omega, w^{-1}\Theta \subset \Phi^+\} \tag{2.5}$$

provides a choice of Weyl group representatives for the double-coset space $P_\Theta\backslash G/P_\Omega$.

When applying the Geometric Lemma along two standard parabolic subgroups P_Θ and P_Ω, associated to $\Theta, \Omega \subset \Delta$ (*cf.* §1.4), we will always use the choice (2.5) of "nice" representatives for the double-coset space $P_\Theta\backslash G/P_\Omega$. Observe that the standard Levi subgroups M_Θ and M_Ω both contain the maximal F-split torus A; moreover, for any $w \in W$, $M_\Theta \cap {}^wM_\Omega$ contains $A = {}^wA$. In particular, W is contained in $S(M_\Theta, M_\Omega)$. We'll write $\mathcal{F}_\Theta^w(\rho)$ to denote $\mathcal{F}_{N_\Theta}^w(\rho)$. We also note the following result, which we will use (often implicitly) in Chapters 4 and 5.

Proposition 2.1.20 ([15, Proposition 1.3.3]). *Let $\Theta, \Omega \subset \Delta$ and let $w \in [W_\Theta\backslash W/W_\Omega]$.*

1. *The standard parabolic subgroup $P_{\Theta\cap w\Omega}$ is equal to $(P_\Theta \cap {}^wP_\Omega)\,{}^wN_\Omega$.*
2. *The unipotent radical of $P_{\Theta\cap w\Omega}$ is generated by ${}^wN_\Omega$ and $N_\Theta \cap {}^wN_\emptyset$, where $N_\emptyset$ is the unipotent radical of the minimal parabolic subgroup corresponding to $\emptyset \subset \Delta$.*
3. *The standard Levi subgroup of $P_{\Theta\cap w\Omega}$ is $M_{\Theta\cap w\Omega} = M_\Theta \cap wM_\Omega w^{-1}$.*
4. *The subgroup $P_\Theta \cap {}^wM_\Omega$ is a $w\Omega$-standard parabolic in $M_{w\Omega} = {}^wM_\Omega$ with unipotent radical $N_\Theta \cap {}^wM_\Omega$ and standard Levi subgroup $M_{\Theta\cap w\Omega} = M_\Theta \cap {}^wM_\Omega$.*

2.1.3 A regularity condition and irreducibility of induced representations

Let Δ be a choice of simple roots for the root system Φ of G with respect to a fixed maximal F-split torus A. Let W be the Weyl group of G with respect to A. For any two subsets Θ and Ω of Δ, define

$$W(\Theta, \Omega) = \{w \in W : w\Omega = \Theta\}. \tag{2.6}$$

If $W(\Theta, \Omega)$ is nonempty, then the subsets Θ and Ω are said to be *associate*. Observe that if w is an element of $W(\Theta, \Omega)$ then ${}^wM_\Omega = M_\Theta$.

Let Θ be a subset of Δ and let P_Θ be the associated standard parabolic subgroup with Levi M_Θ and unipotent radical N_Θ. Let ρ be a smooth representation of M_Θ.

Definition 2.1.21. We say that a smooth representation ρ of M_Θ is *regular* if for any non-trivial element $w \in W(\Theta, \Theta)$, the twist ${}^w\rho = \rho \circ \operatorname{Int} w^{-1}$ is not equivalent to ρ, that is, ${}^w\rho \ncong \rho$.

Note that ρ is regular if and only if ${}^x\rho \ncong \rho$ for every $x \in N_G(M_\Theta)$ with non-trivial image in $N_G(M_\Theta)/M_\Theta$. This allows one to easily extend the definition of regular representations to nonstandard Levi subgroups of (nonstandard) parabolic subgroups.

The following result of Bruhat appears as [15, Theorem 6.6.1].

Theorem 2.1.22 (Bruhat)**.** *Let $P = MN$ be a parabolic subgroup of G with Levi subgroup M and unipotent radical N. If ρ is an irreducible admissible unitary regular representation of M, then the parabolically induced representation $\iota_P^G \rho$ of G is irreducible.*

2.1.4 Supercuspidal and square integrable representations

An irreducible smooth representation (π, V) is *supercuspidal* if every matrix coefficient $\phi_{\widetilde{v},v}$ of π has compact support modulo Z_G. Let ω be a unitary character of Z_G. A smooth ω-representation (π, V) is *square integrable (mod Z_G)* if every matrix coefficient $\phi_{\widetilde{v},v}$ of π is a square integrable function on $Z_G\backslash G$. The discrete spectrum of G consists of those representations that occur as direct summands of the space $L^2(Z_G\backslash G)$ of square integrable functions on $Z_G\backslash G$. A smooth representation π of G is *essentially square integrable* if there exists a quasi-character χ of G such that the tensor product $\pi \otimes \chi$ is square integrable. We refer to a representation of the form $\pi \otimes \chi$ as the *twist* of π by the character χ.

The reason to consider compactly supported and square integrable functions on $Z_G\backslash G$ rather than on G itself is the following. When π is irreducible, Schur's Lemma gives the existence of the central character $\omega_\pi : Z_G \to \mathbb{C}$ of π. Let $\widetilde{v} \in \widetilde{V}$ and $v \in V$ be nonzero vectors and consider the associated matrix coefficient $\phi_{\widetilde{v},v}$. By the irreducibility of π, there exists $g_0 \in G$ such that $\phi_{\widetilde{v},v}(g_0) \neq 0$. Observe that, for any $z \in Z_G$, we have

$$\phi_{\widetilde{v},v}(zg_0) = \omega_\pi(z)\phi_{\widetilde{v},v}(g_0) \neq 0$$

In general, the centre Z_G of G is non-compact; therefore, $\phi_{\widetilde{v},v}$ must be non-compactly supported on G. If π is an ω-representation, where ω is a unitary character, then the function $g \mapsto |\phi_{\widetilde{v},v}(g)|$ is a well-defined on $Z_G\backslash G$. Indeed, for $g \in G$ and $z \in Z_G$, we have

$$|\phi_{\widetilde{v},v}(zg)| = |\omega(z)\phi_{\widetilde{v},v}(g)| = |\phi_{\widetilde{v},v}(g)|,$$

where the final equality uses that ω is unitary. When Z_G is non-compact, the function $g \mapsto |\phi_{\widetilde{v},v}(g)|^2$ is not integrable on G, because it is constant on the Z_G-cosets. The quotient Z_G/A_G is compact; therefore, it suffices to consider integrability of functions on $A_G\backslash G$ rather than $Z_G\backslash G$.

2.1.5 Exponents and Casselman's Criterion

Let (π, V) be a finitely generated admissible representation of G. Recall A_G denotes the F-split component of the centre of G. Let ω be a quasi-character of A_G. For $n \in \mathbb{N}$, $n \geq 1$, define the subspace

$$V_{\omega,n} = \{v \in V : (\pi(z) - \omega(z))^n v = 0 \text{ for all } z \in A_G\}, \tag{2.7}$$

and set

$$V_{\omega,\infty} = \bigcup_{n=1}^{\infty} V_{\omega,n}. \tag{2.8}$$

Each $V_{\omega,n}$ is a G-stable subspace of V and $V_{\omega,\infty}$ is the generalized eigenspace of V for the A_G-action on V by the eigencharacter ω. We let $V_\omega = V_{\omega,1}$ be the ω-eigenspace for the action of A_G on V. Note that if $V = V_\omega$, then (π, V) is an ω-representation. Note that the same analysis as above can be carried out for any closed subgroup Z of Z_G, i.e., we can consider the generalized Z-eigenspaces in V.

By [15, Proposition 2.1.9], we have that

1. V is a direct sum $V = \bigoplus_{\chi} V_{\chi,\infty}$, where χ ranges over quasi-characters of A_G, and
2. since V is finitely generated, there are only finitely many χ such that $V_{\chi,\infty} \neq 0$. Moreover, there exists $n \in \mathbb{N}$ such that $V_{\chi,\infty} = V_{\chi,n}$, for each χ.

Definition 2.1.23. Let $\mathcal{E}xp_{A_G}(\pi)$ be the (finite) set of quasi-characters of A_G such that $V_{\chi,\infty} \neq 0$. The quasi-characters that appear in $\mathcal{E}xp_{A_G}(\pi)$ are called the *exponents* of π.

The second item above implies that V has a finite filtration such that the quotients are χ-representations, for $\chi \in \mathcal{E}xp_{A_G}(\pi)$. From this last observation, we obtain the following lemma.

Lemma 2.1.24. *The characters χ of A_G that appear in $\mathcal{E}xp_{A_G}(\pi)$ are precisely the central quasi-characters of the irreducible subquotients of π.*

Proof. It is straightforward to show that the irreducible subquotients of $V_{\chi,\infty}$ have central quasi-character χ (we give the details in Lemma 3.5.2). We obtain all irreducible subquotients of the finite length (finitely generated admissible) representation (π, V) this way. □

The following lemma will be applied in the next subsection.

Lemma 2.1.25. *Let $Z_1 \supset Z_2$ be two closed subgroups of the centre Z_G of G. The map of exponents $\mathcal{E}xp_{Z_1}(\pi) \to \mathcal{E}xp_{Z_2}(\pi)$ defined by restriction of quasi-characters is surjective.*

Proof. Let $\omega \in \mathcal{E}xp_{Z_2}(\pi)$. It suffices to prove that $\widehat{\omega} \in \mathcal{E}xp_{Z_1}(\pi)$, for some extension $\widehat{\omega}$ of ω to Z_1. By assumption, there exists a nonzero vector $v \in V_{\omega,\infty}$. In particular, there is an irreducible subquotient of $V_{\omega,\infty}$, hence of (π, V), where Z_2 acts by the character ω. On this irreducible subquotient, by Schur's Lemma, the subgroup Z_1 must act by some extension $\widehat{\omega}$ of ω. By Lemma 2.1.24, $\widehat{\omega}$ must occur in $\mathcal{E}xp_{Z_1}(\pi)$. □

Exponents of parabolically induced representations

Let $P = MN$ be a parabolic subgroup of G. Let (ρ, V_ρ) be a finitely generated admissible representation of the Levi M and define $\pi = \iota_P^G \rho$. By Proposition 2.1.9, π is a finitely generated admissible representation of G. We compute the exponents of π in terms of the exponents of ρ. Let V denote the space of π. We may apply [15, Proposition 2.1.9] to both (π, V) and (ρ, V_ρ) to study their exponents, as above.

Lemma 2.1.26. *Let $P = MN$ be a parabolic subgroup of G, let (ρ, V_ρ) be a finitely generated admissible representation of M and let $\pi = \iota_P^G \rho$. The quasi-characters $\chi \in \mathcal{E}xp_{A_G}(\pi)$ are the restriction to A_G of characters μ of A_M appearing in $\mathcal{E}xp_{A_M}(\rho)$.*

Proof. First note that if $P = G$, then $\pi = \rho$ and there is nothing to do. Without loss of generality, assume that $P = MN$ is a proper parabolic subgroup of G. Given $a \in A_G$, we have that $\delta_P(a) = 1$, since a is central in G. It follows that for any $f \in V$ we have

$$(\pi(a)f)(g) = f(ga) = f(ag) = \delta_P^{1/2}(a)\rho(a)f(g) = \rho(a)f(g), \tag{2.9}$$

for all $a \in A_G$ and $g \in G$. Suppose that $\chi \in \mathcal{E}xp_{A_G}(\pi)$ and $f \in V_{\chi,\infty}$ is nonzero. Fix $g_0 \in G$ such that $w_0 = f(g_0)$ is nonzero. There exists $n \in \mathbb{N}$, $n \geq 1$ such that $f \in V_{\chi,n}$. More precisely, $(\pi(a) - \chi(a))^n f = 0_V$, for all $a \in A_G$, where $0_V : G \to V_\rho$ is the zero function. By induction and using (2.9), we see that

$$0 = [(\pi(a) - \chi(a))^n f](g_0) = (\rho(a) - \chi(a))^n (f(g_0)) = (\rho(a) - \chi(a))^n w_0,$$

for any $a \in A_G$. That is, $w_0 \in (V_\rho)_{\chi,\infty}$ and $(V_\rho)_{\chi,\infty}$ is nonzero; moreover, $\chi \in \mathcal{E}xp_{A_G}(\rho)$. By Lemma 2.1.25, the map $\mathcal{E}xp_{A_M}(\rho) \to \mathcal{E}xp_{A_G}(\rho)$ defined by restriction is surjective. In particular, there exists $\mu \in \mathcal{E}xp_{A_M}(\rho)$ such that χ is equal to the restriction of μ to A_G. □

Exponents along parabolic subgroups

Let (π, V) be a finitely generated admissible representation of G. Let $P = MN$ be a parabolic subgroup of G with Levi factor M and unipotent radical N. By Theorem 2.1.16, the Jacquet module (π_N, V_N) of π along P is also finitely generated and admissible. Applying [15, Proposition 2.1.9], we obtain a direct sum decomposition

$$V_N = \bigoplus_{\chi \in \mathcal{E}xp_{A_M}(\pi_N)} (V_N)_{\chi,\infty} \tag{2.10}$$

where the set $\mathcal{E}xp_{A_M}(\pi_N)$ of quasi-characters of A_M, such that $(V_N)_{\chi,\infty} \neq 0$, is finite.

Definition 2.1.27. The quasi-characters of A_M appearing in $\mathcal{E}xp_{A_M}(\pi_N)$ are called the *exponents of π along P*.

Fix a choice of simple roots Δ for the root system of G with respect to a fixed maximal F-split torus A. For any $\epsilon > 0$, define $A_\Theta^-(\epsilon)$ to be the set

$$A_\Theta^-(\epsilon) = \{a \in A_\Theta : |\alpha(a)|_F \leq \epsilon \text{ for all } \alpha \in \Delta \setminus \Theta\} \tag{2.11}$$

and write A_Θ^- for $A_\Theta^-(1)$ (*cf.* (1.8)). The following theorem of Casselman [15, Theorem 4.4.6] completely characterizes square integrable representations in terms of their exponents along parabolic subgroups.

Theorem 2.1.28 (Casselman's Criterion)**.** *Let ω be a quasi-character of Z_G. If (π, V) is an admissible ω-representation of G, then π is square integrable (modulo Z_G) if and only if*

1. *ω is unitary, and*
2. *for every proper subset $\Theta \subset \Delta$ and every exponent $\chi \in \mathcal{E}xp_{A_\Theta}(\pi_{N_\Theta})$ of π along P_Θ, we have that $|\chi(a)| < 1$ for all $a \in A_\Theta^- \setminus A_\Theta(\mathcal{O}_F)A_G$.*

2.2 Distinguished representations

Let (π, V) be a smooth representation of G. Let H be a closed subgroup of G and let χ be a quasi-character of H. We suppress the notation $\mathrm{Res}_H^G \pi$ and let π also denote its restriction to H. We begin this section with a definition.

Definition 2.2.1. The representation π is said to be (H,χ)-*distinguished* if the space $\mathrm{Hom}_H(\pi,\chi)$ is nonzero. If χ is the trivial character of H, then we refer to $(H,1)$-distinguished representations simply as H-*distinguished*.

The elements of $\mathrm{Hom}_H(\pi,1)$ are H-invariant elements of the algebraic dual V^* of V and are not necessarily smooth vectors. By Frobenius Reciprocity, $\mathrm{Hom}_H(\pi,\chi)$ is naturally isomorphic to $\mathrm{Hom}_G(\pi,\mathrm{Ind}_H^G\chi)$. In particular, an irreducible smooth representation π is (H,χ)-distinguished if and only if it is equivalent to a subrepresentation of $\mathrm{Ind}_H^G\chi$. As noted in the introduction, it is exactly the H-distinguished representations of G that contribute to the harmonic analysis on the homogeneous space $H\backslash G$. We will explore this idea further when we discuss *relative matrix coefficients* below (*cf.* 2.2.1). First, we note some simple, but useful, preliminary results on distinguished representations.

Lemma 2.2.2. *If (π,V) is a smooth H-distinguished representation with central character ω_π, then ω_π is trivial on $H\cap Z_G$.*

Proof. By assumption, there exists $\lambda\in\mathrm{Hom}_H(\pi,1)$ and $v\in V$, such that $\langle\lambda,v\rangle\neq 0$. Let $z\in H\cap Z_G$, then we have that

$$0\neq\langle\lambda,v\rangle=\langle\lambda,\pi(z)v\rangle=\langle\lambda,\omega_\pi(z)v\rangle=\omega_\pi(z)\langle\lambda,v\rangle.$$

In particular, $0=(1-\omega_\pi(z))\langle\lambda,v\rangle$ and we see that $\omega_\pi(z)=1$ for all $z\in H\cap Z_G$. □

Lemma 2.2.3. *Let (π,V) be a finitely generated admissible representation of G. If no (irreducible) sub-quotient of (π,V) is H-distinguished, then (π,V) itself cannot be H-distinguished.*

We can also rephrase Lemma 2.2.3, as the contrapositive statement, as follows.

Lemma 2.2.4. *Let (π,V) be a finitely generated admissible representation of G. If (π,V) is H-distinguished, then there exists an (irreducible) H-distinguished sub-quotient of (π,V).*

Proof. If (π,V) is irreducible, then we're done. Without loss of generality, assume that (π,V) is reducible. Suppose that $\lambda\in\mathrm{Hom}_H(\pi,1)$ is nonzero. Let $\{0\}=V_1\subset V_2\subset\ldots\subset V_\ell=V$ be a (finite) filtration of V by G-stable subspaces. Denote each successive quotient by $V^i=V_i/V_{i-1}$, $1\leq i\leq\ell$. Such a filtration exists because (π,V) has finite length [15, Theorem 6.3.10]. Let k, $1\leq k\leq\ell$, be the smallest integer such that $\lambda|_{V_k}\neq 0$ and $\lambda|_{V_{k-1}}=0$. Then λ defines a nonzero H-invariant form on the quotient V^k. □

Suppose that $V'\subset V$ is a subrepresentation of V and that the quotient V/V' is H-distinguished. Let $\bar\lambda\in\mathrm{Hom}_H(V/V',1)$. We can define a nonzero H-invariant form λ on V by declaring $\lambda(v)=\bar\lambda(v+V')$. In particular, we have the following lemma.

Lemma 2.2.5. *If (π,V) has an H-distinguished quotient, then any H-invariant form on the quotient lifts to give an H-invariant form on V. In particular, (π,V) is H-distinguished.*

Let θ be an involution of G. There is a right action of G on the set of involutions given by $\theta\mapsto g\cdot\theta$, where

$$g\cdot\theta(x)=g^{-1}\theta(gxg^{-1})g=\mathrm{Int}\,g^{-1}\circ\theta\circ\mathrm{Int}\,g(x). \tag{2.12}$$

Lemma 2.2.6. *The subgroup $G^{g\cdot\theta}$, of $g\cdot\theta$-fixed points in G, is G-conjugate to G^θ. More precisely, $G^{g\cdot\theta} = g^{-1}(G^\theta)g$. Moreover, a smooth representation (π, V) of G is G^θ-distinguished if and only if π is $G^{g\cdot\theta}$-distinguished.*

Proof. Let $h \in G^\theta$, then we have that

$$g\cdot\theta(g^{-1}hg) = g^{-1}\theta(gg^{-1}hgg^{-1})g = g^{-1}\theta(h)g = g^{-1}hg$$

so $g^{-1}hg$ is $g\cdot\theta$-fixed. It follows that $g^{-1}(G^\theta)g \subset G^{g\cdot\theta}$. Since conjugation by g is an automorphism of G, it follows that $G^{g\cdot\theta} = g^{-1}(G^\theta)g$.

Let λ be a nonzero element of $\mathrm{Hom}_{G^\theta}(\pi, 1)$. Define $\lambda' = \lambda \circ \pi(g)$ and observe that λ' is another nonzero linear functional on V. Moreover, λ' is $G^{g\cdot\theta}$-invariant. It follows that the map $\lambda \mapsto \lambda \circ \pi(g)$ is a bijection from $\mathrm{Hom}_{G^\theta}(\pi, 1)$ to $\mathrm{Hom}_{G^{g\cdot\theta}}(\pi, 1)$ with inverse $\lambda' \mapsto \lambda' \circ \pi(g^{-1})$. In particular, π is G^θ-distinguished if and only if π is $G^{g\cdot\theta}$-distinguished. □

Of course, the second statement of the previous lemma holds more generally. For any closed subgroup H of G, a smooth representation π is H-distinguished if and only if π is ${}^gH$-distinguished, for all $g \in G$. The content of the lemma is that distinction, relative to θ, is determined only by the orbit of θ under the action of G on the set of involutions.

2.2.1 Relative matrix coefficients

Remark 2.2.7. For the remainder of this chapter, we specialize to the symmetric space setting. Fix an F-rational involution θ of $\mathbf{G}$. Let H be the F-points of the θ-fixed point subgroup $\mathbf{H} = \mathbf{G}^\theta$ of $\mathbf{G}$. We retain the notation and conventions of §1.5.

Let (π, V) be a smooth H-distinguished representation of G. Let $\lambda \in \mathrm{Hom}_H(\pi, 1)$ be a nonzero H-invariant linear form on V and let v be a nonzero vector in V. In analogy with the usual matrix coefficients, define a complex-valued function $\varphi_{\lambda,v}$ on G by $\varphi_{\lambda,v}(g) = \langle \lambda, \pi(g)v\rangle$. We refer to the functions $\varphi_{\lambda,v}$ as *relative matrix coefficients* (with respect to λ) or as λ-*relative matrix coefficients*. Since π is a smooth representation, the relative matrix coefficient $\varphi_{\lambda,v}$ lies in $C^\infty(G)$, for every $v \in V$. In addition, since λ is H-invariant, the functions $\varphi_{\lambda,v}$ descend to well-defined functions on the quotient $H\backslash G$. That is, for any H-invariant form λ the relative matrix coefficients $\{\varphi_{\lambda,v} : v \in V\}$ are smooth functions on the homogeneous space $H\backslash G$. Moreover, we have the following result.

Proposition 2.2.8. *Let (π, V) be an H-distinguished representation of G. Fix a nonzero element λ of $\mathrm{Hom}_H(\pi, 1)$. The map that sends a vector v in V to the λ-relative matrix coefficient $\varphi_{\lambda,v}$ is a nonzero intertwining operator from π to the right-regular representation of G on $C^\infty(H\backslash G)$.*

Proof. We have already seen that the map $v \mapsto \varphi_{\lambda,v}$ is a nonzero linear map from V to $C^\infty(H\backslash G)$. We now observe that $v \mapsto \varphi_{\lambda,v}$ is an intertwining operator. Let $g \in G$, then for any $x \in G$ we have

$$\varphi_{\lambda,\pi(g)v}(x) = \langle \lambda, \pi(x)\pi(g)v\rangle = \langle \lambda, \pi(xg)v\rangle = \varphi_{\lambda,v}(xg).$$

That is, $\varphi_{\lambda,\pi(g)v} = R_g\varphi_{\lambda,v}$, for any $v \in V$ and $g \in G$, where R_g denotes the operator on $C^\infty(H\backslash G)$ given by right-translation by g. □

Note that if π is irreducible and H-distinguished, then π may be realized as a sub-representation of the right-regular representation of G on $C^\infty(H\backslash G)$. In fact, this last remark is equivalent to Frobenius Reciprocity. Indeed, the right regular representation on $C^\infty(H\backslash G)$ is equal to the induced representation $\mathrm{Ind}_H^G 1_H$, where 1_H is the trivial character of H. On the other hand, any subrepresentation of $C^\infty(H\backslash G)$ is H-distinguished by the invariant linear form given by evaluating a function $f \in C^\infty(H\backslash G)$ on the identity coset, i.e., $f \mapsto f(He)$.

Let (π, V) be a smooth H-distinguished representation of G. Let $\lambda \in \mathrm{Hom}_H(\pi, 1)$ be a nonzero H-invariant linear form on V. In analogy with the classical case, one makes the following definitions.

Definition 2.2.9. The representation (π, V) is said to be

1. (H, λ)-*relatively supercuspidal* or *relatively supercuspidal with respect to* λ if and only if all of the λ-relative matrix coefficients are compactly supported modulo $Z_G H$.

2. H-*relatively supercuspidal* or simply *relatively supercuspidal* if and only if π is (H, λ)-relatively supercuspidal for every $\lambda \in \mathrm{Hom}_H(\pi, 1)$.

Let ω be a unitary character of Z_G and further suppose that π is an ω-representation.

Definition 2.2.10. The representation (π, V) is said to be

1. (H, λ)-*relatively square integrable* or *relatively square integrable with respect to* λ if and only if all of the λ-relative matrix coefficients are square integrable modulo $Z_G H$.

2. H-*relatively square integrable* or simply *relatively square integrable* if and only if π is (H, λ)-relatively square integrable for every $\lambda \in \mathrm{Hom}_H(\pi, 1)$.

As above (*cf.* §2.1.4), since ω is unitary, the function $g \mapsto |\varphi_{\lambda,v}(g)|$ is well-defined on $Z_G H\backslash G$ for any $v \in V$. To make sense of integrating relative matrix coefficients we need a G-invariant measure on the quotient $Z_G H\backslash G$. The centre Z_G of G is unimodular since it is abelian. The fixed point subgroup H is also reductive [19, Theorem 1.8] and thus unimodular. It follows that there exists a G-invariant measure dg on the quotient $Z_G H\backslash G$ by [64, Proposition 12.8]. If (π, V) is (H, λ)-relatively square integrable then the morphism sending v to the relative matrix coefficient $\varphi_{\lambda,v}$ maps V into the space of square integrable (modulo Z_G) functions on $H\backslash G$. We let $L^2(H\backslash G)$ denote the space of (equivalence classes) of square integrable functions on $Z_G H\backslash G$. Explicitly, a function $f : Z_G H\backslash G \to \mathbb{C}$ lies in $L^2(H\backslash G)$ if and only if the integral

$$\int_{Z_G H\backslash G} |f(g)|^2 \, dg$$

is finite.

Definition 2.2.11. If (π, V) is an irreducible subrepresentation of $L^2(H\backslash G)$, then we say that (π, V) occurs in the *discrete spectrum* of $H\backslash G$. In this case, we say that (π, V) is a *relative discrete series (RDS)* representation.

2.2.2 The work of Kato and Takano

The main result of Kato and Takano's paper [44] is a generalization of Jacquet's Subrepresentation Theorem to the symmetric space setting. In particular, their result [44, Theorem 7.1] recovers Jacquet's theorem when applied to the group case. In [45], Kato and Takano prove a symmetric space version of Casselman's Criterion 2.1.28 that also recovers Casselman's result in the group case.

Invariant forms on Jacquet modules

Let (π, V) be an admissible H-distinguished representation of G. Let $\lambda \in \mathrm{Hom}_H(\pi, 1)$ be nonzero. Let $P = MN$ be a θ-split parabolic subgroup of G with θ-stable Levi subgroup M and unipotent radical N. The main ingredient in Kato and Takano's work is the construction of an M^θ-invariant linear form $r_P\lambda$ on the Jacquet module π_N, canonically associated to the invariant form λ. The construction of $r_P\lambda$, which we now review, was discovered independently by Lagier [48] and Kato–Takano [44].

Fix a maximal θ-stable F-split torus A_0 containing a maximal (θ, F)-split torus S_0. Let Δ_0 be a θ-base of the root system Φ_0 of G with respect to A_0. Assume that $P = P_\Theta$ is a Δ_0-standard θ-split parabolic subgroup of G corresponding to a θ-split subset Θ of Δ_0. In order to define the M^θ-invariant form $r_P\lambda$, we first require the following [44, Lemma 4.3] (*cf.* [15, Proposition 1.4.4]).

Lemma 2.2.12 (Kato–Takano)**.** *There exists a decreasing sequence $\{K_n\}_{n\geq 0}$ of θ-stable compact open subgroups of G satisfying the following properties.*

1. *The family $\{K_n\}_{n\geq 0}$ gives a neighbourhood base of the identity in G.*

2. *For each $n \geq 1$, the group K_n is normal in K_0 and the quotient group K_n/K_{n+1} is a finite abelian p-group, where p is the characteristic of the residue field k_F.*

3. *For each K_n, $n \geq 1$, and each Δ_0-standard θ-split parabolic subgroup P_Θ of G, K_n has Iwahori factorization with respect to P_Θ, and*

$$s(N_\Theta \cap K_n)s^{-1} \subset (N_\Theta \cap K_n), \quad s^{-1}(N_\Theta^{op} \cap K_n)s \subset (N_\Theta^{op} \cap K_n),$$

for all $s \in S_\Theta^-$.

4. *For each Δ_0-standard θ-split parabolic subgroup P_Θ of G, the family $\{M_\Theta \cap K_n\}_{n\geq 0}$ satisfies properties (1)–(3) for the group M_Θ.*

Kato and Takano refer to a family of θ-stable compact open subgroups $\{K_n\}_{n\geq 0}$ satisfying (1)–(4) of Lemma 2.2.12 as *adapted* to the data (S_0, A_0, Δ_0). The construction of $r_P\lambda$ depends on Casselman's Canonical Lifting [15, Proposition 4.1.4] along a family of θ-stable compact open subgroups $\{K_n\}_{n\geq 0}$ adapted to (S_0, A_0, Δ_0). We now recall Casselman's Canonical Lifting following [44, §5.2]. For any compact open subgroup K of G, the projection operator $\mathcal{P}_K$ from V to the subspace of V^K of K-fixed vectors is given by the integral

$$\mathcal{P}_K(v) = \frac{1}{\mathrm{vol}(K)} \int_K \pi(k)v \; dk,$$

for any $v \in V$. One can show that $V(N)$ is equal to the union, over compact open subgroups N_1 of N, of the sets

$$V_0(N_1) = \left\{ v \in V : \int_{N_1} \pi(n)v \; dn = 0 \right\}.$$

Fix an adapted family $\{K_n\}_{n\geq 0}$ as in Lemma 2.2.12. Let $[v] \in V_N$ and take a θ-stable compact open subgroup $K = K_n$ from the adapted family such that $[v] \in (V_N)^{M\cap K}$. There exists a compact open subgroup N_0 of N such that $V^K \cap V(N)$ is contained in $V_0(N_0)$. By [44, Lemma 2.8], there exists a positive real number $0 < \epsilon \leq 1$ such that sN_0s^{-1} is contained in $N \cap K$, for all $s \in S_M^-(\epsilon)$. Note that since $S_M^-(\epsilon) \subset A_M^-(\epsilon)$, one may replace $A_M^-(\epsilon)$ by in $S_M^-(\epsilon)$ Casselman's argument. For any $s \in S_M^-(\epsilon)$, the projection of V onto V_N induces an isomorphism from $\mathscr{P}_K\left(\pi(s)V^K\right)$ to $(V_N)^{M\cap K}$. The vector $v \in \mathscr{P}_K(\pi(s)V^K)$ that satisfies $v + V(N) = [v]$ is called the *canonical lift* of $[v]$ with respect to K. Casselman proves that v depends on the choice of K, but the canonical lift v of $[v]$ does not depend on N_0 nor ϵ. By [15, Proposition 4.1.8], if v' is any other canonical lift of $[v]$ with respect to a subgroup K' of K, then $v' \in V^{(M\cap K)(N^{op}\cap K)}$ and $v = \mathscr{P}_K(v') = \mathscr{P}_{N\cap K}(v')$. The following is [44, Proposition 5.3].

Proposition 2.2.13 (Kato–Takano). *Let λ be an H-invariant linear form on an admissible representation (π, V) of G and let $P = MN$ be a Δ_0-standard θ-split parabolic subgroup.*

1. *For $K = K_n$, $n \geq 1$, in the adapted family $\{K_n\}_{n\geq 0}$ and $v \in V^{(M\cap K)(N^{op}\cap K)}$, we have that $\langle \lambda, v\rangle = \langle \lambda, \mathscr{P}_{N\cap K}(v)\rangle$.*

2. *Given $[v] \in V_N$, for any two canonical lifts $v, v' \in V$, we have $\langle \lambda, v\rangle = \langle \lambda, v'\rangle$.*

Definition 2.2.14 (Kato–Takano, Lagier). Let $P = MN$ be a θ-split parabolic subgroup of G with Levi subgroup M and unipotent radical N. Let $\lambda \in \mathrm{Hom}_H(\pi, 1)$ be nonzero. Define the linear form $r_P\lambda$ on V_N by declaring that

$$\langle r_P\lambda, [v]\rangle = \langle \lambda, v\rangle,$$

where $v \in V$ is any canonical lift of $[v] \in V_N$ with respect to an adapted family $\{K_n\}_{n\geq 0}$.

By Proposition 2.2.13, the linear form $r_P\lambda$ is well defined and does not depend on the choice of $\{K_n\}_{n\geq 0}$, nor on the choice of canonical lift. Kato and Takano go on to prove the following proposition (*cf.* [44, Proposition 5.6]), which guarantees that $r_P\lambda$ has the desired invariance properties.

Proposition 2.2.15 (Kato–Takano). *Let λ be an H-invariant linear form on an admissible representation (π, V) of G and let $P = MN$ be a θ-split parabolic subgroup of G.*

1. *The linear form $r_P\lambda$ on V_N is M^θ-invariant.*

2. *The mapping $r_P : \mathrm{Hom}_H(\pi, 1) \to \mathrm{Hom}_{M^\theta}(\pi_N, 1)$, sending λ to $r_P\lambda$, is linear.*

We omit a discussion of the precise asymptotic results and the transitivity properties of $r_P\lambda$; however, we do note the following two results which are the main theorems of [44]. The first theorem [44, Theorem 6.2] gives a characterization of relative supercuspidality. The second result [44, Theorem 7.1] is a symmetric space version of Jacquet's Subrepresentation Theorem.

Theorem 2.2.16 (Kato–Takano). *Let (π, V) be an admissible H-distinguished representation of G and let λ be a nonzero H-invariant linear form on V. Then, (π, V) is (H, λ)-relatively supercuspidal if and only if $r_P\lambda = 0$ for every proper θ-split parabolic subgroup P of G.*

Theorem 2.2.17 (Relative Jacquet's Subrepresentation Theorem, Kato–Takano). *Let π be an irreducible admissible H-distinguished representation of G. There exists a θ-split parabolic subgroup $P = MN$ of G and an irreducible M^θ-relatively supercuspidal representation ρ of M such that π is equivalent to a subrepresentation of $\iota_P^G\rho$.*

Exponents relative to invariant forms and the Relative Casselman's Criterion

We recall the main result of [45] as Theorem 2.2.18, which is a generalization of Casselman's Criterion 2.1.28 to the symmetric space setting. Kato and Takano's result characterizes (H,λ)-relatively square integrable representations in terms of exponents along proper θ-split parabolic subgroups.

Suppose, that (π, V) is a finitely generated admissible and H-distinguished representation of G. Fix a nonzero H-invariant form λ on V. For any closed subgroup Z of the centre of G, Kato and Takano define

$$\mathcal{E}xp_Z(\pi,\lambda) = \{\chi \in \mathcal{E}xp_Z(\pi) : \lambda|_{V_{\chi,\infty}} \neq 0\}, \tag{2.13}$$

and refer to the set $\mathcal{E}xp_Z(\pi,\lambda)$ as *exponents of* π *relative to* λ. The following result is [45, Theorem 4.7].

Theorem 2.2.18 (The Relative Casselman's Criterion, Kato–Takano)**.** *Let ω be a unitary character of Z_G. Let (π, V) be a finitely generated admissible H-distinguished ω-representation of G. Fix a nonzero H-invariant linear form λ on V. The representation (π, V) is (H,λ)-relatively square integrable if and only if the condition*

$$|\chi(s)| < 1 \qquad \text{for all } \chi \in \mathcal{E}xp_{S_M}(\pi_N, r_P\lambda) \text{ and all } s \in S_M^- \setminus S_G S_M^1 \tag{2.14}$$

is satisfied for every proper θ-split parabolic subgroup $P = MN$ of G.

Remark 2.2.19. In (2.14), we write S_M^1 for the integer points $S_M(\mathcal{O}_F)$ of the (θ, F)-split component of M. For the definition of S_M^-, refer to §1.6.

The following, is an immediate corollary of the Relative Casselman's Criterion.

Corollary 2.2.20 (Kato–Takano)**.** *If (π, V) is an H-distinguished discrete series representation of G, then π is (H,λ)-relatively square integrable with respect to any H-invariant form $\lambda \in \mathrm{Hom}_H(\pi, 1)$.*

2.3 Distinguished discrete series in the linear and Galois cases

In this section, we recall work of Gurevich and Offen [27] and Zhang [76] on producing explicit invariant linear forms on discrete series representations. We are most interested in their results on the linear and Galois cases; therefore, we do not give a comprehensive survey of their work. We also recall several results of Anandavardhanan–Kable–Tandon [2], Anandavardhanan–Rajan [5], Kable [43], and Matringe [50, 51, 53] on the distinguished discrete series representations that form the inducing data for our construction in Theorem 4.2.1. Many of the known results on distinguished discrete series are phrased in terms of the existence of poles of particular L-functions. We also discuss multiplicity-one results, and certain symmetry properties of distinguished representations, in the linear [40] and Galois cases [21].

2.3.1 Invariant forms by integrating matrix coefficients

Let (π, V) be an admissible representation of G. Following [76], define a pairing

$$\begin{aligned} &\mathscr{L} : \widetilde{V} \times V \to \mathbb{C} \\ &(\widetilde{v}, v) \mapsto \mathscr{L}_{(\widetilde{v},v)} = \int_{H/Z_H} \phi_{\widetilde{v},v}(h)\, dh. \end{aligned} \tag{2.15}$$

The quantity $\mathscr{L}_{(\widetilde{v},v)}$ in (2.15) is often called the H-*integral* of the matrix coefficient $\phi_{\widetilde{v},v}$. If $\mathscr{L}_{(\widetilde{v},v)}$ is well-defined, then we say that the matrix coefficient $\phi_{\widetilde{v},v}$ is H-*integrable*. Moreover, if the pairing $\mathscr{L}$ is well-defined on $\widetilde{V} \times V$, then for any $\widetilde{v} \in \widetilde{V}$ define $\ell_{\widetilde{v},H} \in \mathrm{Hom}_H(\pi, 1)$ by $\langle \ell_{\widetilde{v},H}, v\rangle = \mathscr{L}_{(\widetilde{v},v)}$. Define the subspace

$$\mathscr{H}(\pi) = \{\ell_{\widetilde{v},H} : \widetilde{v} \in \widetilde{V}\} \tag{2.16}$$

of $\mathrm{Hom}_H(\pi, 1)$.

Remark 2.3.1. The pairing $\mathscr{L}$ is not well-defined for every (G, H) (*cf.* [76, Remark 1.9]).

In [27], Gurevich and Offen give a list of symmetric pairs (G, H) for which the pairing $\mathscr{L}$ is well defined either on all tempered representations or all discrete series representations of G. In particular, the linear and Galois symmetric spaces appear in this list. However, for these two spaces, Gurevich and Offen do not obtain non-vanishing of the H-integrals (2.15) of the matrix coefficients of discrete series. Zhang proves (*cf.* [76, Proposition 3.3]) that the symmetric pairs that appear in the list of [27] satisfy the assumption of [76, Theorem 1.4], which guarantees the desired non-vanishing result. The next theorem follows directly from this last result.

Theorem 2.3.2 (Gurevich–Offen, Zhang)**.** *Let $H\backslash G$ be either $\mathbf{GL}_n(F) \times \mathbf{GL}_n(F)\backslash\mathbf{GL}_{2n}(F)$ (linear case) or $\mathbf{GL}_n(F)\backslash\mathbf{GL}_n(E)$ (Galois case). If π is a discrete series representation of G, then $\mathrm{Hom}_H(\pi, 1)$ is equal to $\mathscr{H}(\pi)$.*

In either the linear or Galois case, let (π, V) be an H-distinguished discrete series representation of G. By Theorem 2.3.2, one has non-vanishing of $\mathscr{H}(\pi)$ (*cf.* (2.16)) and exhaustion of the H-invariant forms on V by H-integrals of matrix coefficients. For Gurevich and Offen's list of symmetric pairs that satisfy the analogous result, we refer the reader to [27, Corollary to Theorem 4.4].

2.3.2 Distinction of discrete series in the linear case

In this subsection, unless otherwise noted, we let $G = \mathbf{GL}_n(F)$, where n is even, and let $H = G^\theta$ be isomorphic to $\mathbf{GL}_{n/2}(F) \times \mathbf{GL}_{n/2}(F)$. Here, we survey the current state of knowledge on distinction of discrete series in the linear case. We assemble an existence result, Corollary 2.3.17, which we apply in Chapter 4 (*cf.* Proposition 4.2.8). First, for any $m \geq 2$, we note that multiplicity-one holds for representations of $\mathbf{GL}_m(F)$ distinguished by a maximal Levi subgroup. The following is a result of Jacquet and Rallis [40].

Proposition 2.3.3 (Jacquet–Rallis)**.** *Let M be a maximal Levi subgroup of $\mathbf{GL}_m(F)$. If π is an irreducible admissible representation of $\mathbf{GL}_m(F)$, then $\dim \mathrm{Hom}_M(\pi, 1) \leq 1$. Furthermore, if π is M-distinguished, then π is equivalent to its contragredient $\widetilde{\pi}$.*

Let π be a discrete series representation of $\mathbf{GL}_m(F)$, $m \geq 2$. Denote by $L(s, \pi \times \pi)$ the local Rankin–Selberg convolution L-function. It is well known that $L(s, \pi \times \pi)$ has a simple pole at $s = 0$ if and only if π is self-contragredient [38]. By [69, Lemma 3.6], we have a local identity

$$L(s, \pi \times \pi) = L(s, \pi, \wedge^2)L(s, \pi, \mathrm{Sym}^2), \tag{2.17}$$

where $L(s,\pi,\wedge^2)$, respectively $L(s,\pi,\mathrm{Sym}^2)$, denotes the exterior square, respectively symmetric square, L-function of π defined via the Local Langlands Correspondence (LLC). It is also well known, see [12, 46] for instance, that $L(s,\pi,\wedge^2)$ cannot have a pole when m is odd.

Remark 2.3.4. It is now known that for all discrete series, and when n is even all irreducible generic representations, the Jacquet–Shalika and Langlands–Shahidi local exterior square L-functions agree with with the exterior square L-function defined via the LLC [35, Theorem 4.3 in §4.2], [46, Theorems 1.1 and 1.2].

For a positive integer m, let $S_{2m} \cong \mathbf{GL}_m(F) \ltimes N_{(m,m)}$ be the *Shalika subgroup* of $\mathbf{GL}_{2m}(F)$ of the form

$$S_{2m} = \left\{ \begin{pmatrix} g & X \\ 0 & g \end{pmatrix} : g \in \mathbf{GL}_m(F), X \in \mathbf{M}_m(F) \right\}.$$

Fix a non-trivial (additive) character ψ of F and define a character Ψ of S_{2m} by

$$\Psi \begin{pmatrix} g & X \\ 0 & g \end{pmatrix} = \psi(\mathrm{tr}\,(g^{-1}X)).$$

Definition 2.3.5. A smooth representation (π, V) of $\mathbf{GL}_{2m}(F)$ admits a *Shalika model* if it is (S_{2m},Ψ)-distinguished. A nonzero linear form in $\mathrm{Hom}_{S_{2m}}(\pi,\Psi)$ is called a *Shalika functional*. If $m = 1$, then a Shalika model is a Whittaker model (*cf.* Definition 2.3.9).

Remark 2.3.6. It is known that an irreducible square integrable representation π of G is H-distinguished if and only if π admits a Shalika model. It was shown by Jacquet and Rallis [40] that if π is an irreducible admissible representation of G that admits a Shalika model, then π is H-distinguished. In particular, an H-invariant form may be constructed explicitly via the Shalika model of π [40]. For irreducible supercuspidal representations, the converse appears as [42, Theorem 5.5]. Independently, Sakellaridis–Venkatesh and Matringe proved the converse result for relatively integrable and relatively square integrable representations by the technique of "unfolding" [66, Example 9.5.2], [53, Theorem 5.1]. In fact, Sakellaridis–Venkatesh prove that there is an equivariant unitary isomorphism between $L^2(H\backslash G)$ and $L^2(S\backslash G)$, where $S = S_n$ is the Shalika subgroup. Several analogous global results appear in [24].

Next, we record [46, Corollary 7.1].

Theorem 2.3.7 (Kewat–Raghunathan)**.** *Let π be an irreducible smooth square integrable representation of G which has no nonzero Shalika functional (is not H-distinguished). The symmetric square L-function $L(s,\pi,\mathrm{Sym}^2)$ has a pole at $s = 0$ if and only if $\widetilde{\pi} \cong \pi$ is self-contragredient.*

In the paper [53], Matringe discusses the relationship between H-distinction of representations of G, the existence of Shalika models, and the exterior square L-function. We first note the following result [53, Theorem 3.1].

Theorem 2.3.8 (Matringe)**.** *Let π be a discrete series representation of $\mathbf{GL}_m(F)$, let M be a maximal Levi subgroup of $\mathbf{GL}_m(F)$ and let χ be a quasi-character of M. If π is (M,χ)-distinguished, then m is even and M is isomorphic to $\mathbf{GL}_{m/2}(F) \times \mathbf{GL}_{m/2}(F)$.*

Matringe also proves the following generalization [53, Theorem 3.2] of the previous theorem. Before stating Matringe's result, we recall the notion of a generic representation.

Definition 2.3.9. A representation (π, V) of $\mathbf{GL}_m(F)$ is said to be *generic* if it admits a Whittaker model, that is, an embedding into $\operatorname{Ind}_N^G \psi_N$, where N is a maximal unipotent subgroup of $\mathbf{GL}_m(F)$ and ψ_N is a non-degenerate character of N.

Theorem 2.3.10 (Matringe). *Let π be a generic representation of $\mathbf{GL}_m(F)$, let M be a maximal Levi subgroup of $\mathbf{GL}_m(F)$ and let χ be a quasi-character of M. If π is (M,χ)-distinguished, then*

1. *$M \cong M_{(k,k)}$, if $m = 2k$ is even, and*

2. *$M \cong M_{(\frac{m+1}{2},\frac{m-1}{2})}$, if m is odd.*

Two results that we want to highlight in this section, which characterize the representations that form the inducing data in the linear case of Theorem 4.2.1, appear as Proposition 6.1 and Theorem 6.1 of [53], respectively.

Theorem 2.3.11 (Matringe). *Suppose that π is a square integrable representation of G, then π is H-distinguished if and only if the exterior square L-function $L(s, \pi, \wedge^2)$ has a pole at $s = 0$.*

Let ρ be an irreducible unitary supercuspidal representation of $\mathbf{GL}_r(F)$, $r \geq 1$. For an integer $k \geq 2$, write $\mathrm{St}(k, \rho)$ for the unique irreducible (unitary) quotient of the parabolically induced representation $\nu^{\frac{1-k}{2}}\rho \times \nu^{\frac{3-k}{2}}\rho \times \ldots \times \nu^{\frac{k-1}{2}}\rho$ of $\mathbf{GL}_{kr}(F)$ (*cf.* [75, Proposition 2.10, §9.1]), where $\nu(g) = |\det(g)|_F$, for any $g \in \mathbf{GL}_r(F)$. The representations $\mathrm{St}(k, \rho)$ are often called *generalized Steinberg representations*. The representations $\mathrm{St}(k, \rho)$ are exactly the nonsupercuspidal discrete series representations of $\mathbf{GL}_{kr}(F)$ [75, Theorem 9.3]. The usual Steinberg representation St_n of $\mathbf{GL}_n(F)$ is obtained as $\mathrm{St}(n, 1)$. Using Zelevinsky's classification of generic representations [75, Theorem 9.7], the previous result can be refined to obtain the next theorem.

Theorem 2.3.12 (Matringe). *Suppose that $n = kr$ is even. Let ρ be an irreducible supercuspidal representation of $\mathbf{GL}_r(F)$. Let $\pi = \mathrm{St}(k, \rho)$ be a generalized Steinberg representation of G.*

1. *If k is odd, then r must be even, and π is H-distinguished if and only if $L(s, \rho, \wedge^2)$ has a pole at $s = 0$ if and only if ρ is $\mathbf{GL}_{r/2}(F) \times \mathbf{GL}_{r/2}(F)$-distinguished*

2. *If k is even, then π is H-distinguished if and only if $L(s, \rho, \mathrm{Sym}^2)$ has a pole at $s = 0$.*

It is well known that for $G = \mathbf{GL}_2(F)$, an infinite dimensional representation π is H-distinguished if and only if π has trivial central character [60, Lemma 1]. The next lemma follows immediately.

Lemma 2.3.13. *Assume that $n = 2$. There are four pairwise inequivalent $\mathbf{GL}_1(F) \times \mathbf{GL}_1(F)$-distinguished unitary twists of the Steinberg representation St_2 of $\mathbf{GL}_2(F)$.*

Proof. The Steinberg representation St_2 of $\mathbf{GL}_2(F)$ has trivial central character and so it is $\mathbf{GL}_1(F) \times \mathbf{GL}_1(F)$-distinguished. A twist $\chi \otimes \mathrm{St}_2$ of St_2 by a quasi-character χ of $F^\times$ has trivial central character if and only if χ is trivial on $(F^\times)^2$. If χ is square-trivial then χ is effectively a character of the finite group $F^\times/(F^\times)^2$ which is isomorphic to the Klein 4-group [67]. In particular, χ is unitary and there are four distinct such characters. Finally, if χ and η are two distinct characters of $F^\times$, then $\chi \otimes \mathrm{St}_2 \not\cong \eta \otimes \mathrm{St}_2$ [11]. ☐

Lemma 2.3.14. *Let $\eta : F^\times \to \mathbb{C}^\times$ be a character of $F^\times$ that is trivial on $(F^\times)^2$. The twisted Steinberg representation $\eta\,\mathrm{St}_4 = \eta \otimes \mathrm{St}_4$ of $\mathbf{GL}_4(F)$ is $\mathbf{GL}_2(F) \times \mathbf{GL}_2(F)$-distinguished.*

Proof. By [24, Corollary 8.5(ii)], a twisted Steinberg representation $\eta\,\mathrm{St}_4$ admits a Shalika model if and only if $\eta^2 = 1$, that is if and only if η is trivial on $(F^\times)^2$. It follows that $\eta\,\mathrm{St}_4$ is $\mathbf{GL}_2(F) \times \mathbf{GL}_2(F)$-distinguished by Theorem 2.3.12 (*cf.* Remark 2.3.6). ☐

Following the proof of Lemma 2.3.13, there are four pairwise inequivalent twists of the Steinberg representation of $\mathbf{GL}_4(F)$ that are $\mathbf{GL}_2(F) \times \mathbf{GL}_2(F)$-distinguished.

Finally, we note a result on supercuspidal representations, due to Hakim and Murnaghan, key in proving (the linear case of) the existence result Proposition 4.2.8. The following is a corollary of [56, Proposition 10.1] and [29, Theorem 1.3].

Theorem 2.3.15 (Murnaghan, Hakim–Murnaghan)**.** *Let $n \geq 2$ be an even integer. Let $G = \mathbf{GL}_n(F)$ and $H \cong \mathbf{GL}_{n/2}(F) \times \mathbf{GL}_{n/2}(F)$. There exist*

1. *infinitely many equivalence classes of H-distinguished irreducible tame supercuspidal representations of G, and*
2. *infinitely many equivalence classes of self-contragredient irreducible tame supercuspidal representations of G that are not H-distinguished.*

Proof. One can show that [56, Proposition 10.1] may be applied in the linear case. It follows that there is an infinite sequence ρ_i of H-distinguished tame supercuspidal representations (obtained via Howe's construction), such that if $i \neq j$, then the Moy–Prasad depth of ρ_i is different from the depth of ρ_j. In particular, the ρ_i are pairwise inequivalent. The result now follows immediately from [29, Theorem 1.3]. ☐

The following is an immediate corollary of Theorem 2.3.15 and Theorem 2.3.12.

Proposition 2.3.16. *For any even integer $n \geq 6$, there are infinitely many equivalence classes of $\mathbf{GL}_{n/2}(F) \times \mathbf{GL}_{n/2}(F)$-distinguished nonsupercuspidal discrete series representations of $\mathbf{GL}_n(F)$.*

Proof. We give the proof in three cases:

1. $n = 2k$, where k is an odd prime,
2. $n \geq 8$ is a power of two, and
3. $n \geq 12$ is any other even integer.

The main ingredient that we need is the following fact: two generalized Steinberg representations $\mathrm{St}(k_1, \rho_1)$ and $\mathrm{St}(k_2, \rho_2)$ of $\mathbf{GL}_n(F)$ are equivalent if and only if $k_1 = k_2$ and ρ_1 is equivalent to ρ_2 [75, Theorem 9.7(b)]. Of course, the generalized Steinberg representations $\mathrm{St}(k, \rho)$ are nonsupercuspidal discrete series representations.

1. Let $n = 2k$, where k is an odd prime. By Theorem 2.3.15(1), there are infinitely many equivalence classes of $\mathbf{GL}_1(F) \times \mathbf{GL}_1(F)$-distinguished irreducible self-contragredient supercuspidal representations ρ of $\mathbf{GL}_2(F)$. Since k is odd, by Theorem 2.3.12 and [75, Theorem 9.7(b)] there are infinitely many equivalence classes of generalized Steinberg representations of $\mathbf{GL}_n(F)$ of the form $\mathrm{St}(k, \rho)$.

2. Let $n \geq 8$ be a power of two. By Theorem 2.3.15(2), there are infinitely many equivalence classes of irreducible self-contragredient supercuspidal representations ρ of $\mathbf{GL}_{n/2}(F)$ that are not $\mathbf{GL}_{n/4}(F) \times \mathbf{GL}_{n/4}(F)$-distinguished. Given such a ρ, the Rankin–Selberg L-function has a pole at $s = 0$ since $\rho \cong \widetilde{\rho}$; therefore, precisely one of the exterior and symmetric square L-functions of ρ has a pole at $s = 0$. Since ρ is not distinguished, by Theorem 2.3.11, $L(s, \rho, \wedge^2)$ does not have a pole at $s = 0$; therefore, $L(s, \rho, \mathrm{Sym}^2)$ has a pole at $s = 0$. By Theorem 2.3.12 and [75, Theorem 9.7(b)] there are infinitely many equivalence classes of generalized Steinberg representations of $\mathbf{GL}_n(F)$ of the form $\mathrm{St}(2, \rho)$.

3. Let $n \geq 12$ be any even integer that is not a power of two nor twice an odd prime. If $n = 2^m k$ where $m \geq 2$ and $k \geq 3$ is odd, then by Theorem 2.3.12, Theorem 2.3.15(1) and [75, Theorem 9.7(b)] there are infinitely many equivalence classes of $\mathbf{GL}_{n/2}(F) \times \mathbf{GL}_{n/2}(F)$-distinguished generalized Steinberg representations of the form $\mathrm{St}(k, \rho)$, where ρ is an irreducible $\mathbf{GL}_{2^{m-1}}(F) \times \mathbf{GL}_{2^{m-1}}(F)$-distinguished supercuspidal representation of $\mathbf{GL}_{2^m}(F)$. Otherwise, $n = 2kr$ where $k, r \geq 3$ are both odd. By Theorem 2.3.12, Theorem 2.3.15(1) and [75, Theorem 9.7(b)] there are infinitely many equivalence classes of $\mathbf{GL}_{n/2}(F) \times \mathbf{GL}_{n/2}(F)$-distinguished generalized Steinberg representations of the form $\mathrm{St}(k, \rho)$, where ρ is an irreducible $\mathbf{GL}_r(F) \times \mathbf{GL}_r(F)$-distinguished supercuspidal representation of $\mathbf{GL}_{2r}(F)$.

□

Corollary 2.3.17. *For any even integer $n \geq 2$, there are infinitely many equivalence classes of $\mathbf{GL}_{n/2}(F) \times \mathbf{GL}_{n/2}(F)$-distinguished discrete series representations of $\mathbf{GL}_n(F)$. Moreover,*

1. *if $n = 2$, there are exactly four $\mathbf{GL}_1(F) \times \mathbf{GL}_1(F)$-distinguished twists of the Steinberg representation St_2 of $\mathbf{GL}_2(F)$;*

2. *if $n = 4$, there are exactly four $\mathbf{GL}_2(F) \times \mathbf{GL}_2(F)$-distinguished twists of the Steinberg representation St_4 of $\mathbf{GL}_4(F)$.*

3. *if $n \geq 6$, there are infinitely many equivalence classes of $\mathbf{GL}_{n/2}(F) \times \mathbf{GL}_{n/2}(F)$-distinguished nonsupercuspidal discrete series representations of $\mathbf{GL}_n(F)$.*

Proof. The main statement follows from Theorem 2.3.15. Corollary 2.3.17(1) follows from Lemma 2.3.13. Corollary 2.3.17(2) follows from Lemma 2.3.14. Finally, Corollary 2.3.17(3) is exactly Proposition 2.3.16. □

Remark 2.3.18. It is known that an irreducible supercuspidal representation ρ of G is H-distinguished if and only if ρ is a local Langlands functorial transfer from $SO_{n+1}(F)$ [42, Theorem 5.5]. The present author is unaware if this result has yet been extended to square integrable, respectively generic representations.

2.3.3 Distinction of discrete series in the Galois case

In this subsection, unless otherwise noted, let $G = R_{E/F}\mathbf{GL}_n(F)$, where $n \geq 2$. We identify G with $\mathbf{GL}_n(E)$. Let $H = \mathbf{GL}_n(F)$ be the subgroup of Galois fixed points in G for the (entry-wise) Galois involution θ. Here, we survey the current state of knowledge on the distinction of discrete series in the Galois case and assemble an existence result, Proposition 2.3.29, which we apply in Chapter 4 (*cf.* Proposition 4.2.8).

Definition 2.3.19. A smooth representation π of G is *conjugate self-dual* if and only if the contragredient representation $\widetilde{\pi}$ is equivalent to the Galois twist ${}^{\theta}\pi$.

In the Galois case, multiplicity-one is due to Flicker [21, Proposition 11]; moreover, in the same paper Flicker uses methods of [25] to prove the following result.

Proposition 2.3.20 (Flicker)**.** *Let π be an irreducible H-distinguished representation of G, then the dimension of* $\mathrm{Hom}_H(\pi, 1)$ *is one and π is conjugate self-dual, that is, $\widetilde{\pi} \cong {}^{\theta}\pi$.*

Corollary 2.3.21. *Let π be an irreducible admissible H-distinguished representation of G, then we have the equivalence $\pi \cong {}^{\theta}\widetilde{\pi}$.*

Let $\eta : E^{\times} \to \mathbb{C}^{\times}$ be an extension to $E^{\times}$ of the character $\eta_{E/F} : F^{\times} \to \mathbb{C}$ associated to E/F by local class field theory [68]. Let $L(s, r(\pi))$ denote the twisted tensor L-function of π introduced by Flicker in [22]. The following proposition combines Corollary 1.5 and Corollary 1.6 of [2].

Proposition 2.3.22 (Anandavardhanan–Kable–Tandon)**.** *Let π be a discrete series representation of G.*

1. *The representation π is H-distinguished if and only if $L(s, r(\pi))$ has a pole at $s = 0$.*

2. *If π is conjugate self-dual, then π is either H distinguished or $(H, \eta_{E/F})$-distinguished, but cannot be both.*

In the previous result, $\eta_{E/F}$ is also used to denote the character $\eta_{E/F} \circ \det$ of H. The following result is due to Anandavardhanan–Rajan [5, Section 4.4], and also appears as [1, Theorem 1.3] and [50, Corollary 4.2].

Theorem 2.3.23 (Anandavardhanan–Rajan)**.** *Let ρ be an irreducible supercuspidal representation of $\mathbf{GL}_r(E)$, then the generalized Steinberg representation $\pi = \mathrm{St}(k, \rho)$ of $\mathbf{GL}_{kr}(E)$ is $\mathbf{GL}_{kr}(F)$-distinguished if and only if ρ is $(\mathbf{GL}_r(F), \eta_{E/F}^{k-1})$-distinguished.*

In the paper [43], Kable also assumes that F has characteristic zero, but he believes that this assumption can be removed. The following result is [43, Theorem 7].

Theorem 2.3.24 (Kable)**.** *Let π be a conjugate self-dual discrete series representation of G and assume that the central character of π is trivial on Z_H.*

1. *If n is odd, then π is H-distinguished.*

2. *If n is even, then exactly one of π and $\eta \otimes \pi$ is H-distinguished.*

The Rankin–Selberg type Asai L-function $L_{\mathrm{Asai}}(s, \pi)$ of π agrees with the Asai L-function of π defined via the local Langlands correspondence [50, 52]. The next proposition that we note appears as [51, Proposition 3.4].

Proposition 2.3.25 (Matringe)**.** *A discrete series representation π of G is H-distinguished if and only if $L_{\mathrm{Asai}}(s, \pi)$ has a pole at $s = 0$.*

Matringe has also given a classification of the H-distinguished unitary generic representations in terms of the inducing quasi-square integrable representations (*cf.* [52, Theorem 5.2]). Matringe continues his study of Galois distinction for the general linear group in [54] and obtains results on H-distinction for Speh representations, the building blocks of the unitary dual of $\mathbf{GL}_n$ [73].

For $m = 2$ the next result is due to Prasad [59]. Prasad's result was generalized to $m \geq 2$ by Anandavardhanan and Rajan [5, Theorem 1.5].

Theorem 2.3.26 (Prasad, Anandavardhanan–Rajan)**.** *Let $m \geq 2$ be an integer. Let χ be a character of $F^\times$. The Steinberg representation St_m of $\mathbf{GL}_m(E)$ is $(\mathbf{GL}_m(F), \chi \circ \det)$-distinguished if and only if:*

1. *m is odd and $\chi = 1$, or*
2. *m is even and $\chi = \eta_{E/F}$.*

Corollary 2.3.27. *Let $\eta : E^\times \to \mathbb{C}^\times$ be an extension of $\eta_{E/F}$. The twist $\eta \otimes \mathrm{St}_2$ of the Steinberg representation St_2 of $\mathbf{GL}_2(E)$ is $\mathbf{GL}_2(F)$-distinguished.*

Proof. By Theorem 2.3.26, St_2 is not $\mathbf{GL}_2(F)$-distinguished, but St_2 is $(\mathbf{GL}_2(F), \eta_{E/F})$-distinguished. It is straightforward to check that an irreducible admissible representation π of $\mathbf{GL}_n(E)$ is $(\mathbf{GL}_n(F), \eta_{E/F})$-distinguished if and only if $\eta \otimes \pi$ is $\mathbf{GL}_n(F)$-distinguished. The result follows. □

Finally, we note a result on supercuspidal representations, due to Hakim and Murnaghan, key in proving (the Galois case of) the existence result Proposition 4.2.8. The following is a corollary of [56, Proposition 10.1] and [29, Theorem 1.1].

Theorem 2.3.28 (Hakim–Murnaghan)**.** *There are infinitely many equivalence classes of*

1. *irreducible supercuspidal representations of G that are H-distinguished.*
2. *irreducible supercuspidal representations of G that are $(H, \eta_{E/F})$-distinguished.*

Proof. By [56, Proposition 10.1], there exist infinitely many pairwise inequivalent irreducible H-distinguished tame supercuspidal representations of G. The result now follows immediately from [29, Theorem 1.1]. In particular, apply the observation that an irreducible admissible representation π of G is H-distinguished if and only if $\eta \otimes \pi$ is $(H, \eta_{E/F})$-distinguished, where η is any extension of $\eta_{E/F}$ to $E^\times$. Similarly, it is straightforward to check that π is $(H, \eta_{E/F})$-distinguished if and only if $\eta \otimes \pi$ is H-distinguished. □

The following is an immediate corollary of Theorem 2.3.28 and Theorem 2.3.23.

Proposition 2.3.29.
1. *If $n \geq 4$ is not equal to an odd prime, then there are infinitely many equivalence classes of $\mathbf{GL}_n(F)$-distinguished nonsupercuspidal discrete series representations of $\mathbf{GL}_n(E)$.*
2. *If n is equal to an odd prime, then the Steinberg representation St_n of $\mathbf{GL}_n(E)$ is a nonsupercuspidal $\mathbf{GL}_n(F)$-distinguished discrete series.*

Proof. Again, recall that two generalized Steinberg representations $\mathrm{St}(k_1, \rho_1)$ and $\mathrm{St}(k_2, \rho_2)$ of $\mathbf{GL}_n(E)$ are equivalent if and only if $k_1 = k_2$ and ρ_1 is equivalent to ρ_2 [75, Theorem 9.7(b)]. Assume that $n \geq 4$ is not an odd prime. Then $n = kr$ for two integers $k, r \geq 2$. Note that $\eta_{E/F}$ is a quadratic character; in particular, if k is even, then $\eta_{E/F}^{k-1} = \eta_{E/F}$ and if k is odd, then $\eta_{E/F}^{k-1} = 1$. By Theorem 2.3.28, there are infinitely many equivalence classes of irreducible supercuspidal representations ρ of $\mathbf{GL}_r(E)$ that are $(\mathbf{GL}_r(F), \eta_{E/F}^{k-1})$-distinguished. By Theorem 2.3.23 and [75, Theorem 9.7(b)], there are infinitely many equivalence classes of generalized Steinberg representations of $\mathbf{GL}_n(E)$ of the form $\mathrm{St}(k, \rho)$ and that are $\mathbf{GL}_n(F)$-distinguished. Of course, the generalized Steinberg representations $\mathrm{St}(k, \rho)$ are nonsupercuspidal discrete series representations. The second statement follows from Theorem 2.3.26(1). □

Remark 2.3.30. Recently, a conjecture of Flicker and Rallis (*cf.* [21]) relating Galois distinction for the general linear group to quadratic base change from representations of quasi-split unitary groups has been verified for irreducible admissible generic representations [4, Theorem 2.3]. We refer to [5, 4] for a discussion of these ideas. Progress on the conjecture for a wider class of irreducible representations has been made by Gurevich, Ma and Mitra [26].

Remark 2.3.31. The authors of [2] give explicit H-invariant linear functionals on the Whittaker models of distinguished generic (in particular, discrete series) representations of G. The invariant forms are given by integrating functions in the Whittaker model, defined with respect to a suitable additive character of E, over a quotient of the mirabolic subgroup. In particular, in the Galois case, the H-invariant forms on the relative discrete series representations produced via Theorem 4.2.1 arise this way. Test vectors for these explicit H-invariant forms have recently been investigated in [3]. In particular, Anandavardhanan and Matringe show that the value of the H-invariant form on the *essential vector* of [37] is an unramified local Asai L-value.

Chapter 3

Generalities and preliminary results

This chapter contains results needed to prove, or simplify the proof of, the main theorems of Chapters 4 and 5, including some preliminary results on exponents of parabolically induced representations. Unless stated otherwise, we work in the symmetric space setting and retain the notation of Chapter 1, i.e., S_0 is a maximal (θ, F)-split torus of G contained in the θ-stable maximal F-split torus A_0, and Δ_0 is a θ-base for the root system Φ_0 of G with respect to A_0. We reiterate that we will refer to the F-points P of an F-parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ simply as a parabolic subgroup of G (*cf.* Remark 1.4.1). We make a similar abuse of notation regarding the F-split and (θ, F)-split components of Levi subgroups (*cf.* Remark 1.5.4).

3.1 Rationality of products of closed subgroups

In Chapter 4, to prove that any θ-split parabolic P of G is H-conjugate to a θ-split parabolic subgroup in a fixed standard class (*cf.* Proposition 4.1.9), we have to overcome a certain rationality issue. Here we note two results that we require to complete this task. The following lemma is well known and provides a standard argument required to complete the proof of Proposition 4.1.9. The author would like to thank Jason Starr for communicating the proof.

Lemma 3.1.1. *Let $\mathbf{M}$ and $\mathbf{H}$ be closed F-subgroups of a connected reductive group $\mathbf{G}$ defined over F, with F-points M, H and G respectively. If the degree-one Galois cohomology of $\mathbf{M} \cap \mathbf{H}$ over F is trivial, then $(\mathbf{HM})(F) = HM$.*

Proof. Let K/F be a finite Galois extension with Galois group Γ_K. We have that Γ_K acts on $\mathbf{G}$ and the subgroup of Γ_K-fixed points of $\mathbf{G}(K)$ is exactly $G = \mathbf{G}(F)$. Let $m \in \mathbf{M}(K)$ and $h \in \mathbf{H}(K)$ such that $g = hm^{-1} \in G$, that is such that $g \in (\mathbf{HM})(F)$. Consider the non-abelian 1-cocycles for Γ_K in $\mathbf{G}(K)$ given by $\kappa \mapsto m^{-1}\kappa(m)$ and $\kappa \mapsto h^{-1}\kappa(h)$, for $\kappa \in \Gamma_K$. We write these 1-cocycles as $(m^{-1}\kappa(m))_{\kappa \in \Gamma_K}$, respectively $(h^{-1}\kappa(h))_{\kappa \in \Gamma_K}$. Observe that, since $g = hm^{-1} \in G$ is Γ_K-invariant, these two 1-cocycles are equal and they are 1-cocycles in $(\mathbf{M} \cap \mathbf{H})(K)$. Indeed, for any $\kappa \in \Gamma_K$, we have that $g = \kappa(g)$ and so

$$m^{-1}\kappa(m) = m^{-1}g^{-1}\kappa(g)\kappa(m) = m^{-1}mh^{-1}\kappa(hm^{-1})\kappa(m) = h^{-1}\kappa(h).$$

By assumption $H^1(F, \mathbf{M} \cap \mathbf{H}) = 0$, and therefore the 1-cocycle $(m^{-1}\kappa(m))_{\kappa \in \Gamma_K} = (h^{-1}\kappa(h))_{\kappa \in \Gamma_K}$ is actually a 1-coboundary $(r^{-1}\kappa(r))_{\kappa \in \Gamma_K}$, for some $r \in (\mathbf{M} \cap \mathbf{H})(K)$. Defining $\widetilde{m} = mr^{-1}$ and $\widetilde{h} = hr^{-1}$, we have that $\widetilde{h}\widetilde{m}^{-1} = g$; moreover, both $\widetilde{m}$ and $\widetilde{h}$ are Γ_K-invariant, i.e., $\widetilde{m} \in M$ and $\widetilde{h} \in H$. Indeed, first considering $\widetilde{m} \in \mathbf{M}(K)$, for any $\kappa \in \Gamma_K$, we have that

$$\kappa(\widetilde{m}) = \kappa(mr^{-1}) = \kappa(m)\kappa(r^{-1}) = mr^{-1}\kappa(r)\kappa(r^{-1}) = mr^{-1} = \widetilde{m},$$

where we've used that $m^{-1}\kappa(m) = r^{-1}\kappa(r)$, for all $\kappa \in \Gamma_K$. Similarly, one can see that $\widetilde{h} \in H$. Since $g = hm^{-1}$ was arbitrary, we have that $(\mathbf{HM})(F) \subset HM$. On the other hand, the containment $HM \subset (\mathbf{HM})(F)$ is clear; therefore, $(\mathbf{HM})(F) = HM$, as claimed. □

Note. A proof of Lemma 3.1.1 may also be extracted from the long exact sequence in Galois cohomology one obtains from the short exact sequence

$$1 \to \mathbf{M}(\bar{F}) \cap \mathbf{H}(\bar{F}) \to \mathbf{M}(\bar{F}) \times \mathbf{H}(\bar{F}) \to \mathbf{M}(\bar{F})\mathbf{H}(\bar{F}) \to 1$$

of pointed sets, where $\bar{F}$ is the algebraic closure of F.

The following lemma is a result on the Galois cohomology of the restriction of scalars of an algebraic group, which is a special case of Shapiro's Lemma [55, Proposition 27.51], and is used in our proof of Proposition 4.1.9 in the Galois case. Precisely, we use this version of Shapiro's Lemma to prove the triviality of a particular intersection cohomology needed to apply Lemma 3.1.1. In fact, in our proof of Proposition 4.1.9, the group $(\mathbf{T}_0 \cap \mathbf{H})(F)$ is what Milne calls a *quasi-trivial F-torus* (*cf.* [55, pp. 487]).

Lemma 3.1.2 (Shapiro's Lemma)**.** *Let $\mathbf{G}$ be an algebraic group defined over a finite extension K' of the field K, then we have the isomorphism $H^i(K, R_{K'/K}\mathbf{G}) \cong H^i(K', \mathbf{G})$ in Galois cohomology for $i = 0, 1$ and for all i if $\mathbf{G}$ is commutative.*

Recall that $R_{K'/K}\mathbf{G}$ denotes Weil's restriction of scalars of $\mathbf{G}$ from K' to K [72].

3.2 Reduction to maximal standard θ-split parabolic subgroups

In this section, we assume all θ-split parabolic subgroups of G are H-conjugate to standard (with respect to Δ_0) θ-split parabolic subgroups. Under this assumption, we prove Lemma 3.2.3, which states that it is sufficient to check the condition (2.14) of the Relative Casselman's Criterion 2.2.18, along maximal standard θ-split parabolic subgroups. Our argument follows Remark 6.10 and Lemma 2.5(2) of Kato and Takano's paper [44]. When applying this result in Chapter 4, Proposition 4.1.9 gives us that the assumptions of Lemma 3.2.3 are satisfied.

Let π be an irreducible smooth H-distinguished representation, with nonzero H-invariant linear form $\lambda \in \mathrm{Hom}_H(\pi, 1)$. It is straightforward to verify the following.

Lemma 3.2.1. *Let $P = MN$ be a proper θ-split parabolic subgroup of G. Assume that P is H-conjugate to a Δ_0-standard θ-split parabolic subgroup P_Θ. If $P = hP_\Theta h^{-1}$, where $h \in H$, then there is a bijection*

$$\begin{gathered} \mathcal{E}xp_{S_\Theta}(\pi_{N_\Theta}, r_{P_\Theta}\lambda) \longleftrightarrow \mathcal{E}xp_S(\pi_N, r_P\lambda) \\ \chi' \mapsto {}^h\chi', \end{gathered}$$

with inverse given by $\chi \mapsto {}^{h^{-1}}\chi$.

Proof. Let V be the space of π. Let $P = MN$ be a proper θ-split parabolic subgroup of G. By assumption, there exists $h \in H$ and a θ-split subset $\Theta \subset \Delta_0$ such that $P = hP_\Theta h^{-1}$, $M = hM_\Theta h^{-1}$, $N = hN_\Theta h^{-1}$ and $A_M = hA_\Theta h^{-1}$. In particular, the (θ, F)-split component S of P is equal to $hS_\Theta h^{-1}$.

Using that $N = hN_\Theta h^{-1}$, it is immediate that $V(N) = \pi(h)V(N_\Theta)$. The space of π_N is $V_N = V/V(N)$ and the space of π_{N_Θ} is $V_{N_\Theta} = V/V(N_\Theta)$. One can then verify that the map $T : V_N \to V_{N_\Theta}$ given by $T(v + V(N)) = \pi(h^{-1})v + V(N_\Theta)$ is a well-defined invertible linear map. In addition, if $m' \in M_\Theta$, so that $m = hm'h^{-1} \in M$, we have that

$$T(\pi_N(m)v + V(N)) = \pi_{N_\Theta}(m')T(v + V(N)).$$

Moreover, given a quasi-character χ of M (so that $\chi' = {}^{h^{-1}}\chi$ is a quasi-character of M_Θ), the map T restricts to a linear isomorphism from $(V_N)_{\chi,\infty}$ to $(V_{N_\Theta})_{\chi',\infty}$.

Recall that elements of $\mathcal{E}xp_S(\pi_P)$ are given by restricting elements of $\mathcal{E}xp_{A_M}(\pi_P)$ to $S \subset A_M$. Let $\chi \in \mathcal{E}xp_S(\pi_P, r_P\lambda)$; in particular, $\chi \in \mathcal{E}xp_S(\pi_P)$ and there exists $v_0 \in V$ such that $v_0 + V(N)$ is a nonzero element of $(V_N)_{\chi,\infty}$ and $\langle r_P\lambda, v_0 + V(N)\rangle \neq 0$. By the above, we have that $T(v_0 + V(N)) \in (V_{N_\Theta})_{\chi',\infty}$ so that $\chi' \in \mathcal{E}xp_S(\pi_{P_\Theta})$. It remains to show that $r_{P_\Theta}\lambda_G$ is nonzero on $(V_{N_\Theta})_{\chi',\infty}$.

Let K_0' be a good compact open subgroup with Iwahori factorization with respect to P_Θ, chosen from and adapted family as in Lemma 2.2.12. Then $K_0 = hK_0'h^{-1}$ has Iwahori factorization with respect to P. We will use K_0 (respectively K_0') to define the canonical lifts used to evaluate $r_P\lambda$ (respectively $r_{P_\Theta}\lambda$). Let $\widetilde{v}_0$ be a canonical lift of $v_0 + V(N)$, relative to K_0, then by definition

$$\langle r_P\lambda, v_0 + V(N)\rangle = \langle \lambda, \widetilde{v}_0\rangle,$$

and by Proposition 2.2.13 this is independent of the choice of canonical lift. Moreover, we have that $\langle \lambda, \widetilde{v}_0\rangle \neq 0$, since $\langle r_P\lambda, v_0 + V(N)\rangle \neq 0$. It can be readily verified that $\pi(h^{-1})\widetilde{v}_0$ is a canonical lift of the vector

$$T(v_0 + V(N)) = \pi(h^{-1})v_0 + V(N_\Theta)$$

in V_{N_Θ} with respect to K_0'. It follows from the H-invariance of λ that

$$\langle r_{P_\Theta}\lambda, T(v_0 + V(N))\rangle = \langle \lambda, \pi(h^{-1})\widetilde{v}_0\rangle = \langle \lambda, \widetilde{v}_0\rangle = \langle r_P\lambda, v_0 + V(N)\rangle \neq 0.$$

Since $T(v_0+V(N))$ lies in $(V_{N_\Theta})_{\chi',\infty}$, we have that $r_{P_\Theta}\lambda$ is nonzero on $(V_{N_\Theta})_{\chi',\infty}$; in particular, ${}^{h^{-1}}\chi$ is an element of $\mathcal{E}xp_{S_\Theta}(\pi_{N_\Theta}, r_{P_\Theta}\lambda)$. This proves that $\chi \mapsto {}^{h^{-1}}\chi$ is a map from $\mathcal{E}xp_S(\pi_N, \lambda)$ to $\mathcal{E}xp_{S_\Theta}(\pi_{N_\Theta}, \lambda)$; moreover, the above argument also shows that $\chi' \mapsto {}^{h}\chi'$ is the inverse-map. □

Remark 3.2.2. In general, if $\mathbf{M}_0$ is the Levi subgroup of a minimal Δ_0-standard θ-split parabolic and $P = xP_\Theta x^{-1}$ is a proper θ-split parabolic, where $x \in (\mathbf{H}\mathbf{M}_0)(F)$ as in Lemma 1.6.4, then we have a bijection

$$\mathcal{E}xp_{S_\Theta}(\pi_{N_\Theta}) \longleftrightarrow \mathcal{E}xp_S(\pi_N);$$
$$\chi' \mapsto {}^{x}\chi'$$

however, in the proof of Lemma 3.2.1 we need $x \in H$, to use the H-invariance of λ, in order to obtain non-vanishing of $r_P\lambda$ on the appropriate subspace of generalized eigenvectors.

Lemma 3.2.3. *Assume that any θ-spilt parabolic subgroup P of G is H-conjugate to a Δ_0-standard θ-split parabolic. If condition (2.14) holds for all Δ_0-standard θ-split parabolic subgroups of G, then the condition (2.14) holds for all θ-split parabolic subgroups of G.*

Proof. Let $P = MN$ be a proper θ-split parabolic subgroup of G. As above, there exists $h \in H$ and a θ-split subset $\Theta \subset \Delta_0$ such that $P = hP_\Theta h^{-1}$. In particular, the (θ, F)-split component S of P is equal to $hS_\Theta h^{-1}$; moreover, $S^- = hS_\Theta^- h^{-1}$. Let $\chi \in \mathcal{E}xp_S(\pi_N, r_P\lambda)$, by Lemma 3.2.1, there exists $\chi' = {}^{h^{-1}}\chi \in \mathcal{E}xp_{S_\Theta}(\pi_{N_\Theta}, r_{P_\Theta}\lambda)$. Let $s \in S^- \setminus S^1 S_{\Delta_0}$, then $s' = h^{-1}sh \in S_\Theta^- \setminus S_\Theta^1 S_{\Delta_0}$. It follows that

$$|\chi(s)| = |\chi(hs'h^{-1})| = |\chi'(s')| < 1,$$

where the final inequality holds by the assumption that (2.14) holds for P_Θ. □

By [45, Lemma 4.6], to apply the Relative Casselman's Criterion 2.2.18, it is sufficient to consider exponents (with respect to λ) along *maximal* θ-split parabolic subgroups; therefore, we have the following.

Corollary 3.2.4. *Let π be an H-distinguished representation of G and λ a nonzero H-invariant linear form on the space of π. Assume that any θ-spilt parabolic subgroup P of G is H-conjugate to a Δ_0-standard θ-split parabolic. Then π is (H, λ)-relatively square integrable if and only if the condition (2.14) of the Relative Casselman's Criterion holds for all Δ_0-standard maximal θ-split parabolic subgroups of G.*

3.3 Invariant forms on induced representations

Let $Q = LU$ be a θ-stable parabolic subgroup with θ-stable Levi subgroup L and unipotent radical U. Let (τ, V_τ) be a smooth representation of L and let $\pi = \iota_Q^G \tau$. Let V_π denote the space of π. We are interested in constructing a nonzero H-invariant linear form on the space of the representation π. The (identity component of the) subgroup Q^θ of θ-fixed points is a parabolic subgroup of H with Levi decomposition $Q^\theta = L^\theta U^\theta$ [31]. Let μ be a positive quasi-invariant measure on the (compact) quotient $Q^\theta \backslash H$. We direct the reader to [6, Theorem 1.21] for the existence and properties of such measures. Suppose that we have a nonzero element λ of the space $\operatorname{Hom}_{L^\theta}(\delta_Q^{1/2}\tau, \delta_{Q^\theta})$. If $\phi \in V_\pi$, then the function $\lambda \circ \phi : G \to \mathbb{C}$ satisfies

$$\lambda \circ \phi(qh) = \langle \lambda, \delta_Q^{1/2}(q)\tau(q)\phi(h)\rangle = \delta_{Q^\theta}(q)\langle \lambda, \phi(h)\rangle = \delta_{Q^\theta}(q)(\lambda \circ \phi)(h),$$

for all $q \in Q^\theta$ and $h \in H$. Moreover, since H is closed in G, the restriction $\phi|_H$ of ϕ to H is a smooth (locally constant) function. It follows that

$$\lambda \circ \phi|_H \in \text{c-}\operatorname{Ind}_{Q^\theta}^H \delta_{Q^\theta} = \operatorname{Ind}_{Q^\theta}^H \delta_{Q^\theta},$$

which is the space of functions on which $d\mu$ is defined. Define a linear form λ^G on V_π by

$$\langle \lambda^G, \phi\rangle = \int_{Q^\theta \backslash H} \langle \lambda, \phi(h)\rangle \, d\mu(h). \tag{3.1}$$

The following lemma, communicated to the author by F. Murnaghan, describes the properties of λ^G.

Lemma 3.3.1. *The map $\lambda \mapsto \lambda^G$ is an injection of $\mathrm{Hom}_{L^\theta}(\delta_Q^{1/2}\tau, \delta_{Q^\theta})$ into the space $\mathrm{Hom}_H(\pi, 1)$ of H-invariant linear forms on π. In particular, if λ is nonzero then λ^G is nonzero and π is H-distinguished.*

Proof. Let $h' \in H$, then by the quasi-invariance of μ, for any $\phi \in V_\pi$ we have

$$\langle \lambda^G, \pi(h')\phi\rangle = \int_{Q^\theta\backslash H} \langle \lambda, \phi(hh')\rangle \, d\mu(h) = \int_{Q^\theta\backslash H} \langle \lambda, \phi(h'')\rangle \, d\mu(h'') = \langle \lambda^G, \phi\rangle;$$

therefore, $\lambda^G \in \mathrm{Hom}_H(\pi, 1)$. We now prove injectivity of the map $\lambda \mapsto \lambda^G$.

Let V_τ be the space of the representation τ of L. Let $I(\tau, QH)$ denote the space of locally constant functions $\phi : QH \to V_\tau$ satisfying:

$$\phi(luh) = \delta_Q^{1/2}(l)\tau(l)\phi(h),$$

for all $l \in L, u \in U$ and $h \in H$. Restriction of functions in $I(\tau, QH)$ to H gives an equivalence between the H-representations $I(\tau, QH)$ and $\text{c-Ind}_{Q^\theta}^H \rho$, where ρ is the representation $\delta_Q^{1/2}\tau$ restricted to L^θ. Indeed, the function $\phi|_H : H \to V_\tau$ is smooth because ϕ is locally constant; moreover, we have that

$$\phi|_H(luh) = \delta_Q^{1/2}(l)\tau(l)\phi|_H(h) = \rho(l)\phi|_H(h),$$

for all $lu \in Q^\theta$ and $h \in H$. Finally, H acts on both $I(\tau, QH)$ and $\text{c-Ind}_{Q^\theta}^H \rho$ by right translation of functions; therefore $\phi \mapsto \phi|_H$ is an H-morphism from $I(\tau, QH)$ to $\text{c-Ind}_{Q^\theta}^H \rho$. It remains to show that this map is an equivalence. First, we show that the map $\phi \mapsto \phi|_H$ is injective. Let $\phi \in I(\tau, QH)$ and suppose that $\phi|_H = 0$. Then we have that

$$\phi(qh) = \delta_Q^{1/2}(q)\tau(q)\phi(h) = 0,$$

for all $q \in Q$ and $h \in H$. It follows that ϕ is equal to the zero-function. Next, we show that the map $\phi \mapsto \phi|_H$ is surjective. Let $\varphi \in \text{c-Ind}_{Q^\theta}^H \rho$, define $\phi \in I(\tau, QH)$ by

$$\phi(qh) = \delta_Q^{1/2}(q)\tau(q)\varphi(h),$$

for $q \in Q$ and $h \in H$. Note that the function ϕ agrees with φ on $Q^\theta = Q \cap H$. It follows that ϕ is well-defined and $\phi|_H = \varphi$.

Using the H-equivalence $I(\tau, QH) \cong \text{c-Ind}_{Q^\theta}^H \rho$ and applying the exact functor $\mathrm{Hom}_H(\cdot, 1)$ we obtain an isomorphism

$$\mathrm{Hom}_H(I(\tau, QH), 1) \cong \mathrm{Hom}_H(\text{c-Ind}_{Q^\theta}^H \rho, 1). \tag{3.2}$$

Applying a form of Frobenius Reciprocity [6, Proposition 2.29] to (3.2), we obtain an explicit isomorphism

$$\mathrm{Hom}_{L^\theta}(\delta_Q^{1/2}\tau, \delta_{Q^\theta}) = \mathrm{Hom}_{L^\theta}(\delta_{Q^\theta}^{-1}\rho, 1) \cong \mathrm{Hom}_H(I(\tau, QH), 1), \tag{3.3}$$

given by the map that takes $\lambda \in \mathrm{Hom}_{L^\theta}(\delta_Q^{1/2}\tau, \delta_{Q^\theta})$ to the linear functional on $I(\tau, QH)$ defined by

$$\phi \mapsto \int_{Q^\theta\backslash H} \langle \lambda, \phi(h)\rangle \, d\mu(h),$$

where $\phi \in I(\tau, QH)$. Finally, since Q is θ-stable we have that QH is closed in G [31, Lemma 1.7], and the map $f \mapsto f|_{QH}$ is a surjective H-morphism from V_π to $I(\tau, QH)$. Therefore, we have an injection of dual spaces $I(\tau, QH)^*$ into V_π^*, and this remains an injection at the level of H-fixed vectors. Noting that $(I(\tau, QH)^*)^H$ is equal to $\mathrm{Hom}_H(I(\tau, QH), 1)$ and $(V_\pi^*)^H$ is equal to $\mathrm{Hom}_H(\pi, 1)$, we have produced the desired injection $\mathrm{Hom}_{L^\theta}(\delta_Q^{1/2}\tau, \delta_{Q^\theta}) \hookrightarrow \mathrm{Hom}_H(\pi, 1)$ given explicitly by the map $\lambda \mapsto \lambda^G$. $\square$

In Chapters 4 and 5, we will work with θ-stable parabolic subgroups Q such that $\delta_Q^{1/2}$ restricted to L^θ is equal to δ_{Q^θ}. In particular, in this setting, we have an injection from $\mathrm{Hom}_{L^\theta}(\tau, 1)$ into $\mathrm{Hom}_H(\iota_Q^G\tau, 1)$.

3.4 Containment relationships between Levi subgroups

In order to apply the Relative Casselman's Criterion 2.2.18 to an induced representation π, we will give a reduction to Casselman's Criterion for the inducing data (*cf.* Proposition 4.2.20, respectively 5.2.17). Here we record a simple lemma that will simplify the exposition in later chapters.

Lemma 3.4.1. *Let Φ be a root system of a reductive group G defined with respect to a fixed maximal F-split torus A. Fix a choice Φ^+ of positive roots corresponding to a base Δ of Φ. Let $\Omega \subset \Delta$ and let $\Theta \subset \Phi^+$, such that Θ is a subset of the simple roots corresponding to some (possibly different) choice of positive roots. If $\Theta \subset \Phi_\Omega^+ = (\mathrm{span}_{\mathbb{Z}_{\geq 0}} \Omega) \cap \Phi$, then $M_\Theta \subset M_\Omega$. Where we define $M_\Theta = C_G(A_\Theta)$ and*

$$A_\Theta = \left(\bigcap_{\alpha \in \Theta} \ker \alpha \right)^\circ,$$

similarly for M_Ω.

Proof. Suppose $\Theta \subset \Phi_\Omega^+$. The tori A_Ω and A_Θ are both F-split sub-tori of A. It is sufficient to prove that A_Ω is contained in A_Θ, since it then follows that

$$M_\Theta = C_G(A_\Theta) \subset C_G(A_\Omega) = M_\Omega,$$

as desired. Suppose that $t \in A_\Omega$ and $\alpha \in \Theta$. By assumption $\alpha \in \Phi_\Omega^+$, so there exists nonnegative integers $c_\beta \in \mathbb{Z}$, for each $\beta \in \Omega$ such that $\alpha = \sum_{\beta \in \Omega} c_\beta \beta$. It follows that $\alpha(t) = \prod_{\beta \in \Omega} \beta(t)^{c_\beta} = 1$, since $t \in \ker \beta$ for all $\beta \in \Omega$. Since $\alpha \in \Theta$ was arbitrary, it follows that t lies in the intersection $\bigcap_{\alpha \in \Theta} \ker \alpha$ and, since t was an arbitrary element of A_Ω, we have that $A_\Omega \subset A_\Theta$. $\square$

We give the following notation in order to discuss Casselman's Criterion 2.1.28 for the inducing data of a parabolically induced representation of G, while working with a root system of G, rather than working only with the root system of the inducing Levi subgroup.

Definition 3.4.2. Let $\Theta \subset \Omega \subset \Delta$, where Δ is any base for Φ, we define

$$A_\Theta^- = \{a \in A_\Theta : |\alpha(a)| \leq 1, \forall \alpha \in \Delta \setminus \Theta\}$$

and,

$$A_\Theta^{-\Omega} = \{a \in A_\Theta : |\beta(a)| \leq 1, \forall \beta \in \Omega \setminus \Theta\}.$$

The set A_Θ^- is the dominant part of A_Θ in G, while $A_\Theta^{-\Omega}$ is the dominant part of A_Θ in M_Ω.

3.5 Exponents and distinction

For applications in studying exponents and distinction of Jacquet modules, we note the following result. In particular, Proposition 3.5.1 is key to the proofs of Theorem 4.2.1 and Theorem 5.2.22.

Proposition 3.5.1. *Let (π, V) be a finitely generated admissible representation of G. Let $\chi \in \mathcal{E}xp_{Z_G}(\pi)$ and assume that all irreducible subquotients of (π, V) with central character χ are not H-distinguished. Then for any $\lambda \in \mathrm{Hom}_H(\pi, 1)$, the restriction of λ to $V_{\chi,\infty}$ is equal to zero, i.e., $\lambda|_{V_{\chi,\infty}} \equiv 0$.*

Proof. Suppose, by way of contradiction, that $\lambda|_{V_{\chi,\infty}} \neq 0$. Then $(\pi|_{V_{\chi,\infty}}, V_{\chi,\infty})$ is an admissible (finite length by [15, Theorem 6.3.10]) H-distinguished representation of G. By Lemma 2.2.4, some irreducible subquotient (ρ, V_ρ) of $V_{\chi,\infty}$ must be H-distinguished. However, the representation (ρ, V_ρ) is also an irreducible subquotient of (π, V) and has central character χ by Lemma 3.5.2. By assumption, no such (ρ, V_ρ) can be H-distinguished; therefore, we must have that $\lambda|_{V_{\chi,\infty}}$ is identically zero. □

To complete the proof of Proposition 3.5.1 we require the following lemma.

Lemma 3.5.2. *Let (π, V) be a a finitely generated admissible representation of G.*

1. *If (ρ, V_ρ) is an irreducible sub-quotient of (π, V) with central quasi-character χ, then $V_{\chi,\infty} \neq 0$ and (ρ, V_ρ) is a sub-quotient of $V_{\chi,\infty}$.*
2. *Any irreducible sub-quotient of $V_{\chi,\infty}$ has central quasi-character χ.*

Proof. By [15, Proposition 2.1.9], there exist finitely many quasi-characters $\omega \in \mathcal{E}xp_{Z_G}(\pi)$ such that $V_{\omega,\infty}$ is a nonzero G-stable subspace of V and V is the direct sum of the generalized eigenspaces $V_{\omega,\infty}$. Moreover, there is an integer m such that $V_{\omega,\infty} = V_{\omega,m}$, for all $\omega \in \mathcal{E}xp_{Z_G}(\pi)$. In particular, the lemma follows from the second statement.

Without loss of generality, assume that $V = V_{\chi,\infty}$. Then $V = V_{\chi,m}$, as above. If $m = 1$, then V is equal to $V_\chi = V_{\chi,1}$ and π has central quasi-character χ by definition. The category of admissible χ-representations is abelian [15]; in particular, all sub-quotients of V have central quasi-character χ. It remains to consider the case that $m \geq 2$. Since V is finitely generated and admissible, V has finite length [15, Theorem 6.3.10]. We'll argue by induction on the length ℓ of (π, V).

If (π, V) has length one ($\ell = 1$), then (π, V) is irreducible; therefore, by Schur's Lemma π admits a central quasi-character $\omega : Z_G \to \mathbb{C}^\times$. Let $v \in V$, such that $v \neq 0$. We have that $\pi(z)v = \omega(z)v$, for all $z \in Z_G$. In addition, we have that $(\pi(z) - \chi(z))^m v = 0$, for all $z \in Z_G$. It follows that,

$$0 = (\pi(z) - \chi(z))^m v = (\omega(z) - \chi(z))^m v,$$

for all $z \in Z_G$. In particular, since $v \neq 0$, we have that $(\omega(z) - \chi(z))^m = 0$, and we obtain that $\omega(z) = \chi(z)$, for all $z \in Z_G$. In this case, π has central character χ.

Next, assume that (π, V) has (finite) length $\ell \geq 2$ and that the result holds for all admissible finitely generated representations of length less than or equal to $\ell - 1$. Let $0 = V_0 \subset V_1 \subset \ldots \subset V_\ell = V$ be a filtration of V by G-stable subspaces such that each successive quotient $V^k = V_k/V_{k-1}$, $1 \leq k \leq \ell$, is an irreducible G-module. By Schur's Lemma, the irreducible representation V^k admits a central quasi-character ω_k. The result follows if we show that each V^k has central quasi-character χ.

Observe that, $V_{\ell-1}$ is an admissible finitely generated representation of length $\ell - 1$; therefore, by the induction hypothesis $\omega_k = \chi$, for all $1 \leq k \leq \ell - 1$. Therefore, it suffices to show that $\omega_\ell = \chi$. The final claim is precisely Lemma 3.5.3, which will complete the proof. □

The proof of Lemma 3.5.2 is completed by the following lemma.

Lemma 3.5.3. *Let (π, V) be a finitely generated admissible representation of G and let χ be a quasi-character of Z_G. If $V = V_{\chi,\infty}$, then any irreducible quotient of V has central character χ.*

First, we need the following technical fact.

Lemma 3.5.4. *Let (π, V) be a finitely generated admissible representation of G and let χ be a quasi-character of Z_G. If $V = V_{\chi,\infty}$, then the contragredient $(\widetilde{\pi}, \widetilde{V})$ satisfies $\widetilde{V} = (\widetilde{V})_{\chi^{-1},\infty}$.*

Proof. We want to show that

$$\widetilde{V} = \bigcup_{n=1}^{\infty} (\widetilde{V})_{\chi^{-1},n}. \tag{3.4}$$

The containment "$\supset$" in (3.4) holds by definition. It suffices to prove that for any $\widetilde{v} \in \widetilde{V}$, there exists $n \in \mathbb{N}$ such that $\widetilde{v} \in (\widetilde{V})_{\chi^{-1},n}$. Let $\widetilde{v} \in \widetilde{V}$. Since (π, V) is finitely generated and admissible, by [15, Proposition 2.1.9] there is $m \in \mathbb{N}$ such that $V = V_{\chi,\infty} = V_{\chi,m}$. For any $v \in V$, we have that

$$\langle (\widetilde{\pi}(z) - \chi^{-1}(z))\ \widetilde{v}, v\rangle = \langle \widetilde{\pi}(z)\ \widetilde{v} - \chi^{-1}(z)\ \widetilde{v}, v\rangle = \langle \widetilde{\pi}(z)\ \widetilde{v}, v\rangle - \langle \chi^{-1}(z)\ \widetilde{v}, v\rangle = \langle \widetilde{v}, (\pi(z^{-1}) - \chi(z^{-1}))\ v\rangle,$$

for all $z \in Z_G$. It follows that,

$$\langle (\widetilde{\pi}(z) - \chi^{-1}(z))^m\ \widetilde{v}, v\rangle = \langle \widetilde{v}, (\pi(z^{-1}) - \chi(z^{-1}))^m\ v\rangle = \langle \widetilde{v}, 0\rangle = 0,$$

for all $z \in Z_G$ and for any $v \in V$. Since $v \in V$ was arbitrary, the equality $(\widetilde{\pi}(z) - \chi^{-1}(z))^m\ \widetilde{v} = 0$, holds for all $z \in Z_G$. We have shown that $\widetilde{v} \in (\widetilde{V})_{\chi^{-1},m}$, as required. $\square$

Finally, we prove Lemma 3.5.3 which completes the proofs of Lemma 3.5.2 and Proposition 3.5.1.

Proof of Lemma 3.5.3. Let $V' \subset V$ be a G-stable subspace such that the quotient V/V' is an irreducible G-module. We have an exact sequence

$$0 \to V' \to V \to V/V' \to 0,$$

of admissible G-modules. By Proposition 2.1.3, and taking contragredients we obtain the exact sequence

$$0 \to \widetilde{(V/V')} \to \widetilde{V} \to \widetilde{V}' \to 0,$$

where, by Corollary 2.1.4, $\widetilde{(V/V')}$ is an irreducible subrepresentation of $\widetilde{V}$. By Lemma 3.5.4 and the length-one case of the proof of Lemma 3.5.2, we see that $\widetilde{(V/V')}$ has central quasi-character χ^{-1}. Finally, taking contragredients again, and applying Lemma 2.1.2 and Lemma 2.1.6, we see that V/V' has central quasi-character χ. $\square$

3.6 A generalization of the group case

Proposition 3.6.1 is a slight generalization of the characterization of distinction in the group case. The notation in this subsection differs slightly from the rest of the thesis. Let $G = G' \times G'$, where $G' = \mathbf{G}'(F)$

is a connected reductive group defined over F. In this section only, we use θ to denote an arbitrary finite order F-automorphism of G', say $\theta^n = \mathrm{Id}$, for a fixed $n \in \mathbb{N}$. In all other instances, θ denotes an F-involution, as in Chapter 1. Let H be the closed F-subgroup $H = \{(g, \theta(g)) : g \in G'\}$ of G.

Proposition 3.6.1. *An irreducible admissible representation $\pi_1 \otimes \pi_2$ of G is H-distinguished if and only if π_2 is equivalent to ${}^\theta\widetilde{\pi}_1$.*

Proof. Let V_i denote the space of the representation π_i, $i = 1, 2$. First, suppose that $\pi_1 \otimes \pi_2$ is H-distinguished. Let $\lambda \in \mathrm{Hom}_H(\pi_1 \otimes \pi_2, 1)$, such that $\lambda \neq 0$. Given any element $w \in V_2$, define a linear map $T_w : V_1 \to \mathbb{C}$ by

$$\langle T_w, v\rangle = \langle \lambda, v \otimes w\rangle,$$

for $v \in V_1$. Since λ is linear, and the tensor product is bilinear, we have that $T_w \in V_1^*$ is an element of the linear dual of V_1. In fact, T_w lies in the smooth dual of V_1. Indeed, since π_2 is smooth, for $w \in V_2$ there exists a compact open subgroup $K_w \subset G'$ such that $w \in V_2^{K_w}$. Define K'_w to be the θ-stable compact open subgroup

$$K'_w = K_w \cap \theta(K_w) \cap \theta^2(K_w) \ldots \cap \theta^{n-1}(K_w)$$

and note that $w \in V_2^{K'_w}$. For any $k \in K'_w$ and $v \in V_1$, we have

$$\begin{aligned}
\langle \pi_1^*(k) T_w, v\rangle &= \langle T_w, \pi_1\left(k^{-1}\right) v\rangle \\
&= \langle \lambda, \pi_1\left(k^{-1}\right) v \otimes w\rangle \\
&= \langle \lambda, \pi_1\left(k^{-1}\right) v \otimes \pi_2\left(\theta\left(k^{-1}\right)\right) w\rangle && \left(\text{since } w \in V_2^{K'_w}, \theta\left(k^{-1}\right) \in K'_w\right) \\
&= \langle \lambda, (\pi_1 \otimes \pi_2)\left(k^{-1}, \theta\left(k^{-1}\right)\right)(v \otimes w)\rangle \\
&= \langle \lambda, v \otimes w\rangle && \left(\text{since } \left(k^{-1}, \theta\left(k^{-1}\right)\right) \in H\right) \\
&= \langle T_w, v\rangle,
\end{aligned}$$

which shows that $T_w \in \widetilde{V_1}$ is smooth, for all $w \in V_2$. So far, we have that $T : V_2 \to \widetilde{V_1}$, $w \mapsto T_w$ is a nonzero linear map. Next, we show that T is a G'-morphism from π_2 to ${}^\theta\widetilde{\pi}_1$. To this end, let $g \in G', w \in V_2$ and $v \in V_1$ – notice that $(\theta^{-1}\left(g^{-1}\right), g^{-1}) \in H$ – then by the H-invariance of λ, we have

$$\begin{aligned}
\langle T_{\pi_2(g)w}, v\rangle &= \langle \lambda, v \otimes \pi_2(g) w\rangle \\
&= \langle \lambda, (\pi_1 \otimes \pi_2)\left(\theta^{-1}\left(g^{-1}\right), g^{-1}\right)(v \otimes \pi_2(g) w)\rangle \\
&= \langle \lambda, \pi_1\left(\theta^{-1}\left(g^{-1}\right)\right) v \otimes w\rangle \\
&= \langle T_w, \pi_1\left(\theta^{-1}\left(g^{-1}\right)\right) v\rangle \\
&= \langle \widetilde{\pi}_1\left(\theta^{-1}(g)\right) T_w, v\rangle \\
&= \langle \left({}^\theta\widetilde{\pi}_1\right)(g) T_w, v\rangle.
\end{aligned}$$

It follows that $T_{\pi_2(g)w} = \left({}^\theta\widetilde{\pi}_1\right)(g) T_w$; therefore, $w \mapsto T_w$ is a nonzero G'-morphism from π_2 to ${}^\theta\widetilde{\pi}_1$. Since π_2 is irreducible and, by Corollary 2.1.4, $\widetilde{\pi}_1$ is irreducible, we obtain that π_2 is equivalent to ${}^\theta\widetilde{\pi}_1$.

Conversely, assume that π_2 and ${}^\theta\widetilde{\pi}_1$ are equivalent. We show that $\pi_1 \otimes \pi_2$ is H-distinguished. Without loss of generality, we assume that $\pi_2 = {}^\theta\widetilde{\pi}_1$ and $V_2 = \widetilde{V}_1$. Define a linear form λ on simple tensors

$v_1 \otimes v_2 \in V_1 \otimes V_2$ by

$$\langle \lambda, v_1 \otimes v_2 \rangle = \langle v_2, v_1 \rangle, \tag{3.5}$$

and extend λ to be linear. Here the pairing on the right-hand side of (3.5) is the pairing of π_1 with its contragredient. It is then immediate that λ is nonzero. It suffices to show that λ is H-invariant, and it is sufficient to check on simple tensors $v_1 \otimes v_2$. Let $(g, \theta(g)) \in H$ and $v_1 \otimes v_2 \in V_1 \otimes V_2$, then

$$\begin{aligned}
\langle \lambda, (\pi_1 \otimes \pi_2)(g, \theta(g))(v_1 \otimes v_2) \rangle &= \langle \lambda, \pi_1(g)v_1 \otimes \pi_2(\theta(g))v_2 \rangle \\
&= \langle \pi_2(\theta(g))v_2, \pi_1(g)v_1 \rangle \\
&= \langle {}^{\theta}\widetilde{\pi}_1(\theta(g))v_2, \pi_1(g)v_1 \rangle && (\text{since } \pi_2 = {}^{\theta}\widetilde{\pi}_1) \\
&= \langle \widetilde{\pi}_1(\theta^{-1}\theta(g))v_2, \pi_1(g)v_1 \rangle && (\text{since } {}^{\theta}\widetilde{\pi}_1(h) = \widetilde{\pi}_1(\theta^{-1}(h))) \\
&= \langle \widetilde{\pi}_1(g)v_2, \pi_1(g)v_1 \rangle \\
&= \langle v_2, \pi_1\left(g^{-1}\right)\pi_1(g)v_1 \rangle && (\text{action of } \widetilde{\pi}_1) \\
&= \langle v_2, v_1 \rangle \\
&= \langle \lambda, v_1 \otimes v_2 \rangle.
\end{aligned}$$

This completes the proof that λ is H-invariant and the proof of the proposition. □

Remark 3.6.2. Note that we only needed that θ has finite order to guarantee that K'_w is open.

Chapter 4

RDS for $\mathbf{GL}_n(F) \times \mathbf{GL}_n(F) \backslash \mathbf{GL}_{2n}(F)$ and $\mathbf{GL}_n(F) \backslash \mathbf{GL}_n(E)$

4.1 Structure of the linear and Galois symmetric spaces

In this chapter, we take $\mathbf{G}$ to be either an even general linear group $\mathbf{GL}_n$, where $n \geq 4$ is even, or the restriction of scalars $R_{E/F}\mathbf{GL}_n$ (associated to E/F) of a general linear group, such that $n \geq 4$. The restriction on n is only to guarantee the existence of regular distinguished discrete series representations on standard-θ-elliptic Levi subgroups (*cf.* Proposition 4.2.8). We identify the F-points of $R_{E/F}\mathbf{GL}_n$ with the set of E-points of $\mathbf{GL}_n$. We refer to the case that $G = \mathbf{GL}_n(F)$ as the *linear case* and $G = R_{E/F}\mathbf{GL}_n(F)$ as the *Galois case*. For a positive integer r, we'll write $\mathbf{G}_r$ for $\mathbf{GL}_r$ with F-points G_r in the linear case, and similarly for $R_{E/F}\mathbf{GL}_r$ with F-points $G_r \simeq \mathbf{GL}_r(E)$, in the Galois case.

4.1.1 Involutions and notation

In the linear case, let θ denote the inner involution of $\mathbf{G}$ given by conjugation by the matrix

$$w_\ell = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix},$$

that is, for any $g \in \mathbf{G}$, we have

$$\theta(g) = \operatorname{Int} w_\ell(g) = w_\ell g w_\ell^{-1}.$$

In the Galois case, the non-trivial element σ of the Galois group of E over F gives rise to an F-involution θ of $\mathbf{G}$ given by coordinate-wise Galois conjugation

$$\theta((a_{ij})) = (\sigma(a_{ij})),$$

where $(a_{ij}) \in \mathbf{GL}_n(E)$. We'll also use w_ℓ to denote the same matrix with unit anti-diagonal in $\mathbf{GL}_n(E)$. The element w_ℓ is diagonalizable over F. In particular, when n is even there exists $x_\ell \in \mathbf{GL}_n(F)$ such that

$$x_\ell w_\ell x_\ell^{-1} = \mathrm{diag}(1_{n/2}, -1_{n/2}), \tag{4.1}$$

where $1_{n/2}$ denotes the $n/2 \times n/2$ identity matrix.

In both cases, define $\mathbf{H} = \mathbf{G}^\theta$ to be the θ-fixed points of $\mathbf{G}$. As usual, $G = \mathbf{G}(F)$ and $H = \mathbf{H}(F)$. It follows that, in the linear case, H is isomorphic to $G_{n/2} \times G_{n/2}$. Indeed, one can check that $H = x_\ell^{-1}(G_{n/2} \times G_{n/2})x_\ell$, where the subgroup $G_{n/2} \times G_{n/2}$ is embedded in G as block-diagonal matrices. In the Galois case, we have that $H = \mathbf{GL}_n(F)$.

For any positive integer r, we'll write J_r for the $r \times r$-matrix in G_r with unit anti-diagonal

$$J_r = \begin{pmatrix} & & 1 \\ & ⋰ & \\ 1 & & \end{pmatrix}.$$

Note that $w_\ell = J_n$ is a representative of the longest element of the Weyl group of $\mathbf{GL}_n$ (with respect to the diagonal F-split torus). In the linear case, θ_r will denote the inner involution $\mathrm{Int}\, J_r$ of $\mathbf{G}_r$ with fixed points $\mathbf{H}_r$. In the Galois case, we let θ_r denote the F-involution of $\mathbf{G}_r$ given by coordinate-wise Galois conjugation and $H_r = \mathbf{GL}_r(F)$ is the group of F-points of the θ_r-fixed subgroup of $\mathbf{G}_r$. In the Galois case, for any positive integer r, there exists $\gamma_r \in G_r = \mathbf{GL}_r(E)$ such that

$$\gamma_r^{-1}\theta_r(\gamma_r) = J_r = \begin{pmatrix} & & 1 \\ & ⋰ & \\ 1 & & \end{pmatrix} \in H_r.$$

For instance, if r is even, then we may take

$$\gamma_r = \begin{pmatrix} 1 & & & & & 1 \\ & \ddots & & & ⋰ & \\ & & 1 & 1 & & \\ & & -\varepsilon & \varepsilon & & \\ & ⋰ & & & \ddots & \\ -\varepsilon & & & & & \varepsilon \end{pmatrix}, \tag{4.2}$$

where $E = F(\varepsilon)$ (*cf.* §1.1), and if r is odd, then we set

$$\gamma_r = \begin{pmatrix} 1 & & & & & & 1 \\ & \ddots & & & & ⋰ & \\ & & 1 & 0 & 1 & & \\ & & 0 & 1 & 0 & & \\ & & -\varepsilon & 0 & \varepsilon & & \\ & ⋰ & & & & \ddots & \\ -\varepsilon & & & & & & \varepsilon \end{pmatrix}. \tag{4.3}$$

In this chapter, we define $\gamma = \gamma_n \in G = \mathbf{GL}_n(E)$. In particular, $w_\ell = J_n = \gamma^{-1}\theta(\gamma)$ is an order-two element of H.

In the Galois case, we define a second involution ϑ of $\mathbf{G}$, that is G-conjugate to θ, by declaring that $\vartheta = \gamma \cdot \theta$ (*cf.* Definition 1.2.1). Explicitly,

$$\vartheta(g) = \gamma^{-1}\theta(\gamma g \gamma^{-1})\gamma, \tag{4.4}$$

for any $g \in \mathbf{G}$. Since $w_\ell = \gamma^{-1}\theta(\gamma)$ is θ-fixed, we have that

$$\vartheta = \operatorname{Int} w_\ell \circ \theta = \theta \circ \operatorname{Int} w_\ell. \tag{4.5}$$

Similarly, for any positive integer r, we define

$$\vartheta_r = \operatorname{Int} J_r \circ \theta_r = \theta_r \circ \operatorname{Int} J_r = \gamma_r \cdot \theta_r. \tag{4.6}$$

In both cases, define $w_+ \in \mathbf{GL}_n(F) \subset \mathbf{GL}_n(E)$ to be the permutation matrix corresponding to the permutation of $\{1, \ldots, n\}$ given by

$$\begin{cases} 2i-1 & \mapsto i \qquad\qquad 1 \le i \le \lfloor n/2 \rfloor + 1 \\ 2i & \mapsto n+1-i \quad 1 \le i \le \lfloor n/2 \rfloor \end{cases} \qquad (n \text{ odd}), \tag{4.7}$$

when n is odd, and when n is even by

$$\begin{cases} 2i-1 & \mapsto i \qquad\qquad 1 \le i \le n/2 \\ 2i & \mapsto n+1-i \quad 1 \le i \le n/2 \end{cases} \qquad (n \text{ even}). \tag{4.8}$$

Remember that in the linear case we'll always assume that n is even. Finally, define

$$w_0 = \begin{cases} w_+ & \text{in the linear case: } G = \mathbf{GL}_n(F), n \ge 4 \text{ even}, \\ {}^{\gamma}w_+ = \gamma w_+ \gamma^{-1} & \text{in the Galois case: } G = \mathbf{GL}_n(E), \text{ any } n \ge 4. \end{cases} \tag{4.9}$$

4.1.2 Tori and root systems relative to θ

In both cases, we choose a θ-stable maximal F-split torus A_0 of G containing a maximal (θ, F)-split torus S_0 (*cf.* §1.5). In the linear case, we set $\mathbf{A}_0$ equal to the diagonal maximal F-split torus of $\mathbf{G}$, and $A_0 = \mathbf{A}_0(F)$. We have that A_0 is θ-stable with action of θ given by

$$\theta(\operatorname{diag}(a_1, \ldots, a_n)) = \operatorname{diag}(a_n, a_{n-1}, \ldots, a_2, a_1), \tag{4.10}$$

where $a_i \in F^\times$. Let S_0 be the (θ, F)-split component of A_0. It is straightforward from (4.10) to check that

$$S_0 = \{\operatorname{diag}(a_1, \ldots, a_{n/2}, a_{n/2}^{-1}, \ldots, a_1^{-1}) : a_i \in F^\times, 1 \le i \le n/2\}; \tag{4.11}$$

moreover, S_0 is a maximal (θ, F)-split torus of G. Indeed, it is readily verified that the upper-triangular Borel subgroup of G is a minimal θ-split parabolic subgroup with Levi subgroup A_0, then it follows from [31, Proposition 4.7(iv)] that S_0 is a maximal (θ, F)-split torus of G contained in A_0 (*cf.* §1.6.1).

In the Galois case, the torus $\mathbf{T}$ obtained as the restriction of scalars of the diagonal torus of $\mathbf{GL}_n$ is a maximal non-split F-torus of $\mathbf{G}$. Write $T = \mathbf{T}(F)$. We identify T with the diagonal matrices in $\mathbf{GL}_n(E)$ and let A_T denote the F-split component of T. Define $T_0 = {}^{\gamma}T = \gamma T \gamma^{-1}$ and let $A_0 = {}^{\gamma}A_T$ denote the F-split component of T_0. The tori T and T_0 are θ-stable; moreover, their F-split components are θ-stable. We have that θ acts trivially on A_T but non-trivially on A_0. Let S be the torus

$$S = \{\operatorname{diag}(a_1, \ldots, a_{\lfloor n/2 \rfloor}, \widehat{1}, a_{\lfloor n/2 \rfloor}^{-1}, \ldots, a_1^{-1}) : a_i \in F^\times, 1 \le i \le \lfloor n/2 \rfloor\},$$

which is a maximal (ϑ, F)-split torus of G (here the symbol $\widehat{\ }$ denotes that the 1 is omitted when n is even). Indeed, S is the (ϑ, F)-split component of the torus T and T is a Levi subgroup of the upper-triangular Borel subgroup of G, which is ϑ-split, and we may apply [31, Proposition 4.7(iv)], as in the linear case. Now, in the Galois case, we define

$$S_0 = {}^{\gamma}S = \gamma S \gamma^{-1}, \tag{4.12}$$

By Lemma 4.1.6, S_0 is a maximal (θ, F)-split torus of G contained in A_0.

Root systems

Let $\Phi_0 = \Phi(G, A_0)$ be the set of roots of G relative to A_0.

In the linear case, we have

$$\Phi_0 = \{\epsilon_i - \epsilon_j : 1 \le i \ne j \le n\}, \tag{4.13}$$

where $\epsilon_i \in X^*(A_0)$ is the i^{th} coordinate (F-rational) character of A_0 such that

$$\epsilon_i(\operatorname{diag}(a_1, \ldots, a_n)) = a_i,$$

for any $\operatorname{diag}(a_1, \ldots, a_n) \in A_0$. Let

$$\Delta_0 = \{\epsilon_i - \epsilon_{i+1} : 1 \le i \le n-1\} \tag{4.14}$$

be the standard base of Φ_0. The set Φ_0^+ of positive roots (determined by Δ_0) is

$$\Phi_0^+ = \{\epsilon_i - \epsilon_j : 1 \le i < j \le n\}.$$

In the Galois case, we relate Φ_0 to another collection of roots, those relative to A_T. Let $\Phi = \Phi(G, A_T)$ be the root system of G with respect to A_T with standard base Δ given by

$$\Delta = \{\epsilon_i - \epsilon_{i+1} : 1 \le i \le n-1\}, \tag{4.15}$$

where, as above, ϵ_i is the i^{th}-coordinate (F-rational) character of the diagonal F-split torus A_T. We observe that $\Phi_0 = {}^{\gamma}\Phi$, where given a root $\beta \in \Phi$,

$$({}^{\gamma}\beta)(a) = \beta({}^{\gamma^{-1}}a) = \beta(\gamma^{-1} a \gamma),$$

for $a \in A_0$. It is straightforward to verify the equality of root spaces $\mathfrak{g}_{{}^{\gamma}\beta} = \operatorname{Ad}\gamma(\mathfrak{g}_\beta)$ in the Lie algebra $\mathfrak{g}$

of G. Thus, in the Galois case, we take

$$\Delta_0 = {}^{\gamma}\Delta \tag{4.16}$$

as a base for Φ_0. The set of positive roots of Φ_0 with respect to Δ_0 is

$$\Phi_0^+ = \left(\operatorname{span}_{\mathbb{Z}_{\geq 0}} \Delta_0\right) \cap \Phi_0,$$

and it is clear that

$$\Phi_0^+ = {}^{\gamma}\Phi^+, \tag{4.17}$$

where Φ^+ is the set of positive roots determined by Δ, which is equal to

$$\Phi^+ = \{\epsilon_i - \epsilon_j : 1 \leq i < j \leq n\}.$$

Since A_0 is θ-stable we obtain an action of θ on Φ_0, where $\alpha \in \Phi_0$ is sent to $\alpha \circ \theta$. It is readily verified that the set Φ_0^θ of θ-fixed roots in Φ_0 is empty. In particular, $\Delta_0^\theta = \emptyset$, that is, there are no θ-fixed simple roots. In fact, we have the following.

Lemma 4.1.1. *The set of simple roots Δ_0 of Φ_0 is a θ-base for Φ_0. In addition, the subset of θ-fixed roots in Φ_0 is empty.*

Proof. We begin in the Galois case, let ${}^{\gamma}a \in A_0$, where $a = \operatorname{diag}(a_1, \ldots, a_n) \in A_T$. First, note that A_T is contained in H; moreover, $\gamma^{-1}\theta({}^{\gamma}a)\gamma = \vartheta(a) = w_\ell a w_\ell^{-1}$. Let ${}^{\gamma}(\epsilon_i - \epsilon_j) \in \Phi_0$ and compute that

$$\begin{aligned}(\theta \cdot {}^{\gamma}(\epsilon_i - \epsilon_j))({}^{\gamma}a) &= {}^{\gamma}(\epsilon_i - \epsilon_j)(\theta({}^{\gamma}a)) \\ &= (\epsilon_i - \epsilon_j)(\gamma^{-1}\theta({}^{\gamma}a)\gamma) \\ &= (\epsilon_i - \epsilon_j)(w_\ell a w_\ell^{-1}) \\ &= (\epsilon_i - \epsilon_j)(\operatorname{diag}(a_n, \ldots, a_1)) \\ &= {}^{\gamma}(\epsilon_{n+1-i} - \epsilon_{n+1-j})({}^{\gamma}a);\end{aligned}$$

therefore, $\theta \cdot {}^{\gamma}(\epsilon_i - \epsilon_j) = {}^{\gamma}(\epsilon_{n+1-i} - \epsilon_{n+1-j})$. The result will now follow from studying the involution $\operatorname{Int} w_\ell$ acting on the roots Φ of the diagonal F-split torus A_T of $\mathbf{GL}_n(E)$, this will complete the proof in both the Galois and linear cases. As above, we see that

$$(\operatorname{Int} w_\ell)(\epsilon_i - \epsilon_j) = (\epsilon_i - \epsilon_j) \circ \operatorname{Int} w_\ell^{-1} = \epsilon_{n+1-i} - \epsilon_{n+1-j}; \tag{4.18}$$

in particular, $\epsilon_i - \epsilon_j$ is $(\operatorname{Int} w_\ell)$-fixed if and only if $i = n+1-i$ and $j = n+1-j$, which would imply that $i = j = (n+1)/2$ and this is impossible since $i \neq j$. It follows, in both cases, that there are no θ-fixed roots in Φ_0.

It remains to show that for any $\alpha \in \Phi_0^+$, we have $\theta(\alpha) \in \Phi_0^-$. It is enough to prove that $\operatorname{Int} w_\ell(\alpha) \in \Phi^-$ for any $\alpha \in \Phi^+$. Let $\alpha = \epsilon_i - \epsilon_j \in \Phi^+$, that is, such that $i < j$. By (4.18), we have $(\operatorname{Int} w_\ell)(\epsilon_i - \epsilon_j) = \epsilon_{n+1-i} - \epsilon_{n+1-j}$, and since $i < j$, we have that $n+1-j > n+1-j$. It follows that $(\operatorname{Int} w_\ell)(\epsilon_i - \epsilon_j) \in \Phi^-$. This proves the result in the linear case. Twisting by γ we have $\theta \cdot {}^{\gamma}(\epsilon_i - \epsilon_j) = {}^{\gamma}(\epsilon_{n+1-i} - \epsilon_{n+1-j})$. Therefore, in the Galois case, since $\Phi_0^+ = {}^{\gamma}\Phi^+$, we obtain that $\theta(\alpha) \in {\Phi_0}^-$, for any $\alpha \in {\Phi_0}^+$. □

Following §1.5.1, since $\Delta_0^\theta = \emptyset$, the restricted root system is just the image of Φ_0 under the restriction map $p : X^*(A_0) \to X^*(S_0)$. That is, we have $\overline{\Phi}_0 = p(\Phi_0)$ and $\overline{\Delta}_0 = p(\Delta_0)$. Moreover, given $\overline{\Theta} \subset \overline{\Delta}_0$, we obtain the corresponding θ-split subset $[\overline{\Theta}] = p^{-1}(\overline{\Theta})$ of Δ_0. Explicitly, in the linear case,

$$\overline{\Delta}_0 = \{\bar{\epsilon}_i - \bar{\epsilon}_{i+1} : 1 \leq i \leq n/2 - 1\} \cup \{2\bar{\epsilon}_{n/2}\}, \tag{4.19}$$

where $\bar{\epsilon}_i \in X^*(S_0)$, $1 \leq i \leq n/2$, is the i^{th} coordinate character of S_0 given by

$$\bar{\epsilon}_i(\operatorname{diag}(a_1, \ldots, a_{n/2}, a_{n/2}^{-1}, \ldots, a_1^{-1})) = a_i.$$

The $n/2 = |\overline{\Delta}_0|$ maximal θ-split subsets of Δ_0, in the linear case, are:

$$\Theta_k = [\overline{\Delta}_0 \setminus \{\bar{\epsilon}_k - \bar{\epsilon}_{k+1}\}] = p^{-1}(\overline{\Delta}_0 \setminus \{\bar{\epsilon}_k - \bar{\epsilon}_{k+1}\}) = \Delta_0 \setminus \{\epsilon_k - \epsilon_{k+1}, \epsilon_{n-k} - \epsilon_{n-k+1}\}, \tag{4.20}$$

for $1 \leq k \leq n/2 - 1$, and

$$\Theta_{n/2} = [\overline{\Delta}_0 \setminus \{2\bar{\epsilon}_{n/2}\}] = p^{-1}(\overline{\Delta}_0 \setminus \{2\bar{\epsilon}_{n/2}\}) = \Delta_0 \setminus \{\epsilon_{n/2} - \epsilon_{n/2+1}\}. \tag{4.21}$$

Similarly in the Galois case, using that $S_0 = {}^\gamma S$, we have that

$$\overline{\Delta}_0 = \{{}^\gamma\bar{\epsilon}_i - {}^\gamma\bar{\epsilon}_{i+1} : 1 \leq i \leq \lfloor n/2 \rfloor - 1\} \cup \{\bar{\alpha}\}, \tag{4.22}$$

with $\bar{\epsilon}_i \in X^*(S)$ as above, where $\bar{\alpha} = {}^\gamma\bar{\epsilon}_{\lfloor n/2 \rfloor}$ when n is odd, and $\bar{\alpha} = 2{}^\gamma\bar{\epsilon}_{n/2}$ when n is even. The $\lfloor n/2 \rfloor = |\overline{\Delta}_0|$ maximal θ-split subsets of Δ_0, in the Galois case, are:

$$\Theta_k = [\overline{\Delta}_0 \setminus \{{}^\gamma\bar{\epsilon}_k - {}^\gamma\bar{\epsilon}_{k+1}\}] = p^{-1}(\overline{\Delta}_0 \setminus \{{}^\gamma\bar{\epsilon}_k - {}^\gamma\bar{\epsilon}_{k+1}\}) = \Delta_0 \setminus \{{}^\gamma(\epsilon_k - \epsilon_{k+1}), {}^\gamma(\epsilon_{n-k} - \epsilon_{n-k+1})\}, \tag{4.23}$$

for $1 \leq k \leq \lfloor n/2 \rfloor - 1$, and

$$\Theta_{\lfloor n/2 \rfloor} = [\overline{\Delta}_0 \setminus \{\bar{\alpha}\}] = p^{-1}(\overline{\Delta}_0 \setminus \{\bar{\alpha}\}) = \Delta_0 \setminus p^{-1}\{\bar{\alpha}\}. \tag{4.24}$$

When n is odd $\Theta_{\lfloor n/2 \rfloor} = \Delta_0 \setminus \{{}^\gamma(\epsilon_{\lfloor n/2 \rfloor} - \epsilon_{\lfloor n/2 \rfloor + 1}), {}^\gamma(\epsilon_{\lfloor n/2 \rfloor + 1} - \epsilon_{\lfloor n/2 \rfloor + 2})\}$, and when n is even $\Theta_{n/2} = \Delta_0 \setminus \{{}^\gamma(\epsilon_{n/2} - \epsilon_{n/2+1})\}$.

Remark 4.1.2. In the Galois case, the θ-split subsets of Δ_0 are the γ-translates of the $(\operatorname{Int} w_\ell)$-split subsets of the standard base of the root system $\Phi = \Phi(G, A_T)$. In fact, since A_T is Galois fixed, the θ-split subsets of Δ_0 are γ-translates of the ϑ-split subsets of Δ.

4.1.3 Maximal θ-split parabolic subgroups

The following is a corollary of the fact that the set of θ-fixed simple roots $\Delta_0^\theta = \emptyset$ is empty.

Corollary 4.1.3. *The Borel subgroup $P_\emptyset = P_0 = M_0N_0$ corresponding to $\emptyset \subset \Delta_0$ is a minimal θ-split parabolic subgroup of G.*

Proof. The subset Δ_0^θ is a minimal θ-split subset of $\overline{\Delta}_0$; therefore, the parabolic $P_{\Delta_0^\theta}$ is a minimal standard θ-split parabolic subgroup [45]. Since $\Delta_0^\theta = \emptyset$, we have $P_{\Delta_0^\theta} = P_\emptyset = P_0$. In the linear case, $M_0 = A_0$ and in the Galois case $M_0 = C_G(A_0) = T_0$. $\square$

Proposition 4.1.4. *The maximal θ-split subsets Θ_k, $1 \leq k \leq \lfloor n/2 \rfloor$, are described in (4.20) and (4.21) in the linear case, respectively (4.23) and (4.24) in the Galois case. It follows that the Δ_0-standard maximal θ-split parabolic subgroups of G are:*

$$P_k := P_{\Theta_k} = \begin{cases} P_{(k,n-2k,k)}, & 1 \leq k \leq n/2 - 1, \quad \text{in the linear case} \\ {}^{\gamma}P_{(k,n-2k,k)}, & 1 \leq k \leq \lfloor n/2 \rfloor - 1, \quad \text{in the Galois case} \end{cases} \tag{4.25}$$

and

$$P_{\lfloor n/2 \rfloor} := P_{\Theta_{\lfloor n/2 \rfloor}} = \begin{cases} P_{(n/2,n/2)}, & \text{in the linear case, since } n \text{ is even} \\ {}^{\gamma}P_{(n/2,n/2)}, & \text{in the Galois case when } n \text{ is even} \\ {}^{\gamma}P_{(\lfloor n/2 \rfloor,1,\lfloor n/2 \rfloor)}, & \text{in the Galois case when } n \text{ is odd.} \end{cases} \tag{4.26}$$

Proof. We first note, in the Galois case, the correspondence between subsets of Δ_0 and Δ, the associated parabolic subgroups, and γ-conjugacy in G. Given $\Theta_0 \subset \Delta_0$, we have that $\Theta_0 = {}^{\gamma}\Theta$ for some subset Θ of Δ. By definition (*cf.* §1.4), we have

$$A_{\Theta_0} = \left(\bigcap_{\alpha \in \Theta_0} \ker \alpha : A_0 \to F^{\times} \right)^{\circ},$$

and since $\Theta_0 = {}^{\gamma}\Theta$ and $A_0 = {}^{\gamma}A_T$, we have that

$$A_{\Theta_0} = \left(\bigcap_{\beta \in \Theta} \ker({}^{\gamma}\beta : A_0 \to F^{\times}) \right)^{\circ} = \gamma \left(\bigcap_{\beta \in \Theta} \left(\ker(\beta : A_T \to F^{\times}) \right) \right)^{\circ} \gamma^{-1} = {}^{\gamma}A_{\Theta};$$

moreover,

$$M_{\Theta_0} = C_G(A_{\Theta_0}) = C_G({}^{\gamma}A_{\Theta}) = \gamma C_G(A_{\Theta})\gamma^{-1} = {}^{\gamma}M_{\Theta}.$$

Let $E_{ij} \in \mathfrak{g}$ be the matrix with a 1 in the (i,j)-position and zeroes elsewhere. Given $\beta = \epsilon_i - \epsilon_j \in \Phi$, the matrices $E_{\beta} = E_{ij}$ and εE_{β} are $\operatorname{Ad} A_T$-eigenvectors, with eigencharacter β, that span the two dimensional (over F) root space $\mathfrak{g}_{\beta}$. It is straightforward to verify that $X_{\alpha} = X_{ij} = \operatorname{Ad}\gamma(E_{ij}) = {}^{\gamma}E_{ij}$, and εX_{α} are $\operatorname{Ad} A_0$-eigenvectors, with eigencharacter $\alpha = {}^{\gamma}\beta \in \Phi_0$, that span the root space $\mathfrak{g}_{\alpha}$. The A_T-root subgroup N_{β} is the F-subgroup of G generated by $\{e + xE_{\beta}, e + y\varepsilon E_{\beta} : x, y \in F\}$. As an abelian group, N_{β} is isomorphic to the additive group $(E, +)$. Similarly, the A_0-root subgroup N_{α} is generated by $\{e + xX_{\alpha}, e + y\varepsilon X_{\alpha} : x, y \in F\}$; moreover, we have

$$e + X_{\alpha} = \gamma\gamma^{-1} + \gamma E_{\beta}\gamma^{-1} = \operatorname{Ad}\gamma(e + E_{\beta}),$$

similarly, $e + \varepsilon X_{\alpha} = \operatorname{Ad}\gamma(e + \varepsilon E_{\beta})$. It follows that $N_{\alpha} = \gamma N_{\beta}\gamma^{-1} = {}^{\gamma}N_{\beta}$, where $\alpha = {}^{\gamma}\beta \in \Phi_0$. More generally, given $\Theta_0 = {}^{\gamma}\Theta \subset \Phi_0$, we have that

$$N_{\Theta_0} = \prod_{\alpha \in {\Phi_0}^+ \setminus \Phi^+_{\Theta_0}} N_{\alpha} = \prod_{{}^{\gamma}\beta \in {}^{\gamma}\Phi^+ \setminus {}^{\gamma}\Phi^+_{\Theta}} N_{{}^{\gamma}\beta} = \prod_{\beta \in \Phi^+ \setminus \Phi^+_{\Theta}} \gamma N_{\beta}\gamma^{-1} = \gamma N_{\Theta}\gamma^{-1},$$

where $\Phi^+_{\Theta_0} = \left(\operatorname{span}_{\mathbb{Z}_{\geq 0}} \Theta_0 \right) \cap \Phi_0 = {}^{\gamma}\Phi_{\Theta}$. That is, given $\Theta_0 = {}^{\gamma}\Theta \subset \Delta_0$, where $\Theta \subset \Delta$, we have that

$N_{\Theta_0} = {}^{\gamma}N_{\Theta}$. It follows that $P_{\Theta_0} = {}^{\gamma}P_{\Theta}$ for any $\Theta_0 = {}^{\gamma}\Theta \subset \Delta_0$.

The maximal θ-split subsets of Δ_0 give rise to maximal θ-split parabolic subgroups $P_k = P_{\Theta_k}$ [45] (*cf.* §1.6). The descriptions of the parabolic subgroups P_k, $1 \le k \le \lfloor n/2 \rfloor - 1$ given in (4.25) now follow from (4.20) in the linear case, respectively (4.23) in the Galois case. On the other hand, the description of $P_{\lfloor n/2 \rfloor}$ given in (4.25) now follow from (4.21) in the linear case, respectively (4.24) in the Galois case. ☐

In both cases, we'll write the Levi factorization of P_k, $1 \le k \le \lfloor n/2 \rfloor$, as $P_k = M_k N_k$, where M_k is the standard Levi factor and N_k is the unipotent radical of P_k. We write A_k for the F-split component and S_k for the (θ, F)-split component of M_k. Explicitly, in the linear case n is even, and we have

$$A_k = \begin{cases} A_{(k,n-2k,k)} & \text{if } 1 \le k \le n/2 - 1 \\ A_{(n/2,n/2)} & \text{if } k = n/2 \end{cases}, \tag{4.27}$$

$$M_k = \begin{cases} M_{(k,n-2k,k)} & \text{if } 1 \le k \le n/2 - 1 \\ M_{(n/2,n/2)} & \text{if } k = n/2 \end{cases}, \tag{4.28}$$

$$N_k = \begin{cases} N_{(k,n-2k,k)} & \text{if } 1 \le k \le n/2 - 1 \\ N_{(n/2,n/2)} & \text{if } k = n/2 \end{cases}, \tag{4.29}$$

$$S_k = \{\mathrm{diag}(\underbrace{a,\ldots,a}_{k}, \underbrace{1\ldots,1}_{n-2k}, \underbrace{a^{-1},\ldots,a^{-1}}_{k}) : a \in F^{\times}\}, \qquad 1 \le k \le n/2 - 1, \qquad \text{and} \tag{4.30}$$

$$S_{n/2} = \{\mathrm{diag}(\underbrace{a,\ldots,a}_{n/2}, \underbrace{a^{-1},\ldots,a^{-1}}_{n/2}) : a \in F^{\times}\}. \tag{4.31}$$

On the other hand, in the Galois case, we have:

$$A_k = \begin{cases} {}^{\gamma}A_{(k,n-2k,k)} & \text{if } 1 \le k \le \lfloor n/2 \rfloor - 1 \\ {}^{\gamma}A_{(\lfloor n/2 \rfloor,\widehat{1},\lfloor n/2 \rfloor)} & \text{if } k = \lfloor n/2 \rfloor \end{cases}, \tag{4.32}$$

$$M_k = \begin{cases} {}^{\gamma}M_{(k,n-2k,k)} & \text{if } 1 \le k \le \lfloor n/2 \rfloor - 1 \\ {}^{\gamma}M_{(\lfloor n/2 \rfloor,\widehat{1},\lfloor n/2 \rfloor)} & \text{if } k = \lfloor n/2 \rfloor \end{cases}, \tag{4.33}$$

$$N_k = \begin{cases} {}^{\gamma}N_{(k,n-2k,k)} & \text{if } 1 \le k \le \lfloor n/2 \rfloor - 1 \\ {}^{\gamma}N_{(\lfloor n/2 \rfloor,\widehat{1},\lfloor n/2 \rfloor)} & \text{if } k = \lfloor n/2 \rfloor \end{cases}, \tag{4.34}$$

$$S_k = \{{}^{\gamma}\,\mathrm{diag}(\underbrace{a,\ldots,a}_{k}, \underbrace{1\ldots,1}_{n-2k}, \underbrace{a^{-1},\ldots,a^{-1}}_{k}) : a \in F^{\times}\}, \qquad 1 \le k \le \lfloor n/2 \rfloor - 1, \qquad \text{and} \tag{4.35}$$

$$S_{\lfloor n/2 \rfloor} = \{{}^{\gamma}\,\mathrm{diag}(\underbrace{a,\ldots,a}_{\lfloor n/2 \rfloor}, \widehat{1}, \underbrace{a^{-1},\ldots,a^{-1}}_{\lfloor n/2 \rfloor}) : a \in F^{\times}\}, \tag{4.36}$$

where $\widehat{1}$ indicates that the 1 is omitted when n is even.

Remark 4.1.5. Note that in the linear case the block-diagonal Levi subgroup $M_{(n_1,\ldots,n_r)}$ is isomorphic to the product of $\mathbf{GL}_{n_i}(F)$, while in the Galois case this Levi subgroup is isomorphic to a product of $R_{E/F}\mathbf{GL}_{n_i}(F)$, which we identify with $\mathbf{GL}_{n_i}(E)$.

4.1.4 The subgroup of θ-fixed points of M_k, $1 \le k \le \lfloor n/2 \rfloor$

Recall that, in the Galois case, ϑ is the involution $\gamma \cdot \theta = \mathrm{Int}\, w_\ell \circ \theta = \theta \circ \mathrm{Int}\, w_\ell$ (*cf.* (4.4)). In the Galois case, we make the following simple observation, which is a special case of part of Lemma 2.2.6.

Lemma 4.1.6. *Assume that we are in the Galois case. An element of G of the form ${}^\gamma x$ is θ-fixed (respectively θ-split) if and only if x is ϑ-fixed (respectively ϑ-split).*

Proof. Let $x \in G$ and consider the image of ${}^\gamma x$ under θ

$$\theta({}^\gamma x) = \theta(\gamma)\theta(x)\theta(\gamma^{-1}) = \gamma\gamma^{-1}\theta(\gamma)\theta(x)\theta(\gamma^{-1})\gamma\gamma^{-1} = \gamma\ \vartheta(x)\gamma^{-1};$$

in particular, ${}^\gamma x$ is θ-fixed (respectively θ-split) if and only if $x = \vartheta(x)$ (respectively $x^{-1} = \vartheta(x)$). □

Fix an integer $1 \le k \le \lfloor n/2 \rfloor$. If n is even and $k = n/2$ then we simply have to omit the "centre-blocks" of the subgroups below. We saw above that M_k is equal to $M_{(k,n-2k,k)}$ in the linear case (4.28), respectively ${}^\gamma M_{(k,n-2k,k)}$ in the Galois case (4.33).

Galois case

Beginning in the Galois case, we want to determine ${M_k}^\theta$ by using Lemma 4.1.6. Let ${}^\gamma m \in M_k$ where $m \in M_{(k,n-2k,k)}$. For the sake of brevity, we'll write $M_\bullet = M_{(k,n-2k,k)}$. Explicitly, we have $m = \mathrm{diag}(A, B, C)$, where $A, C \in \mathbf{GL}_k(E)$ and $B \in \mathbf{GL}_{n-2k}(E)$. Compute $\vartheta(m)$ as follows:

$$\begin{aligned}
\vartheta(m) &= \mathrm{Int}\, w_\ell \circ \theta(m) \\
&= w_\ell\, \mathrm{diag}(\theta_k(A), \theta_{n-2k}(B), \theta_k(C)) w_\ell^{-1} \\
&= \begin{pmatrix} & & J_k \\ & J_{n-2k} & \\ J_k & & \end{pmatrix} \begin{pmatrix} \theta_k(A) & & \\ & \theta_{n-2k}(B) & \\ & & \theta_k(C) \end{pmatrix} \begin{pmatrix} & & J_k \\ & J_{n-2k} & \\ J_k & & \end{pmatrix}^{-1} \\
&= \begin{pmatrix} J_k\theta_k(C)J_k^{-1} & & \\ & J_{n-2k}\theta_{n-2k}(B)J_{n-2k}^{-1} & \\ & & J_k\theta_k(A)J_k^{-1} \end{pmatrix} \\
&= \mathrm{diag}(\vartheta_k(C), \vartheta_{n-2k}(B), \vartheta_k(A)), \qquad (4.37)
\end{aligned}$$

where, for any positive integer r, we have $\vartheta_r = \mathrm{Int}\, J_r \circ \theta_r = \theta_r \circ \mathrm{Int}\, J_r = \gamma_r \cdot \theta_r$ (*cf.* (4.6)). It follows that

$$(M_\bullet)^\vartheta = \left\{ \begin{pmatrix} A & & \\ & B & \\ & & \vartheta_k(A) \end{pmatrix} : A \in G_k,\ B \in G_{n-2k},\ B = \vartheta_{n-2k}(B) \right\}. \qquad (4.38)$$

In particular, the subgroup of ϑ-fixed points $(M_\bullet)^\vartheta$ of $M_\bullet = M_{(k,n-2k,k)}$ is isomorphic to $G_k \times H_{n-2k}$. Indeed, by conjugating $(M_\bullet)^\vartheta$ by the element $\gamma_\bullet = \mathrm{diag}(\gamma_k, \gamma_{n-2k}, \gamma_k)$ of $M_\bullet$ and applying Lemma 4.1.6, we obtain that $(M_\bullet)^\vartheta$ is $M_\bullet$-conjugate (and F-isomorphic to) the subgroup

$$H_\bullet = \left\{ \begin{pmatrix} A & & \\ & B & \\ & & \theta_k(A) \end{pmatrix} : A \in G_k,\ B \in H_{n-2k} \right\}, \qquad (4.39)$$

In particular, for $1 \le k \le \lfloor n/2 \rfloor - 1$, we have

$$M_k^\theta = {}^\gamma(M_\bullet^\vartheta) \cong G_k \times H_{n-2k}, \qquad (4.40)$$

where the last isomorphism is given by G-conjugacy to the subgroup $H_\bullet$. When $k = \lfloor n/2 \rfloor$, we see that

$$M_{\lfloor n/2 \rfloor}^{\theta} = {}^{\gamma}(M_\bullet^{\vartheta}) \cong \begin{cases} G_{n/2} & n \text{ even} \\ G_{\lfloor n/2 \rfloor} \times F^{\times} & n \text{ odd} \end{cases}. \tag{4.41}$$

Remark 4.1.7. For clarity, in the Galois case, we record the relationships between the fixed-point subgroups M_k^θ, $(M_\bullet)^\vartheta$ and $H_\bullet$. We have that

$$\begin{aligned} H_\bullet &= \gamma_\bullet (M_\bullet)^\vartheta \gamma_\bullet^{-1} & \text{and} \\ {M_k}^\theta &= \gamma (M_\bullet)^\vartheta \gamma^{-1} = \gamma\gamma_\bullet^{-1}(H_\bullet)\gamma_\bullet\gamma^{-1}, \end{aligned}$$

where $\gamma_\bullet = \operatorname{diag}(\gamma_k, \gamma_{n-2k}, \gamma_k) \in M_\bullet = M_{(k,n-2k,k)}$.

Remark 4.1.8. Below, we will use the G-conjugacy of M^θ to $H_\bullet$ to reduce M^θ-distinction to studying $H_\bullet$-distinction. It will be important that $H_\bullet$ is an $M_\bullet$-conjugate of $(M_\bullet)^\vartheta$.

Linear case

The linear case can be obtained by studying the calculation of (4.38). One simply ignores the Galois conjugation in the computation (4.37) of $\vartheta(m)$. Let $m = \operatorname{diag}(A, B, C) \in M_k = M_{(k,n-2k,k)}$, where A, C lie in $\mathbf{GL}_k(F)$, and $B \in \mathbf{GL}_{n-2k}(F)$, then following the above computation of $\vartheta(m)$, we have that

$$\theta(m) = \operatorname{diag}(\theta_k(C), \theta_{n-2k}(B), \theta_k(A)).$$

We obtain that the θ-fixed point subgroup of M_k, in the linear case, is equal to

$$M_k^\theta = \left\{ \begin{pmatrix} A & & \\ & B & \\ & & \theta_k(A) \end{pmatrix} : A \in G_k,\ B \in G_{n-2k}(F),\ B = \theta_{n-2k}(B) \right\}. \tag{4.42}$$

In particular, for $1 \le k \le n/2 - 1$, we have

$$M_k^\theta \cong G_k \times H_{n-2k}, \tag{4.43}$$

and

$$M_{n/2}^\theta = \left\{ \begin{pmatrix} A & \\ & \theta_k(A) \end{pmatrix} : A \in G_{n/2} \right\} \cong G_{n/2}. \tag{4.44}$$

4.1.5 H-conjugacy of θ-split parabolic subgroups

Recall that $\sigma \in \operatorname{Gal}(E/F)$ is the non-trivial element of the Galois group of E over F.

Proposition 4.1.9. *Any θ-split parabolic subgroup P of G is H-conjugate to a Δ_0-standard θ-split parabolic subgroup P_Θ, for some $\Theta \subset \Delta_0$.*

Proof. In the linear case, by Corollary 4.1.3, A_0 is the Levi subgroup of the Δ_0-standard minimal θ-split parabolic subgroup $P_0 = A_0N_0$. Observe that the group of F-points of $\mathbf{A}_0 \cap \mathbf{H}$ is equal to

$$(\mathbf{A}_0 \cap \mathbf{H})(F) = \{\operatorname{diag}(a_1,\ldots,a_{n/2},a_{n/2},\ldots,a_1) : a_i \in F^\times, 1 \le i \le n/2\}.$$

In particular, $\mathbf{A}_0 \cap \mathbf{H}$ is an F-split torus. By Hilbert's Theorem 90, $\mathbf{A}_0 \cap \mathbf{H}$ has trivial Galois cohomology over F in degree 1.

Again by Corollary 4.1.3, in the Galois case, the non-split torus T_0 is the Levi subgroup of the Δ_0-standard minimal θ-split parabolic subgroup $P_0 = T_0N_0$ of G. Moreover, T_0 is isomorphic to the product of n-copies of $(R_{E/F}\mathbb{G}_m)(F) \simeq E^\times$. Observe that the group of F-points of $\mathbf{T}_0 \cap \mathbf{H}$ is equal to

$$(\mathbf{T}_0 \cap \mathbf{H})(F) = \left(\{{}^\gamma \operatorname{diag}(a_1,\ldots,a_{\lfloor n/2\rfloor},\widehat{a},\sigma(a_{\lfloor n/2\rfloor}),\ldots,\sigma(a_1)) : a_i \in E^\times, 1 \le i \le \lfloor n/2\rfloor, a \in F^\times\}\right),$$

where the $a \in F^\times$, is omitted when n is even. Indeed, suppose that

$$t = {}^\gamma \operatorname{diag}(a_1,\ldots,a_{\lfloor n/2\rfloor},\widehat{a},a_{\lfloor n/2\rfloor+1},\ldots,a_n)$$

is a θ-fixed element of T_0. By (4.5) we have $\vartheta = \gamma\cdot\theta = \operatorname{Int} w_\ell \circ \theta$, and we compute that

$$\begin{aligned}
{}^\gamma \operatorname{diag}(a_1,\ldots,a_{\lfloor n/2\rfloor},\widehat{a},a_{\lfloor n/2\rfloor+1},\ldots,a_n) &= \theta({}^\gamma \operatorname{diag}(a_1,\ldots,a_{\lfloor n/2\rfloor},\widehat{a},a_{\lfloor n/2\rfloor+1},\ldots,a_n))\\
&= \gamma\ \vartheta(\operatorname{diag}(a_1,\ldots,a_{\lfloor n/2\rfloor},\widehat{a},a_{\lfloor n/2\rfloor+1},\ldots,a_n))\ \gamma^{-1}\\
&= {}^\gamma \operatorname{diag}(\sigma(a_n),\ldots,\sigma(a_{\lfloor n/2\rfloor+1}),\widehat{\sigma(a)},\sigma(a_{\lfloor n/2\rfloor}),\ldots,\sigma(a_1)).
\end{aligned}$$

Therefore, it must be the case that $a_i = \sigma(a_{n+1-i}) \in E^\times$, for $1 \le i \le \lfloor n/2\rfloor$, and $a = \sigma(a)$ so that $a \in F^\times$. In particular, $\mathbf{T}_0 \cap \mathbf{H}$ is F-isomorphic to the product of $\lfloor n/2\rfloor$ copies of $R_{E/F}\mathbb{G}_m$ and one copy of $\mathbb{G}_m$ when n is odd. Without loss of generality, assume that n is odd so that the $F^\times = \mathbb{G}_m(F)$ factor appears in $(\mathbf{T}_0 \cap \mathbf{H})(F)$. So far, we have observed that

$$\mathbf{T}_0 \cap \mathbf{H} \cong \mathbb{G}_m \times \prod_{i=1}^{\lfloor n/2\rfloor} R_{E/F}\mathbb{G}_m.$$

Computing the first Galois cohomology of $\mathbf{T}_0 \cap \mathbf{H}$ over F, we have

$$H^1(F,\mathbf{T}_0 \cap \mathbf{H}) \cong H^1(F,\mathbb{G}_m) \oplus \bigoplus_{i=1}^{\lfloor n/2\rfloor} H^1(F,R_{E/F}\mathbb{G}_m).$$

By Hilbert's Theorem 90, $H^1(F,\mathbb{G}_m) = 0$ is trivial. By Shapiro's Lemma (Lemma 3.1.2), we have that $H^1(F,R_{E/F}\mathbb{G}_m)$ is isomorphic to $H^1(E,\mathbb{G}_m)$, and hence equal to zero. In particular, we have $H^1(F,\mathbf{T}_0 \cap \mathbf{H}) = 0$.

By Lemma 3.1.1, we have that $(\mathbf{HA}_0)(F) = HA_0$ (respectively $(\mathbf{HT}_0)(F) = HT_0$). The proposition follows from [44, Lemma 2.5(2)] (see Lemma 1.6.4 for a statement). □

4.1.6 A class of θ-elliptic Levi subgroups and θ-stable parabolic subgroups

A minimal θ-elliptic Levi and semi-standard-θ-elliptic Levi subgroups

We define the Levi subgroup L_0 of G to be the centralizer in G of the F-split torus $(A_0^\theta)^\circ$.

Lemma 4.1.10. *The Levi subgroup $L_0 = C_G\left((A_0^\theta)^\circ\right)$ of G is θ-elliptic. Moreover, L_0 is minimal among θ-elliptic Levi subgroup of G that contain A_0.*

Proof. First, we observe that L_0 is θ-stable. Let $l \in L_0$ and $a \in (A_0^\theta)^\circ$, we have

$$\theta(l)a\theta(l)^{-1} = \theta(l)\theta(a)\theta(l)^{-1} = \theta(lal^{-1}) = \theta(a) = a;$$

in particular, $\theta(l)$ centralizes a, so $\theta(l) \in L_0$ and L_0 is θ-stable. It is immediate that A_0 is contained in L_0 since A_0 is abelian; moreover, A_0 is a maximal F-split torus of L_0.

Now, we show that L_0 is θ-elliptic. First, note that the (θ, F)-split component S_G of G is the trivial group. Indeed, in the linear case θ is inner and we have that $A_G \cong F^\times$ is contained in H. It follows from (1.2) that $S_G = (\{\pm e\})^\circ = \{e\}$. Again, in the Galois case, θ acts trivially on the F-split component of the centre A_G of G and $S_G = \{e\}$. In both the linear and Galois cases, it is readily verified that the F-split component of the centre of L_0 is equal to $(A_0^\theta)^\circ$, that is, $A_{L_0} = (A_0^\theta)^\circ$. Moreover, in both cases, we actually have that $(A_0^\theta)^\circ = A_0^\theta$. (We give an explicit description of $(A_0^\theta)^\circ$ immediately following the proof.) In particular, A_{L_0} is contained in H and it follows that $S_{L_0} = \{e\}$ (*cf.* [34, §1.3]). By Lemma 1.6.7 L_0 is θ-elliptic.

Finally, we prove that L_0 is minimal among θ-elliptic Levi subgroups containing A_0. Suppose that $L \subset L_0$ is a proper Levi subgroup of L_0 that contains A_0. We'll argue that L cannot be θ-elliptic. Since L is proper in L_0, we have that $A_{L_0} = (A_0^\theta)^\circ$ is a proper sub-torus of A_L. Following [34, §1.3], we have an almost direct product $A_L = (A_L^\theta)^\circ S_L$. Observe that, since $A_L \subset A_0$, we have

$$(A_L^\theta)^\circ = (A_L \cap A_0^\theta)^\circ = (A_L \cap A_{L_0})^\circ = A_{L_0}.$$

Since $S_G = S_{L_0} = \{e\} \subset A_{L_0} = (A_L^\theta)^\circ$ and $A_L = (A_L^\theta)^\circ S_L$ properly contains A_{L_0}, it must be the case that S_G is a proper subtorus of S_L; in particular, S_L is non-trivial. It follows from Lemma 1.6.7 that L is not θ-elliptic and this completes the proof. $\square$

In both cases, we give the following definition.

Definition 4.1.11. A θ-elliptic Levi subgroup L of G is *semi-standard-θ-elliptic* if L contains L_0.

For completeness, we give explicit descriptions of the subgroups considered in Lemma 4.1.10. In the linear case, we have that

$$(A_0^\theta)^\circ = A_0^\theta = \{\mathrm{diag}(a_1, \ldots, a_{n/2}, a_{n/2}, \ldots, a_1) : a_i \in F^\times, 1 \le i \le n/2\}; \tag{4.45}$$

therefore, $L_0 = C_G(A_0^\theta)$ is isomorphic to a product of $n/2$-copies of $\mathbf{GL}_2(F)$ and $A_{L_0} = A_0^\theta$. While in the Galois case, we have $A_0^\theta = {}^\gamma(A_T^\vartheta)$, where

$$(A_T^\vartheta)^\circ = A_T^\vartheta = \{\mathrm{diag}(a_1, \ldots, a_{\lfloor n/2 \rfloor}, \widehat{a_\bullet}, a_{\lfloor n/2 \rfloor}, \ldots, a_1) : a_\bullet, a_i \in F^\times, 1 \le i \le \lfloor n/2 \rfloor\}. \tag{4.46}$$

Indeed, the action of ϑ on A_T is given by conjugation by the Weyl group element w_ℓ, since A_T is θ-fixed. Observe that γ centralizes A_T^ϑ; therefore, we actually have that $A_0^\theta = A_T^\vartheta$ and $L_0 = C_G(A_0^\theta) = C_G(A_T^\vartheta)$.

It follows that, in the Galois case,

$$L_0 \cong \widehat{R_{E/F}\mathbb{G}_m}(F) \times \prod_{i=1}^{\lfloor n/2 \rfloor} (\mathbb{R}_{E/F}\mathbf{GL}_2)(F) \simeq \widehat{\mathbb{G}_m(E)} \times \prod_{i=1}^{\lfloor n/2 \rfloor} \mathbf{GL}_2(E), \tag{4.47}$$

and F-split component of the centre of L_0 is $A_0^\theta = A_T^\vartheta$. In fact, we observe below that in both cases L_0 is the w_+-conjugate of $M_{(2,\ldots,2,\widehat{1})}$, that is $L_0 = w_+ M_{(2,\ldots,2,\widehat{1})} w_+^{-1}$.

Observation 4.1.12. If L is a ϑ-elliptic Levi subgroup of G, then $L = {}^\gamma L$ is a θ-elliptic Levi subgroup. In fact, since γ centralizes $A_0^\theta = A_T^\vartheta$ it is the case that standard-θ-elliptic and standard-ϑ-elliptic Levi subgroups coincide. Where standard-ϑ-elliptic Levi subgroups are ϑ-elliptic Levi subgroups containing $L_0' = C_G(A_T^\vartheta)$. We have that $L_0' = L_0$, since γ centralizes A_T^ϑ and $A_0^\theta = A_T^\vartheta$. In fact, we observe that A_0^θ is equal to the w_+-conjugate the of the F-split torus

$$A_{(2,\ldots,2,\widehat{1})} = \{\mathrm{diag}(a_1, a_1, a_2, a_2, \ldots, a_{\lfloor n/2 \rfloor}, a_{\lfloor n/2 \rfloor}, \widehat{a}) : a, a_i \in F^\times\},$$

corresponding to the partition $(2, \ldots, 2, \widehat{1})$ of n. Moreover, L_0 is the w_+-conjugate of the Levi subgroup $M_{(2,\ldots,2,\widehat{1})}$ corresponding to the same partition. The last remark also holds in the linear case, A_0 is the w_+-conjugate of $A_{(\underline{2})}$ and L_0 is the w_+-conjugate of $M_{(\underline{2})}$, where $(\underline{2})$ is the partition $(2, \ldots, 2)$ of the even number n.

θ-stable parabolic subgroups with semi-standard-θ-elliptic Levi factors

The parabolic subgroups, with semi-standard-θ-elliptic Levi subgroups, described below will form the inducing subgroups in the subsequent construction of relative discrete series representations for $H\backslash G$. Our first goal in this subsection is to describe a choice Δ^{ell} of simple roots in Φ_0, obtained as a Weyl group translate of Δ_0, such that certain Δ^{ell}-standard Levi subgroups are semi-standard-θ-elliptic.

Recall, above we defined w_+ to be the permutation matrix corresponding to the permutation of $\{1, \ldots, n\}$ defined by

$$\begin{cases} 2i-1 & \mapsto i \qquad\qquad 1 \le i \le \lfloor n/2 \rfloor + 1 \\ 2i & \mapsto n+1-i \quad 1 \le i \le \lfloor n/2 \rfloor \end{cases}, \qquad (n \text{ odd}),$$

when n is odd (*cf.* (4.7)), and when n is even (*cf.* (4.8)) by

$$\begin{cases} 2i-1 & \mapsto i \qquad\qquad 1 \le i \le n/2 \\ 2i & \mapsto n+1-i \quad 1 \le i \le n/2 \end{cases} \qquad (n \text{ even}).$$

We defined $w_0 = w_+$ in the linear case, and $w_0 = \gamma w_+ \gamma^{-1} = {}^\gamma w_+$ in the Galois case (*cf.* (4.9)). First, we prove a lemma needed for the Galois case.

Lemma 4.1.13. *In the Galois case, conjugation by γ maps $N_G(A_T)$ to $N_G(A_0)$ and induces an explicit isomorphism of the Weyl group $W_T = W(G, A_T)$, with respect to A_T, with the Weyl group $W_0 = W(G, A_0)$, with respect to A_0. Moreover, we identify W_T with the group of permutation matrices in G (isomorphic to the symmetric group $\mathfrak{S}_n$) and W_0 with the γ-conjugates of these permutation matrices.*

Proof. Recall that $A_0 = {}^\gamma A_T$. Let $t = {}^\gamma a \in A_0$, where $a \in A_T$, and let $w \in N_G(A_T)$. We have that

$waw^{-1} = b \in A_T$. We observe that

$$^{\gamma}w\ t\ ^{\gamma}w^{-1} = \gamma w\gamma^{-1}(\gamma a\gamma^{-1})\gamma w^{-1}\gamma^{-1} = \gamma waw^{-1}\gamma^{-1} = \gamma b\gamma^{-1} = {}^{\gamma}b \in A_0;$$

in particular, we have that ${}^{\gamma}w \in N_G(A_0)$. Of course, $\mathrm{Int}\,\gamma : N_G(A_T) \to N_G(A_0)$ is a bijective homomorphism. Note that $T_0 = C_G(A_0)$. Consider the composition

$$\begin{array}{c} N_G(A_T) \to N_G(A_0) \twoheadrightarrow N_G(A_0)/T_0 \cong W_0 \\ w \mapsto {}^{\gamma}w \quad {}^{\gamma}w \mapsto [{}^{\gamma}w] \end{array}$$

where $[{}^{\gamma}w]$ is the equivalence class of ${}^{\gamma}w$ in W_0. Suppose that $[{}^{\gamma}w]$ is in the kernel of the composite homomorphism, then ${}^{\gamma}w = t \in T_0$, where $t = {}^{\gamma}a$ for some $a \in T$. It follows that $w = a$ is an element of T; in particular, the kernel of the composite map is T. The homomorphism $w \mapsto [{}^{\gamma}w]$ descends to an isomorphism from W_T to W_0 sending $[w]$ to $[{}^{\gamma}w]$. $\square$

In both the linear and Galois cases, define

$$\Delta^{ell} = w_0\Delta_0 \tag{4.48}$$

and call this choice of simple roots the *θ-elliptic simple* roots of G with respect to A_0. Of course, since Δ^{ell} is a Weyl group translate of Δ_0, we have that Δ^{ell} is a base of Φ_0. We next describe why this is the correct choice for our purposes. In the Galois case, we explore the relationship with $\Delta \subset \Phi = \Phi(G, A_T)$. In both cases, set

$$\Delta_{odd} = \{\epsilon_i - \epsilon_{i+1} : i \text{ is odd}\}, \tag{4.49}$$

and in the Galois case further denote

$$\Delta_{0,odd} = {}^{\gamma}\Delta_{odd} \subset \Delta_0 = {}^{\gamma}\Delta. \tag{4.50}$$

In the linear case, the define the subset $\Delta^{ell}_{\min}$ of Δ^{ell} by

$$\Delta^{ell}_{\min} = w_0\Delta_{odd} \tag{4.51}$$

and in the Galois case, define

$$\Delta^{ell}_{\min} = w_0\Delta_{0,odd}. \tag{4.52}$$

In both cases, the subset $\Delta^{ell}_{\min}$ is exactly the subset of θ-elliptic simple roots that cuts out the torus A_0^{θ} from A_0. In particular,

$$A_0^{\theta} = A_{\Delta^{ell}_{\min}} = \left(\bigcap_{\beta \in \Delta^{ell}_{\min}} \ker(\beta : A_0 \to F^{\times}) \right)^{\circ}, \tag{4.53}$$

and

$$L_0 = L_{\Delta^{ell}_{\min}} = C_G\left(A_{\Delta^{ell}_{\min}}\right).$$

In the Galois case, set $\Delta^{\vartheta-\mathrm{ell}} = w_+\Delta$ and $\Delta^{\vartheta-\mathrm{ell}}_{\min} = w_+\Delta_{odd}$. Then $\Delta^{\vartheta-\mathrm{ell}}$ is a base for Φ and $\Delta^{\vartheta-\mathrm{ell}}_{\min}$ is the subset that cuts out the torus A_T^{ϑ} from A_T. Essentially, $\Delta^{\vartheta-\mathrm{ell}}$ is the ϑ-analog of Δ^{ell}. That is,

$$A_T^{\vartheta} = A_{\Delta^{\vartheta-\mathrm{ell}}_{\min}} := \left(\bigcap_{\alpha\in\Delta^{\vartheta-\mathrm{ell}}_{\min}} \ker(\alpha : A_T \to F^\times)\right)^{\circ};$$

moreover, since γ centralizes A_T^{ϑ}, we have $A_T^{\vartheta} = A_0^{\theta}$. We observe that in the Galois case,

$$\Delta^{ell} = w_0\Delta_0 = (\gamma w_+\gamma^{-1}\gamma)\cdot\Delta = {}^{\gamma}(w_+\Delta) = {}^{\gamma}(\Delta^{\vartheta-\mathrm{ell}}). \tag{4.54}$$

Explicitly, we have

$$\Delta^{\vartheta-\mathrm{ell}} = \begin{cases} \{\epsilon_{n+1-i} - \epsilon_{i+1} : 1 \le i \le \lfloor n/2\rfloor\} \cup \Delta^{\vartheta-\mathrm{ell}}_{\min} & \text{if } \ n \text{ is odd} \\ \{\epsilon_{n+1-i} - \epsilon_{i+1} : 1 \le i \le \lfloor n/2\rfloor - 1\} \cup \Delta^{\vartheta-\mathrm{ell}}_{\min} & \text{if } \ n \text{ is even} \end{cases},$$

where $\Delta^{\vartheta-\mathrm{ell}}_{\min} = \{\epsilon_i - \epsilon_{n+1-i} : 1 \le i \le \lfloor n/2\rfloor\}$. We'll write $\Delta_{even} = \{\epsilon_i - \epsilon_{i+1} : 2 \le i \le n-1,\ i \text{ is even}\}$ and note that $\Delta^{\vartheta-\mathrm{ell}} \setminus \Delta^{\vartheta-\mathrm{ell}}_{\min} = w_+(\Delta_{even})$.

Definition 4.1.14. A Levi subgroup L of G is a standard-θ-elliptic Levi subgroup if and only if L is Δ^{ell}-standard and contains L_0.

Observe that a standard-θ-elliptic Levi subgroup L is semi-standard-θ-elliptic (*cf.* Definition 4.1.11).

Proposition 4.1.15. *Let $\Omega^{ell} \subset \Delta^{ell}$ such that Ω^{ell} contains $\Delta^{ell}_{\min}$.*

1. *The Δ^{ell}-standard parabolic subgroup $Q = Q_{\Omega^{ell}}$, associated to Ω^{ell}, is θ-stable.*
2. *In particular, the unipotent radical $U = U_{\Omega^{ell}}$ is θ-stable.*
3. *The Levi subgroup $L = L_{\Omega^{ell}} = C_G(A_{\Omega^{ell}})$ is a standard-θ-elliptic Levi of G.*
4. *The modular function δ_Q of Q satisfies $\left.\delta_Q^{1/2}\right|_{L^\theta} = \delta_{Q^\theta}$.*
5. *We have that*

$$L \cong \prod_{i=1}^{k} G_{m_i} \qquad \text{and} \qquad L^\theta \cong \prod_{i=1}^{k} H_{m_i} \tag{4.55}$$

where $\sum_{i=1}^{k} m_i = n$, such that when n is odd exactly one m_i is odd, and when n is even all of the m_i are even.

Proof. As noted above (*cf.* Observation 4.1.12), we have that A_0^{θ} is equal to the w_+-conjugate the of the F-split torus

$$A_{(2,\dots,2,\widehat{1})} = \{\mathrm{diag}(a_1, a_1, a_2, a_2, \dots, a_{\lfloor n/2\rfloor}, a_{\lfloor n/2\rfloor}, \widehat{a}) : a, a_i \in F^\times\},$$

corresponding to the partition $(2, \ldots, 2, \widehat{1})$ of n. Moreover, L_0 is the w_+-conjugate of the block diagonal Levi subgroup $M_{(2,\ldots,2,\widehat{1})}$ corresponding to the same partition.

Given $\Omega^{ell} \subset \Delta^{ell}$ above, such that Ω^{ell} contains $\Delta^{ell}_{\min}$, it is immediate from (4.53) that

$$A_L = A_{\Omega^{ell}} = \left(\bigcap_{\beta \in \Omega^{ell}} \ker(\beta : A_0 \to F^\times) \right)^\circ$$

is contained in $A_{L_0} = A_0^\theta$. Since $A_{\Omega^{ell}} \subset A_0^\theta$, we see that L_0 is contained in L. By Lemma 1.6.6 it follows that L is a θ-elliptic Levi subgroup of G. In particular, L is θ-stable and standard-θ-elliptic. By Proposition 1.6.8, Q is θ-stable. We can realize $\Omega^{ell} = w_+\Omega$, where $\Omega \subset \Delta_0$ such that $\Delta_{odd} \subset \Omega$, in the linear case, and $\Omega^{ell} = \gamma w_+ \underline{\Omega}$, for some $\underline{\Omega} \subset \Delta$ such that $\underline{\Omega}$ contains Δ_{odd}, in the Galois case. It follows that L is w_+-conjugate to a block-diagonal Levi subgroup $M_{(m_1,\ldots,m_k)}$, corresponding to a partition $(m_1, \ldots, m_k)$ of n, and giving the first isomorphism in (4.55), i.e.,

$$L = w_+ M_{(m_1,\ldots,m_k)} w_+^{-1} \cong \prod_{i=1}^k G_{m_i}.$$

In the Galois case, we're using that γ centralizes $w_+ A_{\underline{\Omega}} w_+^{-1} \subset A_T^\vartheta = w_+ A_{\Delta_{odd}} w_+^{-1}$. Since $L_0 \subset L$ and $L_0 = w_+ M_{(2,\ldots,2,\widehat{1})} w_+^{-1}$ we must have that $M_{(2,\ldots,2,\widehat{1})} \subset M_{(m_1,\ldots,m_k)}$. In particular, $(2, \ldots, 2, \widehat{1})$ is a refinement of the partition $(m_1, \ldots, m_k)$; therefore, when n is odd exactly one m_j is odd, when n is even all of the m_i are even.

Next, we explicitly determine L^θ. Realize L as $w_+ M w_+^{-1}$, where $M = M_{(m_1,\ldots,m_k)}$, as above. Begin in the linear case; given $m \in M$, $l = w_+ m w_+^{-1} \in L$ is θ-fixed, if and only if m is fixed by the involution $\theta_+ = \mathrm{Int}(w_+^{-1} w_\ell w_+)$. So, in the linear case, we have $L^\theta = w_+(M^{\theta_+})w_+^{-1}$. It is straightforward to compute that $w_+^{-1} w_\ell w_+$ is the permutation matrix of $(1\ 2)(3\ 4)\ldots(n-1\ n)$, i.e.,

$$w_+^{-1} w_\ell w_+ = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix} \in M_{(\underline{2})} = M_{(2,\ldots,2)} \subset G.$$

In particular, since m_j is even for all $1 \le j \le k$, we see that θ_+ acts on j^{th}-block, a copy of $G_{m_j} = \mathbf{GL}_{m_j}(F)$, of M as conjugation by the matrix

$$w_{m_j} = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix} \in G_{m_j};$$

moreover, w_{m_j} is G_{m_j}-conjugate to J_{m_j}. It follows that θ_+, acting on M, is M-conjugate to the product-involution $\theta_{m_1} \times \ldots \times \theta_{m_k}$; therefore, we have that

$$L^\theta = w_+(M^{\theta_+})w_+^{-1} \cong M^{\theta_+} \cong \prod_{j=1}^{k}(\mathbf{GL}_{m_j}(F))^{\theta_{m_j}} = \prod_{j=1}^{k} H_{m_j},$$

and the second isomorphism of (4.55) follows.

In the Galois case, note that w_+ is θ-fixed; moreover, note that M is θ-stable. Given $l = w_+mw_+^{-1} \in L$, $m \in M$, we have that l is θ-fixed if and only if m is θ-fixed. It follows that $L^\theta = w_+M^\theta w_+^{-1}$. Since θ acts on M by Galois conjugation in each matrix entry, and

$$M = M_{(m_1,\ldots,m_k)} = \prod_{i=1}^{k} G_{m_i},$$

embedded block-diagonally in G, it is immediate that

$$M^\theta = \prod_{i=1}^{k}(G_{m_i})^{\theta_{m_i}} = \prod_{i=1}^{k} H_{m_i};$$

therefore, we have that

$$L^\theta = w_+M^\theta w_+^{-1} \cong \prod_{i=1}^{k} H_{m_i},$$

as claimed. In both cases, we have $L^\theta \cong \prod_{i=1}^k H_{m_i}$ where L is realized as $L = w_+M_{(m_1,\ldots,m_k)}w_+^{-1}$.

In the Galois case, the statement about modular functions is [27, Lemma 5.5], a proof is given in [49, Lemma 2.5.1]. In the linear case, one may compute the modular functions by hand to verify the desired equality. We omit the straightforward computation. This completes the proof of Proposition 4.1.15. $\square$

4.2 Constructing RDS in the linear and Galois cases

We are now in a position to give our construction of relative discrete series representations in the linear and Galois cases. Recall that we made a particular choice $\Delta^{ell} = w_0\Delta_0$ of simple roots for Φ_0 (*cf.* (4.48)); moreover, we identified a subset $\Delta^{ell}_{\min}$ of Δ^{ell} that determines the minimal θ-elliptic Levi subgroup L_0 (*cf.* (4.51) and (4.52)). We remind the reader that a Levi subgroup L is standard-θ-elliptic if L is Δ^{ell}-standard and contains $L_0 = L_{\Delta^{ell}_{\min}}$ (*cf.* Definition 4.1.14). Our main theorem is the following; we spend the remainder of this chapter proving this result.

Theorem 4.2.1. *Let $\Omega^{ell} \subset \Delta^{ell}$ such that Ω^{ell} contains $\Delta^{ell}_{\min}$. The proper Δ^{ell}-standard parabolic subgroup $Q = Q_{\Omega^{ell}}$ is θ-stable with standard-θ-elliptic Levi subgroup $L = L_{\Omega^{ell}}$. Let τ be a regular L^θ-distinguished discrete series representation of L. The parabolically induced representation $\pi = \iota_Q^G\tau$ is an H-distinguished relative discrete series representation of G. Moreover, π is in the complement of the discrete series representations of G.*

Remark 4.2.2. The RDS representations constructed in Theorem 4.2.1 are irreducibly induced from discrete series and are thus tempered generic representations of G [41].

Remark 4.2.3. If in Theorem 4.2.1 we further suppose that τ is a unitary supercuspidal representation, then π is a non-supercuspidal relatively supercuspidal representation of G. A proof of this result can be reconstructed from our proof of the main theorem. Indeed, in Lemma 4.2.15, when $P_\Theta \cap {}^wM_\Omega$ is proper in ${}^wM_\Omega$ the representation $\mathscr{F}^y_\Theta(\tau)$ vanishes because τ is supercuspidal. Then by Proposition 4.2.23, all nonzero irreducible subquotients of the Jacquet modules of π along proper Δ_0-standard θ-split parabolic subgroups are not distinguished. By Proposition 3.5.1 and Proposition 4.1.9 (*cf.* [44, Remark 6.10]), we obtain that $r_P\lambda^G = 0$, for every proper θ-split parabolic subgroup P of G and any $\lambda^G \in \mathrm{Hom}_H(\pi, 1)$. By Theorem 2.2.16 (Kato–Takano), π is relatively supercuspidal; moreover, since π is parabolically induced, π is not supercuspidal. Note that this modification of Theorem 4.2.1 can be obtained by more direct methods; see, for instance, the work of Murnaghan [57] for such results in a more general setting.

Remark 4.2.4. Let $P = MN$ be a θ-split parabolic subgroup of G with θ-stable Levi subgroup M. It is natural to ask if one may construct RDS representations of G via parabolic induction from P. Let δ be an M^θ-distinguished discrete series representation of M. Define $\pi_\delta = \iota_P^G\delta$. The representation π_δ is H-distinguished by the meromorphic continuation argument due to Blanc and Delorme [8]. In particular, an H-invariant form λ on π_δ is obtained as a limit of invariant forms on a holomorphic family of representations of G with limit π_δ. In this sense, π_δ appears "continuously" in $C^\infty(H\backslash G)$. It is exactly for this reason that we do not expect π_δ to occur in the discrete spectrum of $H\backslash G$. In addition, if one considers the Jacquet module of π_δ along $P = MN$, then the M^θ-distinguished unitary representation δ appears as an irreducible subquotient of $(\pi_\delta)_N$. Thus, we expect that π_δ fails the Relative Casselman's Criterion 2.2.18. However, we do not currently have a method to detect non-vanishing of $r_P\lambda$ and we cannot currently determine if π_δ is a RDS.

4.2.1 Characterization of inducing representations

Here and for the remainder of this chapter, we fix a proper Δ^{ell}-standard θ-stable parabolic subgroup $Q = Q_{\Omega^{ell}}$, for some $\Omega^{ell} \subset \Delta^{ell}$ containing $\Delta^{ell}_{\min}$. As in Proposition 4.1.15, the subgroup Q admits a standard-θ-elliptic Levi subgroup $L = L_{\Omega^{ell}}$. The parabolic subgroup Q has a Levi factorization $Q = LU$ with unipotent radical $U = U_{\Omega^{ell}}$. We refer to §4.1.2 for the definitions of Δ_0 (linear case: (4.14), Galois case: (4.16)), Δ^{ell} (4.48), $\Delta^{ell}_{\min}$ (linear case: (4.51), Galois case: (4.52)), Δ_{odd} (4.49), and $\Delta_{0,odd}$ (4.50).

In the linear case, we have $L = w_+M_\Omega w_+^{-1}$, where $\Omega \subset \Delta_0$ contains Δ_{odd}. Moreover, the block-diagonal subgroup M_Ω is equal to $M_{(m_1,\dots,m_k)}$, for some partition $(m_1, \dots, m_k)$ of n, where each m_i is even, $1 \le i \le k$.

In the Galois case, we have $L = w_0M_\Omega w_0^{-1} = w_+M_{\underline{\Omega}}w_+^{-1}$, where $\Omega^{ell} = w_0\Omega = (\gamma w_+)\underline{\Omega}$, and $\underline{\Omega} \subset \Delta$ contains Δ_{odd}, and $\Omega = {}^\gamma\underline{\Omega} \subset \Delta_0$ contains $\Delta_{0,odd}$. As above, $M_{\underline{\Omega}} = M_{(m_1,\dots,m_k)}$, for some partition $(m_1, \dots, m_k)$ of n, refined by the partition $(2, \dots, 2, \widehat{1})$. In both cases, we have that $Q = w_0P_\Omega w_0^{-1}$ is a Weyl group conjugate of a Δ_0-standard parabolic subgroup P_Ω. Moreover, in the Galois case, we have $w_0P_\Omega w_0^{-1} = w_+P_{\underline{\Omega}}w_+^{-1}$.

Note. In the Galois case, it is helpful to note that $M_\Omega = {}^\gamma M_{\underline{\Omega}}$, where $\underline{\Omega}$ contains Δ_{odd} and $\Omega = {}^\gamma\underline{\Omega}$ (since γ centralizes A_T^ϑ).

Let M denote $M_\Omega = M_{(m_1,\dots,m_k)}$ in the linear case and $M_{\underline{\Omega}} = M_{(m_1,\dots,m_k)}$ in the Galois case. An irreducible admissible representation τ' of M is the (external) tensor product

$$\tau' = \bigotimes_{i=1}^{k} \tau_i, \tag{4.56}$$

where τ_i is an irreducible admissible representation of G_{m_i}, $1 \leq i \leq k$. In the Galois case, we further obtain a representation τ_0 of the Δ_0-standard Levi subgroup M_Ω by conjugating τ' by γ. That is, the representation $\tau_0 = {}^\gamma\tau'$ of M_Ω on the space of τ' with action given by

$$\tau_0(m) = \tau'(\gamma^{-1} m \gamma) \qquad\qquad \forall m \in M_\Omega.$$

In the linear case, $\tau_0 = \tau'$ is a representation of the Δ_0-standard Levi subgroup M. In both cases, we set

$$\tau = {}^{w_0}\tau_0, \tag{4.57}$$

which is the corresponding representation of L (remember that $w_0 = w_+$, in the linear case, *cf.* (4.9)).

Remark 4.2.5. In the Galois case, we have $\tau = {}^{w_0}\tau_0 = {}^{\gamma w_+}\tau'$ so going through M_Ω isn't really necessary; however, we ultimately want to work with representations of Δ_0-standard Levi subgroups. On the other hand, it is also helpful to think about the usual block-diagonal Levi subgroups of G.

As in the proof of Proposition 4.1.15, in the linear case $L^\theta = w_0 M^{\theta_+} w_0^{-1}$, where $\theta_+ = w_0 \cdot \theta = \mathrm{Int}(w_+^{-1} w_\ell w_+)$, and in the Galois case, we have $L^\theta = w_0 M^\theta w_0^{-1}$. It follows that τ is L^θ-distinguished if and only if τ' is M^{θ_+}-distinguished in the linear case, respectively M^θ-distinguished in the Galois case. It is straightforward to show, using (4.56), that τ' is M^{θ_+}-distinguished, respectively M^θ-distinguished, if and only if τ_i is H_{m_i}-distinguished, for all $1 \leq i \leq k$. We record this as a lemma. Note that, in the Galois case, $\gamma \in L_0 \subset L$ (*cf.* (4.2) and (4.3)).

Lemma 4.2.6. *Let $\tau' = \bigotimes_{i=1}^k \tau_i$ be an irreducible admissible representation of M, where τ_i is an irreducible admissible representation of G_{m_i}, $1 \leq i \leq k$.*

1. *The corresponding irreducible admissible representation τ of L is L^θ-distinguished if and only if τ_i is H_{m_i}-distinguished, for all $1 \leq i \leq k$.*
2. *If τ is L^θ-distinguished, then the space $\mathrm{Hom}_{L^\theta}(\tau, 1)$ is one-dimensional.*

Proof. In the linear case, given $\lambda' \in \mathrm{Hom}_{M^{\theta_+}}(\tau', 1)$, one may verify that $\lambda = \lambda' \in \mathrm{Hom}_{L^\theta}(\tau, 1)$. Indeed, given $l = w_0 m w_0^{-1}$, where $m \in M^{\theta_+}$, we have that for any $v \in V_\tau = V_{\tau'}$,

$$\langle \lambda, \tau(l)v \rangle = \langle \lambda', {}^{w_0}\tau'(w_0 m w_0^{-1})v \rangle = \langle \lambda', \tau'(m)v \rangle = \langle \lambda', v \rangle = \langle \lambda, v \rangle.$$

In the Galois case, given $\lambda' \in \mathrm{Hom}_{M^\theta}(\tau', 1)$, we have that $\lambda = \lambda' \circ \tau(\gamma)$ is an element of $\mathrm{Hom}_{L^\theta}(\tau, 1)$. (This is well defined since $\gamma \in L_0 \subset L$, and the representations τ', τ_0, and τ are on the same space.) In both cases, we see that the L^θ-distinction of τ is equivalent to M^{θ_+}-, respectively M^θ-distinction of τ'.

In the proof of Proposition 4.1.15, we saw that M^{θ_+}, respectively M^θ, is M-conjugate to the block-diagonal subgroup $\prod_{i=1}^k H_{m_i}$ (we actually have equality in the Galois case). Without loss of generality, we'll study M^{θ_+}-distinction of τ' in the linear case. In fact, in the proof of Proposition 4.1.15, we observed that θ_+ acting on M is M-equivalent to the product involution $\theta_{m_1} \times \ldots \times \theta_{m_k}$. By Lemma 2.2.6, distinction of τ' by M^{θ_+} (respectively M^θ) is equivalent to $\prod_{i=1}^k H_{m_i}$-distinction. Finally, it is immediate that τ' is $\prod_{i=1}^k H_{m_i}$-distinguished if and only if τ_i is H_{m_i}-distinguished for all $1 \leq i \leq k$. Multiplicity-one in the linear case holds by Proposition 2.3.3, and by Proposition 2.3.20 in the Galois case. □

Remark 4.2.7. In particular, if we take each τ_i to be H_{m_i}-distinguished irreducible admissible square integrable (discrete series) representations of G_{m_i}, then the associated representation τ of L is an L^θ-distinguished discrete series representation of L. The known results on the H_{m_i}-distinguished discrete series representations τ_i are discussed in sections §2.3.2 and §2.3.3; in particular, such representations exist.

Proposition 4.2.8. *Let $L = L_{\Omega^{ell}}$ be a standard-θ-elliptic Levi subgroup of G corresponding to a subset Ω^{ell} of Δ^{ell} that contains $\Delta^{ell}_{\min}$. There exist infinitely many equivalence classes of regular non-supercuspidal L^θ-distinguished discrete series representations of L.*

Proof. By assumption $n \geq 4$ and n is always taken to be even in the linear case. We have that L is isomorphic to a product $\prod_{i=1}^k G_{m_i}$ of smaller general linear groups, for some partition $(m_1, \ldots, m_k)$ of n. Let τ_i be an irreducible admissible representation of G_{m_i}, $1 \leq i \leq k$. The representation $\tau_1 \otimes \ldots \otimes \tau_k$ of L is regular if and only if $\tau_i \ncong \tau_j$ for all $1 \leq i \neq j \leq k$ (Lemma 4.2.22). Moreover, $\tau_1 \otimes \ldots \otimes \tau_k$ is supercuspidal (square integrable) if and only if every τ_i is supercuspidal (square integrable). To prove the proposition, it is sufficient to prove that for any relevant partition of n (*cf.* Proposition 4.1.15), there exist pairwise inequivalent H_{m_i}-distinguished discrete series representations τ_i of G_{m_i}, $1 \leq i \leq k$, such that at least one τ_i is not supercuspidal.

In the linear case, by Proposition 4.1.15, each m_i, $1 \leq i \leq k$ is even. By Theorem 2.3.15, for $m_i \geq 2$, there are infinitely many equivalence classes of H_{m_i}-distinguished irreducible supercuspidal representations of G_{m_i}. By Corollary 2.3.17, for $m_i \geq 2$ there exists at least one non-supercuspidal H_{m_i}-distinguished discrete series representation of G_{m_i} (infinitely many when $m_i \geq 6$). It follows from Lemma 4.2.6 that there exist infinitely many equivalence classes of regular non-supercuspidal L^θ-distinguished discrete series representations of L, in the linear case.

In the Galois case, by Proposition 4.1.15, at most one m_i is odd. Without loss of generality, assume that m_k is odd. By Theorem 2.3.26(1), the Steinberg representation St_{m_k} of G_{m_k} is a non-supercuspidal H_{m_k}-distinguished discrete series. By Theorem 2.3.28, for all $m_i \geq 2$, there are infinitely many equivalence classes of H_{m_i}-distinguished irreducible supercuspidal representations of G_{m_i}. By Corollary 2.3.27 and Proposition 2.3.29, for all $m_i \geq 2$ there exists at least one non-supercuspidal H_{m_i}-distinguished discrete series representation of G_{m_i} (infinitely many when $m_i \geq 4$ is not an odd prime). It follows from Lemma 4.2.6 that there exist infinitely many equivalence classes of regular non-supercuspidal L^θ-distinguished discrete series representations of L, in the Galois case. □

Corollary 4.2.9. *There are infinitely many equivalence classes of H-distinguished relative discrete series representations of G of the form constructed in Theorem 4.2.1. In particular, there are infinitely many classes of such representations where the discrete series τ is not supercuspidal.*

Proof. This is immediate from Proposition 4.2.8 and [75, Theorem 9.7(b)] applied to the representations constructed in Theorem 4.2.1. □

On the other hand, not every θ-elliptic Levi subgroup of G admits distinguished discrete series representations. To give an example, in the linear case, we prove the following.

Lemma 4.2.10. *In the linear case, let $Q = LU$ be the parabolic subgroup $x_\ell^{-1} P_{(n_1,\ldots,n_r)} x_\ell$ of G, for some partition $(n_1, \ldots, n_r)$ of n, with Levi factor $L = x_\ell^{-1} M_{(n_1,\ldots,n_r)} x_\ell$. Then Q is θ-stable and the Levi subgroup L is θ-elliptic. There are no L^θ-distinguished discrete series representations of the non-semi-standard-θ-elliptic Levi L.*

Proof. By (4.1), we have that $x_\ell w_\ell x_\ell^{-1} = \operatorname{diag}(1_{n/2}, -1_{n/2})$, where $1_{n/2}$ is the $n/2 \times n/2$-identity matrix. It is straightforward to verify that the parabolic subgroups $x_\ell^{-1} P_\Theta x_\ell$, with $\Theta \subset \Delta_0$, are all θ-stable. Using that, in this case, all maximal (θ, F)-split tori of G are H-conjugate (this follows from Proposition 4.1.9), it is not difficult to show that L does not contain any non-trivial semisimple (θ, F)-split elements. In particular, L is θ-elliptic since θ acts trivially on A_L.

We'll now show that L does not admit any L^θ-distinguished discrete series. There are two cases to consider, first when $(n_1, \ldots, n_r)$ is a refinement of the partition $(n/2, n/2)$ and second when this is not the case. When $(n_1, \ldots, n_r)$ is a refinement of the partition $(n/2, n/2)$, L is contained in $x_\ell^{-1} M_{(n/2,n/2)} x_\ell$, and the involution θ acts trivially on $L = x_\ell^{-1} M_{(n_1,\ldots,n_r)} x_\ell$. If this is the case, $L^\theta = L$ and a representation τ of L is L-distinguished if and only if τ admits the trivial representation as a quotient. If $L^\theta = L$, then the only L^θ-distinguished irreducible representation of L is the trivial representation. In particular, there are no L^θ-distinguished representations in the discrete spectrum of L. Now assume that $(n_1, \ldots, n_r)$ does not refine $(n/2, n/2)$. There is an integer k, $1 \le k \le r$, such that $n_1 + \ldots + n_{k-1} < n/2 < n_1 + \ldots + n_k$. In this case, it is straightforward to check that

$$L^\theta \cong G_{n_1} \ldots \times G_{n_{k-1}} \times M_{({}^1n_k, {}^2n_k)} \times G_{n_{k+1}} \ldots \times G_{n_r}$$

where $({}^1n_k, {}^2n_k)$ is a partition of n_k, i.e., $n_k = {}^1n_k + {}^2n_k$. It follows that a representation of L is L^θ-distinguished if and only if it is of the form $1 \otimes \ldots \otimes 1 \otimes \tau_k \otimes 1 \otimes \ldots \otimes 1$, where τ_k is a representation of $\mathbf{GL}_{n_k}(F)$ distinguished by the maximal Levi subgroup $M_{({}^1n_k, {}^2n_k)}$. Again there are no L^θ-distinguished discrete series representations of L. □

Assume that $Q = LU$ is as in Lemma 4.2.10. Suppose that n_k is even and τ_k is an $M_{(n_k/2, n_k/2)}$-distinguished discrete series. One may ask, in analogy with Appendix A, if the representation $\pi = \iota_Q^G(1 \otimes \ldots \otimes 1 \otimes \tau_k \otimes 1 \otimes \ldots \otimes 1)$ is a relative discrete series for $H \backslash G$. Note that L is the x_ℓ^{-1}-conjugate of the Levi $M_{(n_1,\ldots,n_r)}$, which is contained in the maximal Δ_0-standard θ-split parabolic subgroup $P = MN$ corresponding to the partition $\left(n - \frac{n_k}{2}, n_k, n - \frac{n_k}{2}\right)$ of n. For simplicity, assume that $M_{(n_1,\ldots,n_r)} = M$. The unitary representation $1 \otimes \tau_k \otimes 1$ occurs as a subquotient of the Jacquet module π_N; moreover, $1 \otimes \tau_k \otimes 1$ is M^θ-distinguished by Corollary 4.2.21. The exponent of π_N contributed by $1 \otimes \tau_k \otimes 1$ (*cf.* Lemma 2.1.24) will fail the Relative Casselman's Criterion (2.14). We do not expect that π is a RDS for G; however, we do not have a proof. The main obstruction is our lack of knowledge regarding non-vanishing the invariant forms of Lagier and Kato–Takano [48, 44] (see Definition 2.2.14).

Remark 4.2.11. We briefly explain the restriction that n is even in the linear case. Assume that $n = 2m + 1$ is odd. Let $G = \mathbf{GL}_n(F)$ and let $\theta = \operatorname{Int} w_\ell$. The θ-fixed point subgroup H of G is isomorphic to $\mathbf{GL}_{m+1}(F) \times \mathbf{GL}_m(F)$. Following Proposition 4.1.15, one can see that a standard-θ-elliptic Levi subgroup L is isomorphic to $\prod_{i=1}^k \mathbf{GL}_{m_i}(F)$, where exactly one m_i is odd. By Theorem 2.3.8 (Matringe), there are no L^θ-distinguished discrete series representations of L. In particular, RDS representations of $H \backslash G$ cannot be produced via a direct analog of Theorem 4.2.1. For a construction of RDS for $\mathbf{GL}_2(F) \times \mathbf{GL}_1(F) \backslash \mathbf{GL}_3(F)$, we direct the reader to Appendix A where we describe all RDS for $\mathbf{GL}_{n-1}(F) \times \mathbf{GL}_1(F) \backslash \mathbf{GL}_n(F)$ by adapting [74].

4.2.2 H-distinction, multiplicity-one and π not discrete for G

Let $Q = LU$ be as in the previous subsection §4.2.1. Applying Lemma 3.3.1 and Proposition 4.1.15, we obtain an injection of $\mathrm{Hom}_{L^\theta}(\tau, 1)$ into the space $\mathrm{Hom}_H(\pi, 1)$ of H-invariant linear forms on π. In particular, we have the following proposition.

Proposition 4.2.12. *Let τ be an irreducible admissible representation of L. If τ is L^θ-distinguished, then the representation $\pi = \iota_Q^G\tau$ is H-distinguished.*

By Proposition 2.3.3 (Jacquet–Rallis), we have that multiplicity-one holds in the linear case. In the Galois case, multiplicity-one is given by Proposition 2.3.20 (Flicker)

Proposition 4.2.13. *(Jacquet–Rallis, Flicker) Let π be an irreducible admissible representation of G, then $\dim \mathrm{Hom}_H(\pi, 1)$ is at most one-dimensional*

In particular, the H-invariant form λ^G in $\mathrm{Hom}_H(\pi, 1)$, that we produce via Lemma 3.3.1 from $\lambda \in \mathrm{Hom}_{L^\theta}(\tau, 1)$, is unique up to scalar multiples. In fact, Proposition 4.2.13 holds for the inducing data as well and λ itself is unique up to scalar multiples. Since Q is a proper parabolic, we have that π is in the complement of the discrete series of G [75]. Moreover, when τ is a regular discrete series (unitary) representation of L, we have that the induced representation $\pi = \iota_Q^G\tau$ is irreducible by Theorem 2.1.22 (Bruhat). We record these facts as the following proposition.

Proposition 4.2.14. *If τ is a regular discrete series representation of L, then $\pi = \iota_Q^G\tau$ is irreducible and does not occur in the discrete spectrum of G.*

To complete the proof of Theorem 4.2.1, it only remains to show that π satisfies the Relative Casselman's Criterion 2.2.18 [45].

4.2.3 Exponents along maximal standard θ-split parabolic subgroups

We now work under the notation and hypotheses of Theorem 4.2.1.

Let P_Θ be a maximal Δ_0-standard θ-split parabolic subgroup of G, given by a θ-split $\Theta \subset \Delta_0$. Let M_Θ be the standard Levi subgroup of P_Θ and let N_Θ be the unipotent radical P_Θ. Recall that such subgroups are characterized in Proposition 4.1.4. By the Geometric Lemma 2.1.17, there exists a filtration of the Jacquet module π_{N_Θ} of π along P_Θ such that the associated graded object is given by

$$\mathrm{gr}(\pi_{N_\Theta}) = \bigoplus_{y \in M_\Theta \backslash S(M_\Theta, L)/L} \mathcal{F}_\Theta^y(\tau),$$

where $M_\Theta\backslash S(M_\Theta, L)/L$ is in bijection with $P_\Theta\backslash G/Q$ (*cf.* (2.3)) and

$$\mathcal{F}_\Theta^y(\tau) = \iota_{M_\Theta \cap {}^yQ}^{M_\Theta}\left(({}^y\tau)_{N_\Theta \cap {}^yL}\right).$$

By Lemma 2.1.24, the exponents of π_{N_Θ} are exactly the central characters of the irreducible subquotients of the representations $\mathcal{F}_\Theta^y(\tau)$ of M_Θ. We work with representatives for $P_\Theta\backslash G/Q$ given by $[W_\Theta\backslash W_0/W_\Omega] \cdot w_0^{-1}$ where, as in Lemma 2.1.19, the set of Weyl group elements

$$[W_\Theta\backslash W_0/W_\Omega] = \{w \in W_0 : w\Omega > 0, w^{-1}\Theta > 0\},$$

gives a set of representatives for $P_\Theta\backslash G/P_\Omega$. Let $w \in [W_\Theta\backslash W_0/W_\Omega]$. By Proposition 2.1.20(3), we have that $P_\Theta \cap {}^wM_\Omega$ is a $(w\Omega)$-standard parabolic subgroup of ${}^wM_\Omega = M_{w\Omega}$ with unipotent radical $N_\Theta \cap {}^wM_\Omega$. Similarly, $M_\Theta \cap {}^wP_\Omega = M_\Theta \cap P_{w\Omega}$ is a Θ-standard parabolic subgroup of M_Θ with unipotent radical $M_\Theta \cap {}^wN_\Omega$. In addition, $M_{\Theta\cap w\Omega} = M_\Theta \cap {}^wM_\Omega$ is a standard Levi factor of both parabolic subgroups. With the choice $[W_\Theta\backslash W_0/W_\Omega] \cdot w_0^{-1}$ of representatives for $P_\Theta\backslash G/Q$ and using that $\tau = {}^{w_0}\tau_0$ (4.57), we have that for $y = ww_0^{-1}$, with $w \in [W_\Theta\backslash W_0/W_\Omega]$, the representation

$$\mathcal{F}^y_\Theta(\tau) = \iota^{M_\Theta}_{M_\Theta\cap {}^wP_\Omega}\left(({}^w\tau_0)_{N_\Theta\cap {}^wM_\Omega}\right), \tag{4.58}$$

of M_Θ can be computed using Δ_0-standard parabolic and Levi subgroups. We are assuming that τ_0 is an irreducible regular unitary (square integrable) representation of M_Ω. When $M_\Theta \cap {}^wP_\Omega$ is a proper parabolic of M_Θ, by Lemma 2.1.26 the exponents of $\mathcal{F}^y_\Theta(\tau)$ are the restriction to A_Θ of the exponents of ${}^w\tau_0$ along $P_\Theta \cap {}^wM_\Omega$. In addition, since ${}^w\tau_0$ is a square integrable representation of ${}^wM_\Omega$, the exponents of ${}^w\tau_0$ along $P_\Theta \cap {}^wM_\Omega$ satisfy Casselman's Criterion for square-integrability, see Theorem 2.1.28. We record this here as a lemma.

Lemma 4.2.15. *The exponents of $\mathcal{F}^y_\Theta(\tau)$ are the restriction to A_Θ of the exponents of ${}^w\tau_0$ along the parabolic subgroup $P_\Theta \cap {}^wM_\Omega$ of ${}^wM_\Omega$. If $\tau = {}^{w_0}\tau_0$ is a discrete series representation of L, and the parabolic subgroup $P_\Theta \cap {}^wM_\Omega$ of ${}^wM_\Omega$ is proper, then for any exponent $\chi \in \mathcal{E}xp_{A_\Theta}(\mathcal{F}^y_\Theta(\tau))$ of $\mathcal{F}^y_\Theta(\tau)$, the inequality $|\chi(a)|_F < 1$ is satisfied for every $a \in A^{-w\Omega}_{\Theta\cap w\Omega} \setminus A^1_{\Theta\cap w\Omega}A_{w\Omega}$.*

Proof. This is a special case of Lemma 2.1.26 and Theorem 2.1.28 applied to the discrete series representation ${}^w\tau_0$ of ${}^wM_\Omega$. The dominant cone $A^{-w\Omega}_{\Theta\cap w\Omega} \setminus A^1_{\Theta\cap w\Omega}A_{w\Omega}$ is described in Definition 3.4.2. ☐

In the case that $P_\Theta \cap {}^wM_\Omega$ not a proper parabolic of ${}^wM_\Omega$, we make the following observation.

Lemma 4.2.16. *Assume, as above, that τ is a regular unitary irreducible admissible representation of L. If $y = ww_0^{-1}$ is such that $P_\Theta \cap {}^wM_\Omega = {}^wM_\Omega$, then $\mathcal{F}^y_\Theta(\tau)$ is irreducible and the central character $\chi_{\Theta,y}$ of $\mathcal{F}^y_\Theta(\tau)$ is unitary.*

Proof. If $P_\Theta \cap {}^wM_\Omega = {}^wM_\Omega$, then $N_\Theta \cap {}^wM_\Omega = \{e\}$ and ${}^wM_\Omega \subset M_\Theta$. It follows that the representation $({}^w\tau_0)_{N_\Theta\cap {}^wM_\Omega}$ is equal to ${}^w\tau_0$ and is irreducible and unitary. Moreover, since τ is a regular representation of L, it follows that ${}^w\tau_0$ is regular as a representation of ${}^wM_\Omega$ regarded as a Levi subgroup of M_Θ. By Theorem 2.1.22 (Bruhat), the representation $\mathcal{F}^y_\Theta(\tau)$ is irreducible and unitary. By the irreducibility of $\mathcal{F}^y_\Theta(\tau)$, the only exponent is its central character $\chi_{\Theta,y}$. Since $\mathcal{F}^y_\Theta(\tau)$ is unitary, the character $\chi_{\Theta,y}$ of A_Θ is unitary. ☐

Remark 4.2.17. We find ourselves in the situation of Lemma 4.2.16 in two cases

(A) when ${}^wM_\Omega = M_\Theta$ if and only if Θ and Ω are associate and $w \in [W_\Theta\backslash W_0/W_\Omega] \cap W(\Theta, \Omega)$,

(B) when $w \in [W_\Theta\backslash W_0/W_\Omega]$ is such that ${}^wM_\Omega \subsetneq M_\Theta$ is a proper Levi subgroup of M_Θ.

Below, we will use the assumption that τ is regular to show that, in these two cases, we do not have M^θ_Θ-distinction of the irreducible subquotient $\mathcal{F}^y_\Theta(\tau)$ of π_{N_Θ}. In particular, we can safely ignore the unitary exponents contributed by $\mathcal{F}^y_\Theta(\tau)$ when applying the Relative Casselman's Criterion 2.2.18 [45], since the invariant form $r_P\lambda^G$ must vanish on the corresponding generalized eigenspaces (*cf.* Proposition 3.5.1).

A technical lemma

In order to apply the Relative Casselman's Criterion 2.2.18 to $\pi = \iota_Q^G \tau$, we will give a reduction to Casselman's Criterion 2.1.28 for the inducing data. Here we prove a technical result, Lemma 4.2.19, required for this reduction (below we're implicitly using Proposition 2.1.20).

Remark 4.2.18 (Motivation for Lemma 4.2.19). The left-hand side of (4.59) is the (θ, F)-split cone on which we must evaluate exponents in order to verify the relative version of Casselman's Criterion (Theorem 2.2.18) for π. The right-hand side of (4.59) is the cone on which we evaluate the exponents of the representation ${}^w\tau_0$ along the parabolic $P_\Theta \cap {}^w M_\Omega$ in Casselman's Criterion (Theorem 2.1.28).

Lemma 4.2.19. *Let Ω be a subset of Δ_0 such that $\Omega^{ell} = w_0 \Omega$ contains $\Delta_{\min}^{ell}$ and determines a standard-θ-elliptic Levi subgroup of G. Let Θ be a maximal θ-split subset of Δ_0. Let $w \in [W_\Theta \backslash W_0 / W_\Omega]$ such that $M_{\Theta \cap w\Omega} = M_\Theta \cap w M_\Omega w^{-1}$ is a proper Levi subgroup of $M_{w\Omega} = w M_\Omega w^{-1}$. Then we have the containment*

$$S_\Theta^- \setminus S_\Theta^1 S_{\Delta_0} \subset A_{\Theta \cap w\Omega}^{-w\Omega} \setminus A_{\Theta \cap w\Omega}^1 A_{w\Omega}. \tag{4.59}$$

See Definition 3.4.2, (2.14) and (1.8) for the notation used in (4.59). Recall that $S_\Theta^1 = S_\Theta(\mathcal{O}_F)$ and $A_{\Theta \cap w\Omega}^1 = A_{\Theta \cap w\Omega}(\mathcal{O}_F)$ (*cf.* Remark 2.2.19).

Proof. First, S_Θ is contained in A_Θ and since $\Theta \cap w\Omega$ is a subset of Θ, we have that

$$\mathbf{A}_\Theta = \left(\bigcap_{\alpha \in \Theta} \ker \alpha \right)^\circ \subset \left(\bigcap_{\alpha \in \Theta \cap w\Omega} \ker \alpha \right)^\circ = \mathbf{A}_{\Theta \cap w\Omega}.$$

At the level of F-points, we have $A_\Theta \subset A_{\Theta \cap w\Omega}$, and similarly for the integer points $A_\Theta^1 \subset A_{\Theta \cap w\Omega}^1$. It follows that $S_\Theta \subset A_{\Theta \cap w\Omega}$, and $S_\Theta^1 \subset A_{\Theta \cap w\Omega}^1$. Also, we have $S_{\Delta_0} \subset A_{\Delta_0} \subset A_\Omega$, and since $S_{\Delta_0} = S_G$ is central in G, we have $S_{\Delta_0} = w S_{\Delta_0} w^{-1} \subset w A_\Omega w^{-1} = A_{w\Omega}$. We now observe that $A_\Theta^- \subset A_{\Theta \cap w\Omega}^{-w\Omega}$ and in particular that $S_\Theta^- \subset A_{\Theta \cap w\Omega}^{-w\Omega}$; it is clear that $S_\Theta^- \subset A_\Theta^-$. Note that $w\Omega$ is a base for the root system of $M_{w\Omega}$ relative to the maximal F-split torus A_0. Suppose that, $a \in A_\Theta^-$, then $|\alpha(a)| \leq 1$, for all $\alpha \in \Delta_0 \setminus \Theta$; moreover, since $a \in A_\Theta$ we have that $|\alpha(a)| = 1$, for $\alpha \in \Theta$ as well. Let $\beta \in w\Omega \setminus (\Theta \cap w\Omega)$, then $\beta = w\alpha$ for some $\alpha \in \Omega$. Since $w \in [W_\Theta \backslash W_0 / W_\Omega]$, we have that $\beta = w\alpha \in \Phi_0^+$. Write

$$\beta = \sum_{\epsilon \in \Delta_0} c_\epsilon \cdot \epsilon,$$

where $c_\epsilon \geq 0$. Then we have that

$$|\beta(a)| = \left| \prod_{\epsilon \in \Delta_0} \epsilon(a)^{c_\epsilon} \right| = \prod_{\epsilon \in \Delta_0} |\epsilon(a)|^{c_\epsilon} \leq 1,$$

since $|\epsilon(a)| \leq 1$, for all $\epsilon \in \Delta_0$, and $c_\epsilon \geq 0$. In particular, $a \in A_{\Theta \cap w\Omega}^{-w\Omega}$. Putting this together, we see that $S_\Theta^1 S_{\Delta_0} \subset S_\Theta^- \cap A_{\Theta \cap w\Omega}^1 A_{w\Omega}$; therefore, to prove the desired result, it suffices to prove the opposite inclusion.

It is at this point that we specialize to the two explicit cases. By assumption $\Theta = \Theta_k$, for some $1 \leq k \leq \lfloor n/2 \rfloor$, as in Proposition 4.1.4. Suppose that s is an element of $S_\Theta^- \cap A_{\Theta \cap w\Omega}^1 A_{w\Omega}$, we want to show that $s \in S_\Theta^1 S_{\Delta_0}$. Recall that $S_{\Delta_0} = S_G = \{e\}$; therefore, it is sufficient to prove that $s \in S_\Theta^1 = S_\Theta(\mathcal{O}_F)$. By assumption, $s = tz$ where $t \in A_{\Theta \cap w\Omega}^1$ and $z \in A_{w\Omega}$. Since $w \in [W_\Theta \backslash W_0 / W_\Omega]$, we have that $w\Omega \subset \Phi_0^+$;

moreover, by the assumption that $M_{\Theta\cap w\Omega}$ is a proper Levi subgroup of $M_{w\Omega}$, we have that $\Theta\cap w\Omega\subsetneq w\Omega$ is a proper subset. By Lemma 3.4.1, $w\Omega$ cannot be contained in $\Phi_\Theta^+=\left(\mathrm{span}_{\mathbb{Z}_{\geq 0}}\Theta\right)\cap\Phi_0$. It follows that there exists $\alpha\in w\Omega\setminus(\Theta\cap w\Omega)$ such that $\alpha\in\Phi_0^+$ and $\alpha\notin\Phi_\Theta^+$. There is a unique expression

$$\alpha=\sum_{j=1}^{n-1}c_j{}^{\gamma}(\epsilon_j-\epsilon_{j+1})\qquad\qquad c_j\in\mathbb{Z},c_j\geq 0,$$

such that, since $\Theta=\Theta_k$ and $\alpha\notin\Phi_\Theta^+$, c_k or c_{n-k} is nonzero ($c_{n/2}\neq 0$, when n even, $k=n/2$). First, observe that

$$\alpha(s)=\alpha(t)\alpha(z)=\alpha(t)\in\mathcal{O}_F^\times,$$

since $z\in A_{w\Omega}$, so $z\in\ker\alpha$, and $t\in A^1_{\Theta\cap w\Omega}=A_{\Theta\cap w\Omega}(\mathcal{O}_F)$. On the other hand, in the Galois case, writing s explicitly as $s={}^{\gamma}s'$, where

$$s'=\begin{cases}\mathrm{diag}(\underbrace{a,\dots,a}_{k},\underbrace{1\dots,1}_{\lfloor n/2\rfloor-2k},\underbrace{a^{-1},\dots,a^{-1}}_{k}), & 1\leq k\leq\lfloor n/2\rfloor\\ \mathrm{diag}(\underbrace{a,\dots,a}_{n/2},\underbrace{a^{-1},\dots,a^{-1}}_{n/2}), & n\text{ even}, k=n/2\end{cases},$$

and applying α, we have that

$$\begin{aligned}\alpha(s)&=\left(\sum_{j=1}^{n-1}c_j{}^{\gamma}(\epsilon_j-\epsilon_{j+1})\right)({}^{\gamma}s')\\&=\prod_{j=1}^{n-1}{}^{\gamma}(\epsilon_j-\epsilon_{j+1})(\gamma s'\gamma^{-1})^{c_j}\\&=\prod_{j=1}^{n-1}(\epsilon_j-\epsilon_{j+1})(s')^{c_j}\\&=(\epsilon_k-\epsilon_{k+1})(s')^{c_k}(\epsilon_{n-k}-\epsilon_{n-k+1})(s')^{c_{n-k}}\\&=a^{c_k}a^{c_{n-k}}\\&=a^c\end{aligned}$$

where $c=c_k+c_{n-k}$ (or $2c_{n/2}$ when n is even and $k=n/2$). In the linear case, we may simply take $s=s'$, and we have

$$\alpha(s)=\left(\sum_{j=1}^{n-1}c_j(\epsilon_j-\epsilon_{j+1})\right)(s')=(\epsilon_k-\epsilon_{k+1})(s')^{c_k}(\epsilon_{n-k}-\epsilon_{n-k+1})(s')^{c_{n-k}}=a^{c_k}a^{c_{n-k}}=a^c.$$

In both cases, we have that $a^c\in\mathcal{O}_F^\times$, since $c=c_k+c_{n-k}$ (respectively $c=2c_{n/2}$) is a positive integer. We have $|a|_F^c=1$, where $c\in\mathbb{N}$, $c\geq 1$ a positive integer, that is, $|a|_F\in\mathbb{Q}$ is a root of unity; therefore, $|a|_F=1$. In particular, we have that $a\in\mathcal{O}_F^\times$ and $s\in S^1_\Theta=S_\Theta(\mathcal{O}_F)$, as desired. $\square$

We now give the reduction of verifying the Relative Casselman's Criterion 2.2.18 for $\pi=\iota_Q^G\tau$ to

Casselman's Criterion 2.1.28 on the inducing representation.

Proposition 4.2.20. *If $y = ww_0^{-1} \in [W_\Theta\backslash W_0/W_\Omega]\cdot w_0^{-1}$ is such that $P_\Theta \cap {}^wM_\Omega$ is a proper parabolic subgroup of ${}^wM_\Omega$, then the exponents $\chi \in \mathcal{E}xp_{S_\Theta}(\mathcal{F}^y_\Theta(\tau))$ of $\mathcal{F}^y_\Theta(\tau)$ satisfy*

$$|\chi(s)|_F < 1,$$

for all $s \in S^-_\Theta \setminus S^1_\Theta S_{\Delta_0}$.

Proof. First, by Lemma 4.2.19, we have that $S^-_\Theta \setminus S^1_\Theta S_{\Delta_0} \subset A^{-w\Omega}_{\Theta\cap w\Omega} \setminus A^1_{\Theta\cap w\Omega}A_{w\Omega}$. By Lemma 2.1.25, any exponent $\chi \in \mathcal{E}xp_{S_\Theta}(\mathcal{F}^y_\Theta(\tau))$ is the restriction to S_Θ of an exponent $\widehat{\chi} \in \mathcal{E}xp_{A_\Theta}(\mathcal{F}^y_\Theta(\tau))$; therefore, the result follows from Lemma 4.2.15. □

4.2.4 Regularity and distinction of Jacquet modules

Again, we let $Q = LU$ be as in §4.2.1. Unless otherwise specified let $\pi = \iota_Q^G\tau$, as in Theorem 4.2.1. In the linear case, we realize $L = w_+M_\Omega w_+^{-1}$, where $\Omega \subset \Delta_0$ contains Δ_{odd} (and $w_0 = w_+$). In the Galois case, we have $L = w_0M_\Omega w_0^{-1} = w_+M_{\underline{\Omega}}w_+^{-1}$, where $\Omega^{ell} = w_0\Omega = (\gamma w_+)\underline{\Omega}$, and $\underline{\Omega} \subset \Delta$ contains Δ_{odd}. Let M denote

$$M = \begin{cases} M_\Omega = M_{(m_1,\ldots,m_k)} & \text{in the linear case} \\ M_{\underline{\Omega}} = M_{(m_1,\ldots,m_k)} & \text{in the Galois case} \end{cases} \tag{4.60}$$

Let P_Θ be a maximal Δ_0-standard parabolic subgroup, corresponding to a maximal θ-split subset Θ of Δ_0 (*cf.* Proposition 4.1.4). In this subsection, we study M^θ_Θ-distinction of the Jacquet module π_{N_Θ}. In preparation for this, we characterized the θ-fixed points of the standard Levi M_Θ of P_Θ (*cf.* §4.1.4). For the Galois case, see (4.40) and (4.41). For the linear case, see (4.42) and (4.44).

To begin, in the Galois case, we characterize $H_\bullet$-distinction for representations of $M_\bullet$ (*cf.* Remark 4.1.7 and (4.39)).

Corollary 4.2.21. *Assume that we are in the Galois case. Fix an integer $1 \le k \le \lfloor n/2\rfloor$. Let $\Pi = \pi_1\otimes\pi_2\otimes\pi_3$ be an irreducible admissible representation of $M_\bullet = M_{(k,n-2k,k)}$. Then Π is $H_\bullet = H_{(k,n-2k,k)}$-distinguished if and only if π_2 is H_{n-2k}-distinguished and $\pi_3 \cong {}^{\theta_k}\widetilde{\pi}_1$ (cf. §4.1.1).*

Proof. Without loss of generality, assume that n is odd and $1 \le k \le \lfloor n/2\rfloor$. Let 1_m denote the $m\times m$ identity matrix in G_m. Let V_Π denote the space of Π and V_{π_i} denote the space of π_i, $1 \le i \le 3$.

By Proposition 3.6.1, the condition $\pi_3 \cong {}^{\theta_k}\widetilde{\pi}_1$ is equivalent to $\{(A,\theta_k(A)) : A \in G_k\}$-distinction of the representation $\pi_1\otimes\pi_3$. Assume that Π is $H_\bullet$-distinguished. To show that $\pi_1\otimes\pi_3$ is $\{(A,\theta_k(A)) : A \in G_k\}$-distinguished, define $\langle\lambda_{\pi_1\otimes\pi_3}, v_1\otimes v_3\rangle = \langle\lambda, v_1\otimes w_2\otimes v_3\rangle$, where $w_2 \in V_{\pi_2}$ is fixed and extend $v_1\otimes v_3 \mapsto \langle\lambda, v_1\otimes w_2\otimes v_3\rangle$ to be a linear form. Since λ is nonzero on some simple tensor in V_Π, there exists a vector w_2, such that $\lambda_{\pi_1\otimes\pi_3}$ is nonzero. Moreover, $\lambda_{\pi_1\otimes\pi_3}$ is invariant under $\{\mathrm{diag}(A, 1_{n-2k}, \theta_k(A))\} \subset H_\bullet$. To show that π_2 is H_{n-2k}-distinguished, choose $v_1 \in V_{\pi_1}$ and $v_3 \in V_{\pi_3}$ such that $\langle\lambda, v_1\otimes v_2\otimes v_3\rangle \neq 0$ for some $v_2 \in V_{\pi_2}$. Define $\lambda_{\pi_2} \in V^*_{\pi_2}$, by $\langle\lambda_{\pi_2}, v\rangle = \langle\lambda, v_1\otimes v\otimes v_3\rangle$ for $v \in V_{\pi_2}$. The H_{n-2k}-invariance of λ_{π_2} follows from $H_\bullet$-invariance of λ since $\{1_k\}\times H_{n-2k}\times\{1_k\} \subset H_\bullet$.

Conversely, suppose that π_2 is H_{n-2k}-distinguished and $\pi_1\otimes\pi_3$ is $\{(A,\theta_k(A)) : A \in G_k\}$-distinguished, choose nonzero invariant linear forms $\lambda_2 \in V^*_{\pi_2}$ and $\lambda_{1\times 3} \in (V_{\pi_1}\otimes V_{\pi_3})^*$ respectively. Define a linear form on the space V_Π of Π on simple tensors by

$$\langle \lambda, v_1 \otimes v_2 \otimes v_3 \rangle = \langle \lambda_2 \otimes \lambda_{1\times 3}, v_2 \otimes (v_1 \otimes v_3) \rangle = \langle \lambda_2, v_2 \rangle \langle \lambda_{1\times 3}, v_1 \otimes v_3 \rangle,$$

then extend λ to all of V_Π by linearity. We have that λ is nonzero since λ_2 and $\lambda_{1\times 3}$ are both nonzero. The $H_\bullet$-invariance of λ is immediate from the invariance properties of λ_2 and $\lambda_{1\times 3}$. □

It is clear from Definition 2.1.21 that $\tau = {}^{w_0}\tau_0$ (*cf.* (4.57)) is a regular representation of L if and only if τ_0 is a regular representation of M_Ω. Moreover, in the Galois case, τ_0 is a regular representation of M_Ω if and only if τ' is a regular representation of M. Of course, in the linear case $\tau_0 = \tau'$ and γ does not appear (*cf.* §4.2.1). Let $\tau' = \bigotimes_{i=1}^k \tau_i$ be an irreducible admissible representation of M and let τ_0 be the corresponding representation of the Δ_0-standard Levi subgroup M_Ω. Recall that $W_0 = W(G, A_0)$ is the Weyl group of G with respect to A_0. In the linear case, we realize W_0 as the subgroup of permutation matrices in G. In the Galois case, we realize W_0 as in Lemma 4.1.13. We make the following observation.

Lemma 4.2.22. *Let τ_i be an irreducible admissible representation of G_{m_i}, for $1 \le i \le k$. The representation $\tau_1 \otimes \ldots \otimes \tau_k$ is regular if and only if $\tau_i \not\cong \tau_j$, for all $1 \le i \ne j \le k$.*

Let $y = ww_0^{-1}$, be as above (*cf.* §4.2.3), where $w \in [W_\Theta\backslash W_0/W_\Omega]$. There are two cases (*cf.* Remark 4.2.17) where $\mathscr{F}_\Theta^y(\tau)$ is irreducible. In these cases, the central characters $\chi_{\Theta,y}$ of $\mathscr{F}_\Theta^y(\tau)$ are unitary. The two cases are:

(A) when ${}^wM_\Omega = M_\Theta$ if and only if Θ and Ω are associate and $w \in [W_\Theta\backslash W_0/W_\Omega] \cap W(\Theta, \Omega)$,

(B) when $w \in [W_\Theta\backslash W_0/W_\Omega]$ is such that ${}^wM_\Omega \subsetneq M_\Theta$ is a proper Levi subgroup of M_Θ.

In Case (A), $\mathscr{F}_\Theta^y(\tau) = {}^w\tau_0$, while in Case (B), $\mathscr{F}_\Theta^y(\tau) = \iota_{M_\Theta \cap {}^wP_\Omega}^{M_\Theta} {}^w\tau_0$, where $M_\Theta \cap {}^wP_\Omega$ is a proper parabolic subgroup of M_Θ. In both cases (A) and (B), we will argue that when τ is regular, $\mathscr{F}_\Theta^y(\tau)$ cannot be M_Θ^θ-distinguished. This largely amounts to a careful application of Corollary 4.2.21 and Lemma 4.2.22, together with Proposition 2.3.3 and Corollary 2.3.21, which describe a symmetry property of distinguished representations in the linear and Galois cases respectively.

Proposition 4.2.23. *Let $\tau' = \bigotimes_{i=1}^k \tau_i$ be an irreducible admissible regular representation of M. Let τ be the corresponding irreducible admissible regular representation of L (cf. (4.57)). Assume that τ is L^θ-distinguished. Let P_Θ be a maximal Δ_0-standard parabolic subgroup, corresponding to a maximal θ-split subset Θ of Δ_0. Let $y = ww_0^{-1}$, where $w \in [W_\Theta\backslash W_0/W_\Omega]$. In both Case (A) and Case (B), $\mathscr{F}_\Theta^y(\tau)$ cannot be M_Θ^θ-distinguished.*

Proof. By Proposition 4.2.6 τ is L^θ-distinguished if and only if each τ_i is H_{m_i}-distinguished, $1 \le i \le k$. By Proposition 4.1.4, M_Θ is equal to $M_{(r,n-2r,r)}$ in the linear case, respectively ${}^\gamma M_{(r,n-2r,r)}$ in the Galois case, for some $1 \le r \le \lfloor n/2 \rfloor$. Without loss of generality, we assume that the "central block" of M_Θ appears. As above, we'll use the shorthand $M_\bullet$ for $M_{(r,n-2r,r)}$. In §4.1.4, we showed that in the Galois case M_Θ^θ is equal to ${}^\gamma(M_\bullet^\vartheta)$. Recall that $M_\bullet^\vartheta$ is $M_\bullet$-conjugate by $\gamma_\bullet = \operatorname{diag}(\gamma_r, \gamma_{n-2r}, \gamma_r)$ to $H_\bullet$, where

$$H_\bullet = \left\{ \begin{pmatrix} A & & \\ & B & \\ & & \theta_r(A) \end{pmatrix} : A \in G_r,\ B \in H_{n-2r} \right\} = \gamma_\bullet (M_\bullet)^\vartheta \gamma_\bullet^{-1}.$$

In the linear case, we simply have that $M_\Theta^\theta = M_\bullet^\theta = H_\bullet$ (4.42). In the Galois case, by Lemma 2.2.6, $H_\bullet$-distinction of an irreducible admissible representation ρ of $M_\bullet$ is equivalent to M_Θ^θ-distinction of the M_Θ representation ${}^\gamma\rho$. It follows that:

1. in the linear case, $\mathscr{F}_\Theta^y(\tau)$ is M_Θ^θ-distinguished if and only if $\mathscr{F}_\Theta^y(\tau)$ is $H_\bullet$-distinguished, and
2. in the Galois case, $\mathscr{F}_\Theta^y(\tau)$ is M_Θ^θ-distinguished if and only if ${}^{\gamma^{-1}}\mathscr{F}_\Theta^y(\tau)$ is $H_\bullet$-distinguished.

Without loss of generality, we complete the proof in the Galois case. We'll use that H_{m_i}-distinction of τ_i implies that $\tau_i \cong {}^\theta\widetilde{\tau}_i$ (*cf.* Corollary 2.3.21). To obtain the proof in the linear case, instead use that H_{m_i}-distinction of τ_i, implies that τ_i is self-contragredient (*cf.* Proposition 2.3.3).

Case (A)

Suppose that Θ and Ω are associate and $w \in [W_\Theta\backslash W_0/W_\Omega] \cap W(\Theta,\Omega)$. Then $M_\Theta = {}^wM_\Omega$ and $\mathscr{F}_\Theta^y(\tau) = {}^w\tau_0 = {}^{w\gamma}(\tau_1 \otimes \tau_2 \otimes \tau_3)$, where τ_1, τ_3 are representations of G_r and τ_2 is a representation of G_{n-2r}. By our conventions, $\gamma^{-1}w\gamma$ is a permutation matrix (*cf.* Lemma 4.1.13). Moreover, ${}^{\gamma^{-1}}(\mathscr{F}_\Theta^y(\tau)) = {}^{\gamma^{-1}w\gamma}(\tau_1 \otimes \tau_2 \otimes \tau_3)$ is equal to $\tau_{x(1)} \otimes \tau_{x(2)} \otimes \tau_{x(3)}$, for some compatible permutation x of $\{1,2,3\}$. By Corollary 4.2.21, ${}^{\gamma^{-1}}\mathscr{F}_\Theta^y(\tau)$ is $H_\bullet$-distinguished if and only if $\tau_{x(3)} \cong {}^{\theta_r}\widetilde{\tau_{x(1)}}$ and $\tau_{x(2)}$ is H_{n-2r}-distinguished. By Corollary 2.3.21, each τ_i satisfies $\tau_i \cong {}^\theta\widetilde{\tau}_i$ and $\tau_{x(2)}$ is H_{n-2r}-distinguished. However, since τ is regular, by Lemma 4.2.22, the τ_i are pairwise inequivalent. In particular, we have that

$$ {}^{\theta_r}\widetilde{\tau_{x(1)}} \cong \tau_{x(1)} \ncong \tau_{x(3)}; $$

therefore, ${}^{\gamma^{-1}}(\mathscr{F}_\Theta^y(\tau))$ is not $H_\bullet$-distinguished and $\mathscr{F}_\Theta^y(\tau)$ is not M_Θ^θ-distinguished.

Case (B)

Suppose that $w \in [W_\Theta\backslash W_0/W_\Omega]$ is such that $wM_\Omega w^{-1} \subset M_\Theta$ is a proper Levi subgroup. In this case, $\mathscr{F}_\Theta^y(\tau) = \iota_{M_\Theta \cap {}^wP_\Omega}^{M_\Theta} {}^w\tau_0$ and τ_0 is a regular irreducible unitary representation. By Theorem 2.1.22, the representation ${}^{\gamma^{-1}}(\mathscr{F}_\Theta^y(\tau))$ is an irreducible unitary representation of $M_\bullet$. Indeed, $M_\Theta = {}^\gamma M_\bullet$ and $M_\Omega = {}^\gamma M$, with $M_\bullet = M_{(r,n-2r,r)}$ and $M = M_{(m_1,\ldots,m_k)}$; moreover, $w' = \gamma^{-1}w\gamma$ is an element of $[W_{M_\bullet}\backslash W/W_M]$. Writing P_* for $P_{(m_1,\ldots,m_k)}$, we have

$$ \begin{aligned} {}^{\gamma^{-1}}(\mathscr{F}_\Theta^y(\tau)) &= {}^{\gamma^{-1}}\left(\iota_{M_\Theta \cap {}^wP_\Omega}^{M_\Theta} {}^w\tau_0\right) \\ &= {}^{\gamma^{-1}}\left(\iota_{{}^\gamma M_\bullet \cap {}^{w\gamma}P_*}^{{}^\gamma M_\bullet} {}^{w\gamma}(\tau_1 \otimes \ldots \otimes \tau_k)\right) \\ &\cong \iota_{M_\bullet \cap {}^{w'}P_*}^{M_\bullet} {}^{w'}(\tau_1 \otimes \ldots \otimes \tau_k) \\ &= \pi_1 \otimes \pi_2 \otimes \pi_3 \qquad \text{(Lemma 4.2.24)}, \end{aligned} $$

where π_1, π_2 are irreducible admissible representations of G_r and π_2 is an irreducible admissible representation of G_{n-2r}. Since $w' \in [W_{M_\bullet}\backslash W/W_M]$, by Proposition 2.1.20, the group $M_\bullet \cap {}^{w'}P_*$ is a parabolic subgroup of $M_\bullet$ (a product of parabolic subgroups on each block of $M_\bullet$). It follows that each of the π_j, $j = 1,2,3$, are irreducibly induced representations of the form $\tau_{a_1} \times \ldots \times \tau_{a_r}$ (Bernstein–Zelevinsky notation [7, 75]), for some subset of the representations $\{\tau_1,\ldots,\tau_k\}$. Again, by Corollary 4.2.21 ${}^{\gamma^{-1}}(\mathscr{F}_\Theta^y(\tau))$ is $H_\bullet$-distinguished if and only if π_2 is G_{n-2r}-distinguished and $\pi_3 \cong {}^{\theta_r}\widetilde{\pi_1}$. Suppose that $\pi_1 = \tau_{a_1} \times \ldots \times \tau_{a_l}$, and $\pi_3 = \tau_{b_1}\times\ldots\times\tau_{b_s}$, then we have that

$$\begin{aligned}
\widetilde{\pi}_1 &\cong \widetilde{\tau_{a_1}}\times\ldots\times\widetilde{\tau_{a_l}} && \text{(Proposition 2.1.9)}\\
&\cong {}^{\theta_{m_{a_1}}}\tau_{a_1}\times\ldots\times{}^{\theta_{m_{a_l}}}\tau_{a_l} && \text{(Corollary 2.3.21)}\\
&\cong {}^{\theta_r}\pi_1 && \text{(Lemma 4.2.24).}
\end{aligned}$$

Moreover, we have that $\pi_1 \cong {}^{\theta_r}\widetilde{\pi}_1$. Since τ is regular, by Lemma 4.2.22, the τ_i are pairwise inequivalent. By [75, Theorem 9.7(b)], we have that $\pi_1 \ncong \pi_3$. Indeed, the inequivalent "segments" π_1 and π_2 are unitary, hence not "linked", so the work of Zelevinsky applies. That is, we have ${}^{\theta}\widetilde{\pi_1}\cong\pi_1\ncong\pi_3$. In particular, ${}^{\gamma^{-1}}(\mathscr{F}^y_\Theta(\tau))$ is not $H_\bullet$-distinguished, and $\mathscr{F}^y_\Theta(\tau)$ is not M^θ_Θ-distinguished. □

We prove Lemma 4.2.24 in a general setting to complete the proof of Proposition 4.2.23.

Lemma 4.2.24. *Let $P = MN$ be a parabolic subgroup of a reductive subgroup G' of a reductive p-adic group G. Let $\psi\in\mathrm{Aut}(G)$ be an automorphism of G and ρ an irreducible smooth representation of M. Define $\pi = \iota_P^{G'}\rho$ and $\pi' = \iota_{\psi(P)}^{\psi(G')}({}^\psi\rho)$. The representations ${}^\psi\pi$ and π' are equivalent.*

Proof. The space V_π of π is also the space of ${}^\psi\pi$. The modular function $\delta_{\psi(P)}$ is equal to $\delta_P\circ\psi^{-1}$. Define a linear operator $T: V_\pi\to V_{\pi'}$ by $T(f) = f\circ\psi^{-1} = {}^\psi f$. It is straightforward to verify that ${}^\psi f\in V_{\pi'}$ and that T is an invertible intertwining operator from ${}^\psi\pi$ to π'. □

Corollary 4.2.25. *We use the terminology of the proof of Proposition 4.2.23, Case (B). In the Galois case, we have that*

$$^{\gamma^{-1}}\left(\iota^{\,^\gamma M_\bullet}_{\,^\gamma M_\bullet\cap\,^{w\gamma}P_*}\,{}^{w\gamma}(\tau_1\otimes\ldots\otimes\tau_k)\right)\cong\iota^{M_\bullet}_{M_\bullet\cap\,^{w'}P_*}\,{}^{w'}(\tau_1\otimes\ldots\otimes\tau_k),$$

where $w' = \gamma^{-1}w\gamma$ and

$${}^{\theta_r}(\tau_{a_1}\times\ldots\times\tau_{a_l})\cong{}^{\theta_{m_{a_1}}}\tau_{a_1}\times\ldots\times{}^{\theta_{m_{a_l}}}\tau_{a_l}.$$

4.2.5 The proof of Theorem 4.2.1

Finally, we assemble the proof of the main result of this chapter.

Proof of Theorem 4.2.1. By Proposition 4.2.14, since τ is unitary and regular, the unitary representation π is irreducible; In addition, π does not occur in the discrete series of G. The representation π is H-distinguished by Proposition 4.2.12. Let λ^G denote a fixed H-invariant linear form on π. By Proposition 4.2.13, λ^G is unique up to scalar multiples. To complete the proof, it remains to show that π satisfies the Relative Casselman's Criterion.

By Corollary 3.2.4, it is sufficient to verify that the condition (2.14) is satisfied for every Δ_0-standard maximal θ-split parabolic P_Θ. By the Geometric Lemma 2.1.17 and Lemma 2.1.24, the exponents of π along P_Θ are given by the union

$$\mathscr{E}xp_{A_\Theta}(\pi_{N_\Theta}) = \bigcup_{y\in[W_\Theta\backslash W_0/W_\Omega]\cdot w_0^{-1}}\mathscr{E}xp_{A_\Theta}(\mathscr{F}^y_\Theta(\tau)),$$

where the exponents on the right-hand side are the central characters of the irreducible subquotients of $\mathscr{F}^y_\Theta(\tau)$. By Lemma 2.1.25, the map from $\mathscr{E}xp_{A_\Theta}(\mathscr{F}^y_\Theta(\tau))$ to $\mathscr{E}xp_{S_\Theta}(\pi_{N_\Theta}, r_{P_\Theta}\lambda^G)$ defined by restriction of characters is surjective. Set $y = ww_0^{-1}$, where $w\in[W_\Theta\backslash W_0/W_\Omega]$.

In the two cases:

(A) when ${}^wM_\Omega = M_\Theta$ if and only if Θ and Ω are associate and $w \in [W_\Theta\backslash W_0/W_\Omega] \cap W(\Theta, \Omega)$,

(B) when $w \in [W_\Theta\backslash W_0/W_\Omega]$ is such that ${}^wM_\Omega \subsetneq M_\Theta$ is a proper Levi subgroup of M_Θ.

the parabolic subgroup $P_\Theta \cap {}^wM_\Omega$ of ${}^wM_\Omega$ is equal to ${}^wM_\Omega$. By Lemma 4.2.16, in cases (A) and (B), the central character (exponent) $\chi_{\Theta,y}$ of the irreducible representation $\mathcal{F}^y_\Theta(\tau)$ is unitary. In cases (A) and (B), by Proposition 4.2.23, $\mathcal{F}^y_\Theta(\tau)$ cannot be M^θ_Θ-distinguished. In particular, the exponents $\chi_{\Theta,y}$ of $\mathcal{F}^y_\Theta(\tau)$ do not occur in $\mathcal{E}xp_{S_\Theta}(\pi_{N_\Theta}, r_{P_\Theta}\lambda^G)$. Indeed, by Proposition 3.5.1, $r_{P_\Theta}\lambda^G$ must vanish on the generalized eigenspaces associated to the central characters $\chi_{\Theta,y}$ of $\mathcal{F}^y_\Theta(\tau)$. By the definition of the exponents relative to λ^G (*cf.* (2.13)), the characters $\chi_{\Theta,y}$ do not occur in $\mathcal{E}xp_{S_\Theta}(\pi_{N_\Theta}, r_{P_\Theta}\lambda^G)$.

Otherwise, outside of Cases (A) and (B), we have that $P_\Theta \cap {}^wM_\Omega$ is a proper parabolic subgroup of ${}^wM_\Omega$. By Proposition 4.2.20, we have that

$$|\chi(s)|_F < 1, \text{ for all } \chi \in \mathcal{E}xp_{S_\Theta}(\mathcal{F}^y_\Theta(\tau)), \text{ and all } s \in S^-_\Theta \setminus S^1_\Theta S_{\Delta_0}.$$

In particular, (2.14) holds for all exponents $\chi \in \mathcal{E}xp_{S_\Theta}(\pi_{N_\Theta}, r_{P_\Theta}\lambda^G)$ relative to λ^G along all maximal Δ_0-standard θ-split parabolic subgroups P_Θ. By Theorem 2.2.18 (Kato–Takano), we conclude that π is (H, λ^G)-relatively square integrable and hence occurs in the discrete spectrum of $H\backslash G$. □

This completes the proof of our main result, in the linear and Galois cases, and provides a novel construction of relative discrete series representations.

Remark 4.2.26. Outside of cases (A) and (B), we currently do not know if $r_{P_\Theta}\lambda^G$ vanishes on the generalized eigenspaces $(V_{N_\Theta})_{\chi,\infty})$, for the exponents χ of π along P_Θ.

4.2.6 Exhaustion of the discrete spectrum

At present, we do not know if our construction, outlined in Theorem 4.2.1, exhausts all relative discrete series, that are not discrete series for G, in the linear and Galois cases. In fact, we face the same issue for the unitary case and Theorem 5.2.22. An auxiliary question, along the same lines, is if there are *enough* RDS produced via Theorem 4.2.1 and Theorem 5.2.22 to exhaust $L^2_{disc}(H\backslash G)$ via further constructing toric families of discrete series following Sakellaridis and Venkatesh [66]. Addressing the question of exhaustion of the discrete spectrum, by our construction and/or extended via producing toric families of RDS, is the topic of future work.

It appears that the major obstruction to showing that an irreducible H-distinguished representation (π, V) of G is not (H, λ)-relatively square integrable is establishing non-vanishing of the invariant forms $r_P\lambda$. To show that π is not a RDS, one would be required to either

1. explicitly produce an (H, λ)-relative matrix coefficient that is not square integrable modulo Z_GH,

2. find a proper θ-split parabolic subgroup $P = MN$ and an exponent $\chi \in \mathcal{E}xp_{S_M}(\pi_N)$ that fails (2.14) such that $r_P\lambda$ restricted to the generalized χ-eigenspace in V_N is nonzero.

Establishing that $r_P\lambda$ is nonzero is obfuscated by the use of Casselman's Canonical Lifting (*cf.* §2.2.2) in the definition of $r_P\lambda$ (*cf.* Definition 2.2.14). The former method to show that π is not a RDS is perhaps the most approachable. In particular, we may be able to study the matrix coefficients when the invariant form on π is produced using Lemma 3.3.1. Currently, we cannot exclude the possibility that certain representations are RDS. For instance, it may be possible to relax the regularity condition imposed in Theorem 4.2.1; however, the author has not yet succeeded in doing so.

A separate unresolved issue is giving a combinatorial classification of all θ-elliptic Levi subgroups in G, up to H-conjugacy, and further determining *exactly* which proper θ-elliptic Levi subgroups admit L^θ-distinguished discrete series representations. The existence of L^θ-distinguished discrete series appears to be related to the existence of semisimple L-regular θ-split elements in L.

The problem of showing that a representation $\pi = \iota_Q^G \tau$ is not relatively supercuspidal is discussed in Chapter 6, under the additional assumption that the inducing representation τ is not (L^θ, λ)-relatively supercuspidal. We directly study the relative matrix coefficients, with respect to a linear functional λ^G produced by Lemma 3.3.1. The analysis of (H, λ^G)-relative matrix coefficients is facilitated by the explicit nature of the invariant form λ^G as an integral over $Q^\theta\backslash H$. There may be some hope of addressing the extra assumption on the (L^θ, λ)-relative matrix coefficients in the linear and Galois cases since the invariant form λ may be obtained as an L^θ-integral of a matrix coefficient (*cf.* Theorem 2.3.2, Remarks 2.3.6 and 2.3.31). We have yet to undertake this study.

Chapter 5

RDS for $\mathbf{U}_{E/F}(F)\backslash\mathbf{GL}_n(E)$, n even

In this chapter, we construct relative discrete series representations for $H\backslash G$ in the case that $G = \mathbf{GL}_n(E)$, $n \geq 4$ is even, and H is the group of F-points of a quasi-split unitary group. Again, the construction is via parabolic induction from a regular distinguished discrete series representation of a θ-elliptic Levi subgroup. We obtain representations induced from the block upper-triangular maximal parabolic subgroup $P_{(n/2,n/2)}$ of G. The major technical hurdle in this chapter is dealing with non-standard θ-split parabolic subgroups. We are able to show that the methods of Chapter 4 extend beyond the case when all θ-split parabolic subgroups are H-conjugate to those in a fixed standard class. It is work in progress to extend the results of Chapter 4 and Chapter 5 to any symmetric quotient of $\mathbf{GL}_n$.

5.1 Structure of the symmetric space $\mathbf{U}_{E/F}(F)\backslash\mathbf{GL}_n(E)$

For now, let $n \geq 2$ be any integer. We will indicate when we restrict our attention to the case that n is even (*cf.* Remark 5.2.8). Let $\mathbf{G} = R_{E/F}\mathbf{GL}_n$ be the restriction of scalars from E to F of $\mathbf{GL}_n$. As in Chapter 4, we identify the group $G = \mathbf{G}(F)$ with the set of E-points of $\mathbf{GL}_n$. The non-trivial element σ of the Galois group $\mathrm{Gal}(E/F)$ of E over F gives rise to an F-involution of $\mathbf{G}$ given by entry-wise Galois conjugation. By abuse of notation, we denote the Galois involution of $\mathbf{G}$ by σ and write

$$\sigma(g) = (\sigma(g_{ij})), \qquad \text{where } g = (g_{ij}) \in G.$$

Recall that for $a, b \in F$, we have $\sigma(a + b\varepsilon) = a - b\varepsilon$. Let $\mathbf{X}$ denote the F-variety of Hermitian matrices in $\mathbf{G}$,

$$\mathbf{X} = \{x \in \mathbf{G} : {}^t\sigma(x) = x\}.$$

Here ${}^tg$ denotes the transpose of $g \in \mathbf{G}$. There is a right action of $\mathbf{G}$ on $\mathbf{X}$ given by $x \cdot g = {}^t\sigma(g)xg$, where $x \in \mathbf{X}$ and $g \in \mathbf{G}$. Write $X = \mathbf{X}(F)$ for the F points of $\mathbf{X}$. There is a finite set X/G of G-orbits in X indexed by $F^\times/N_{E/F}(E^\times)$ [20]. By Local Class Field Theory, $F^\times/N_{E/F}(E^\times)$ is isomorphic to $\mathrm{Gal}(E/F)$, which has order 2.

Remark 5.1.1. Elsewhere in this thesis, we used to denote X the F-points of a spherical variety or $H\backslash G$. In this chapter, we use $X = \mathbf{X}(F)$ to denote the variety of Hermitian matrices in G following the notation of [20].

Given $x \in X$, define an F-involution θ_x of G by

$$\theta_x(g) = x^{-1\,t}\sigma(g)^{-1}x, \tag{5.1}$$

for $g \in \mathbf{G}(F)$. Let $\mathbf{H}^x = \mathbf{G}^{\theta_x}$ be the subgroup of θ_x-fixed elements. The group of F-points $H^x = \mathbf{H}^x(F)$ is a *unitary group* associated to x and E/F.

Remark 5.1.2. In the literature, $\mathbf{U}_{E/F,x}$ is often used to denote the unitary group $\mathbf{H}^x$ associated to E/F and x. We will use this notation for unitary groups that appear as subgroups of Levi subgroups of G.

Remark 5.1.3. Two involutions θ_{x_1} and θ_{x_2} are G-equivalent (*cf.* Definition 1.2.1) if and only if x_1 and x_2 lie in the same G-orbit in X/G. Indeed, if x and y are in the same G-orbit in X, then there exists $g \in G$ such that $y = x\cdot g = {}^t\sigma(g)xg$, and one can check that θ_y is equal to the involution $g\cdot\theta_x = \operatorname{Int} g^{-1}\circ\theta_x\circ\operatorname{Int} g$. Note that the G-action $\theta \mapsto g\cdot\theta$ on involutions is also a right-action. Since X/G has order two, there are two G-equivalence classes of involutions of the form θ_x. It is well known that when n is odd, $\mathbf{H}^x$ is always quasi-split over F. When n is even there are two isomorphism classes of unitary group associated to E/F, one of which is quasi-split over F.

We fix $\theta = \theta_{w_\ell}$, where w_ℓ is the permutation matrix in G with unit anti-diagonal, and write $\mathbf{H} = \mathbf{G}^\theta$. It is well known that $\mathbf{H}$ is quasi-split over F. As usual, write $H = \mathbf{H}(F)$ for the group of F-points of $\mathbf{H}$.

As in Chapter 4, let J_r be the $r\times r$ matrix with unit anti-diagonal and note that $w_\ell = J_n$. For any positive integer r, there exists $\gamma_r \in \mathbf{GL}_r(E)$ such that ${}^t\sigma(\gamma_r)J_r\gamma_r$ lies in the diagonal F-split torus of $\mathbf{GL}_r(E)$. For instance if r is even, we set

$$\gamma_r = \begin{pmatrix} 1 & & & & & 1 \\ & \ddots & & & \iddots & \\ & & 1 & 1 & & \\ & & 1 & -1 & & \\ & \iddots & & & \ddots & \\ 1 & & & & & -1 \end{pmatrix}$$

and if r is odd, we take

$$\gamma_r = \begin{pmatrix} 1 & & & & & & 1 \\ & \ddots & & & & \iddots & \\ & & 1 & 0 & 1 & & \\ & & 0 & 1 & 0 & & \\ & & 1 & 0 & -1 & & \\ & \iddots & & & & \ddots & \\ 1 & & & & & & -1 \end{pmatrix}.$$

In this chapter, we set $\gamma = \gamma_n$ and notice that

$${}^t\sigma(\gamma)w_\ell\gamma = \operatorname{diag}(\underbrace{2,\ldots,2}_{\lfloor\frac{n}{2}\rfloor},\widehat{1},\underbrace{-2,\ldots,-2}_{\lfloor\frac{n}{2}\rfloor}), \tag{5.2}$$

lies in the diagonal F-split torus of G. In fact, it is worth noting that, when n is even, ${}^t\sigma(\gamma)w_\ell\gamma$ lies in the torus $A_{(n/2,n/2)}$, the F-split component of the block-diagonal Levi subgroup $M_{(n/2,n/2)}$.

5.1.1 Tori and root systems and relative to θ

First, we describe the (θ_x, F)-split component of G for any Hermitian matrix $x \in \mathbf{X}(F)$. Write $S_{G,x}$ for the (θ_x, F)-split component of G. We recall that $S_{G,x}$ is defined to be the F-split torus $\left(\{z \in A_G : \theta_x(z) = z^{-1}\}\right)^\circ$. For the involution $\theta = \theta_{w_\ell}$, we'll write S_G for S_{G,w_ℓ}, as usual.

Lemma 5.1.4. *For any $x \in X$, the torus $S_{G,x}$ is equal to the F-split component A_G of the centre of G.*

Proof. Let $z \in A_G$ and compute that

$$\theta_x(z) = x^{-1}{}^t\sigma(z)^{-1}x = x^{-1}z^{-1}x = z^{-1},$$

where ${}^t\sigma(z) = z$ since z is diagonal and has entries in $F^\times$. The last equality holds because z^{-1} is central in G. It follows that $S_{G,x} = (A_G)^\circ = A_G$, for all $x \in X$. □

The maximal (non-split) F-torus $\mathbf{T}$ of $\mathbf{G}$ is obtained by restriction of scalars of the diagonal torus of $\mathbf{GL}_n$. Write $T = \mathbf{T}(F)$, identify T with the diagonal matrices in $\mathbf{GL}_n(E)$ and let A_T be the F-split component of T. Define $T_0 = {}^\gamma T$ and let $A_0 = {}^\gamma A_T$ denote the F-split component of T_0. The tori T and T_0 are θ-stable; moreover, their F-split components are θ-stable. Observe that A_0 is a maximal F-split torus of G that is θ-split. In particular, A_0 is a maximal (θ, F)-split torus of G. Indeed, we have that ${}^t\sigma(\gamma)w_\ell\gamma$ lies in the abelian subgroup A_T; therefore, for any $\gamma t\gamma^{-1} \in A_0$, we have

$$\theta(\gamma t\gamma^{-1}) = w_\ell^{-1}{}^t\sigma(\gamma)^{-1}({}^t\sigma(t)^{-1}){}^t\sigma(\gamma)w_\ell = \gamma({}^t\sigma(\gamma)w_\ell\gamma)^{-1}t^{-1}({}^t\sigma(\gamma)w_\ell\gamma)\gamma^{-1} = (\gamma t\gamma^{-1})^{-1},$$

where we've used that ${}^t\sigma(t)^{-1} = t^{-1}$, for any $t \in A_T$.

As in Chapter 4, let $\Phi = \Phi(G, A_T)$ be the root system of G with respect to A_T with standard base Δ. Let $\Phi_0 = \Phi(G, A_0)$ be the root system of G with respect to A_0 and observe that $\Phi_0 = {}^\gamma\Phi$. We take $\Delta_0 = {}^\gamma\Delta$ for a set of simple roots in Φ_0. The set of positive roots of Φ_0 with respect to Δ_0 is denoted Φ_0^+. We have that $\Phi_0^+ = {}^\gamma\Phi^+$, where Φ^+ is the set of positive roots in Φ determined by Δ.

Lemma 5.1.5. *Every root $\alpha \in \Phi_0$ is sent to its negative $-\alpha$ by θ.*

Proof. Let $\alpha \in \Phi_0$. For any $a \in A_0$, we have that $\theta(a) = a^{-1}$; therefore,

$$(\theta\alpha)(a) = \alpha(\theta(a)) = \alpha(a^{-1}) = \alpha(a)^{-1} = (-\alpha)(a).$$

Since $a \in A_0$ was arbitrary, we have that $\theta\alpha = -\alpha$. □

The following two corollaries of Lemma 5.1.5 follow immediately. Refer to Definition 1.5.2 for the definition of a θ-base.

Corollary 5.1.6. *The set Φ_0^θ of θ-fixed roots in Φ_0 is empty*

Corollary 5.1.7. *Any set of simple roots in Φ_0 is a θ-base for Φ_0. In particular, Δ_0 is a θ-base.*

Explicitly, $\Delta_0 = \{{}^\gamma(\epsilon_i - \epsilon_{i+1}) : 1 \le i \le n-1\}$ and, by Corollary 5.1.7, the set of simple roots Δ_0 is a θ-base for Φ_0. Since the maximal F-split torus A_0 is a maximal (θ, F)-split torus, the restricted root system of $H\backslash G$ is just the root system Φ_0 of G [30] (*cf.* §1.5.1).

5.1.2 Δ_0-standard θ-split parabolic subgroups

Proposition 5.1.8. *Every parabolic subgroup of G standard with respect to Δ_0 is a θ-split parabolic subgroup. Any such parabolic subgroup is the γ-conjugate of the usual block-upper triangular parabolic subgroups of G.*

The first statement of Proposition 5.1.8 follows from the fact that $\overline{\Delta}_0 = \Delta_0$ (*cf.* §1.6.1), the second statement follows as in the proof of Proposition 4.1.4.

H-conjugacy of θ-split parabolic subgroups

The torus T_0 is a Levi subgroup of a minimal Δ_0-standard θ-split parabolic subgroup. One may readily verify that $\mathbf{T}_0 \cap \mathbf{H}$ is isomorphic to the direct product of n-copies of $\mathbf{U}_{E/F,1}$, where $\mathbf{U}_{E/F,1}$ is the one-dimensional unitary group associated to E/F. It follows that the first Galois cohomology $H^1(F, T_0 \cap H)$ is isomorphic to $\oplus_1^n H^1(F, \mathbf{U}_{E/F,1})$. It is well known that the Galois cohomology group of $\mathbf{U}_{E/F,1}$ over F, in degree one, is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. In particular, $H^1(F, T_0 \cap H) \cong (\mathbb{Z}/2\mathbb{Z})^n$ is non-trivial. For this reason, we cannot use Lemma 1.6.4(2), to prove that all θ-split parabolic subgroups of G are H-conjugate to Δ_0-standard parabolic subgroups. Presently, the author does not know if all θ-split parabolic subgroups are H-conjugate to the Δ_0-standard ones. In the current chapter, we are forced to consider both standard and non-standard θ-split parabolic subgroups in our analysis. By Lemma 1.6.4(1), we do have that any θ-split parabolic subgroup of G is $(\mathbf{HT}_0)(F)$-conjugate to a Δ_0-standard θ-split parabolic subgroup.

5.1.3 Δ-standard θ-stable parabolic subgroups

Let $(\underline{n}) = (n_1, \ldots, n_r)$ be a partition of n, we say that $(\underline{n})$ is *balanced* if $n_i = n_{r+1-i}$, for $1 \leq i \leq r$.

Lemma 5.1.9. *The θ-stable block upper-triangular parabolic subgroups of G correspond to balanced partitions of n. The only such parabolic that has a θ-stable θ-elliptic Levi subgroup is $P_{(n/2,n/2)}$, in the case that n is even.*

Proof. Let $(\underline{n}) = (n_1, \ldots, n_r)$ be a partition of n. Let $A = A_{(\underline{n})}$ be the diagonal F-split torus corresponding to $(\underline{n})$, and let $M = C_G(A)$ be the standard Levi subgroup of $P = P_{(\underline{n})}$. Let N be the unipotent radical of P. The parabolic subgroup P is θ-stable if and only if M and N are θ-stable; moreover, M is θ-stable if and only if A is θ-stable. Observe that A is θ-stable if and only if $w_\ell \in N_G(A)$. Indeed, $\theta(a) = w_\ell^{-1}{}^t\sigma(a)^{-1} w_\ell$ and A is stable under the involution $a \mapsto {}^t\sigma(a)^{-1} = a^{-1}$. Since $w_\ell^{-1}\operatorname{diag}(a_1, \ldots, a_n) w_\ell = \operatorname{diag}(a_n, \ldots, a_1)$, it follows that $\theta(A) = A_{(\underline{n})^{op}}$, where $(\underline{n})^{op}$ is the opposite partition $(n_r, \ldots, n_1)$ to $(\underline{n})$. It is clear that A is θ-stable if and only if $(\underline{n})^{op} = (\underline{n})$; this is equivalent to asking that $(\underline{n})$ is balanced. Therefore, it suffices to show that if N is θ-stable if and only if $(\underline{n})$ is balanced. First note that N is stable under the map $n \mapsto \sigma(n)^{-1}$, on the other hand, the transpose map sends N to the opposite unipotent radical N^{op}. In particular, N is θ-stable if and only if $N = w_\ell^{-1} N^{op} w_\ell$. By a simple matrix computation, this occurs if and only if $(\underline{n})^{op} = (\underline{n})$.

Let $(\underline{n}) = (n_1, \ldots, n_{\lfloor r/2 \rfloor}, \widehat{n_\bullet}, n_{\lfloor r/2 \rfloor}, \ldots, n_1)$ be a balanced partition of n. Now, we show that M is θ-elliptic if and only if $(\underline{n}) = (n/2, n/2)$. An element $a = \operatorname{diag}(a_1, \ldots, a_n)$ of A_T is θ-split if and only if a centralizes w_ℓ. Indeed, since A_T is pointwise fixed by taking the transpose-Galois conjugates, we have that

$$\theta(a) = w_\ell^{-1}\operatorname{diag}(a_1^{-1}, \ldots, a_n^{-1}) w_\ell = \operatorname{diag}(a_n^{-1}, \ldots, a_1^{-1}),$$

which is equal to a^{-1} if and only if $a_i = a_{n+1-i}$, for all $1 \le i \le n$. It follows that $a^{-1} = \theta(a)$ if and only if $a \in C_{A_T}(w_\ell)$, where

$$C_{A_T}(w_\ell) = \{\mathrm{diag}(a_1, \ldots, a_{\lfloor n/2 \rfloor}, \widehat{a_\bullet}, a_{\lfloor n/2 \rfloor}, \ldots, a_1) : a_i \in F^\times, 1 \le i \le \lfloor n/2 \rfloor\}.$$

The (θ, F)-split component S_M of M is the identity component of $A \cap C_{A_T}(w_\ell)$. We have that

(5.3)

$$A \cap C_{A_T}(w_\ell) = \left\{ \mathrm{diag}(\underbrace{a_1, \ldots, a_1}_{n_1}, \ldots, \underbrace{a_{\lfloor r/2 \rfloor}, \ldots, a_{\lfloor r/2 \rfloor}}_{n_{\lfloor r/2 \rfloor}}, \underbrace{\widehat{a_\bullet, \ldots, a_\bullet}}_{n_\bullet}, \underbrace{a_{\lfloor r/2 \rfloor}, \ldots, a_{\lfloor r/2 \rfloor}}_{n_{\lfloor r/2 \rfloor}}, \ldots, \underbrace{a_1, \ldots, a_1}_{n_1}) \right\};$$

in particular, $S_M = A \cap C_{A_T}(w_\ell)$. By Lemma 5.1.4, we have $S_G = A_G$ and observe that S_M is equal to A_G if and only if $r = 2$, that is, n is even and $(\underline{n}) = (n/2, n/2)$, as claimed. □

Remark 5.1.10. When n is even, we set $L = M_{(n/2,n/2)}$ and reiterate that L is the only proper block-diagonal θ-elliptic Levi subgroup of G.

Corollary 5.1.11. *The minimal parabolic (Borel) subgroup Q_0 of G consisting of the upper-triangular matrices is a θ-stable minimal parabolic of G. In particular, $Q_0 = (R_{E/F}\mathbf{B})(F)$, where $\mathbf{B}$ is the upper-triangular Borel subgroup of $\mathbf{GL}_n$.*

Proof. The partition $(1, \ldots, 1)$ is balanced, and we may apply Lemma 5.1.9. □

Corollary 5.1.12. *The F-subgroup $Q_0 \cap H$ of H, consisting of the upper-triangular elements of H, is a Borel subgroup of H.*

Proof. See, for instance, [27, Lemma 3.1] for the relationship between parabolic subgroups of H and θ-stable parabolic subgroups of G. □

5.1.4 θ-elliptic Levi subgroups

In this subsection, we first observe that $L_0 = C_G((A_0^\theta)^\circ)$ is equal to G. In particular, the method used in Chapter 4 to produce proper θ-elliptic Levi subgroups (*cf.* Proposition 4.1.15) does not apply in the current setting. Secondly, we prove that any θ-stable parabolic subgroup of G is H-conjugate to a θ-stable parabolic subgroup containing the diagonal F-split torus A_T. Using this result, we determine all θ-elliptic Levi subgroups of G that contain A_T.

Lemma 5.1.13. *There are no proper θ-elliptic Levi subgroups of G that contain A_0.*

Proof. The maximal F-split torus A_0 of G is (θ, F)-split. The lemma follows from Lemma 1.6.2 and Lemma 1.6.3. □

The Levi subgroup $L_0 = C_G((A_0^\theta)^\circ)$ is still a minimal θ-elliptic Levi subgroup that contains A_0; however, in this setting $L_0 = G$. To see this directly, one may check that $(A_0^\theta)^\circ = \{e\}$.

On the other hand, there exist proper θ-elliptic Levi subgroups of G that contain the diagonal F-split torus A_T (at least when n is even). Recall that a parabolic subgroup P of G is called A_T-semi-standard if P contains A_T. If P is an A_T-semi-standard parabolic subgroup, then there is a unique Levi factor M of P that contains A_T. We refer to M as the A_T-semi-standard Levi factor of P.

Recall from Corollary 5.1.11 that the upper-triangular Borel subgroup of Q_0 of G is a minimal θ-stable parabolic subgroup of G; moreover, $Q_0 \cap H$ is a Borel subgroup of H. The diagonal torus A_T is a θ-stable maximal F-split torus of G. By [31, Lemma 3.5], the torus $(A_T \cap H)^\circ$ is a maximal F-split torus of H.

Lemma 5.1.14. *Let $\mathbf{P}$ be any θ-stable parabolic subgroup of $\mathbf{G}$, then $P = \mathbf{P}(F)$ is H-conjugate to a θ-stable A_T-semi-standard parabolic subgroup.*

Proof. Let $\mathbf{P} = \mathbf{MN}$ be a θ-stable parabolic subgroup with the indicated Levi factorization, where $\mathbf{M}$ and $\mathbf{N}$ are both θ-stable. Let $\mathbf{P}_\bullet$ be a minimal θ-stable parabolic subgroup of $\mathbf{G}$ contained in $\mathbf{P}$. Let $\mathbf{A}_\bullet$ be a θ-stable maximal F-split torus contained in $\mathbf{P}_\bullet$ [31, Lemma 2.5]. By [31, Corollary 5.8], there exists $g = nh \in (N_{\mathbf{G}}(\mathbf{A}_\bullet) \cap N_{\mathbf{G}}((\mathbf{A}_\bullet \cap \mathbf{H})^\circ))(F)\mathbf{H}(F)$ such that $g^{-1}\mathbf{P}_\bullet g = \mathbf{Q}_0$. Note that n normalizes $(\mathbf{A}_\bullet \cap \mathbf{H})^\circ$ and all of $\mathbf{A}_\bullet$, while h is θ-fixed. Observe that

$$g^{-1}A_\bullet g = h^{-1}n^{-1}A_\bullet nh = h^{-1}A_\bullet h \subset Q_0; \tag{5.4}$$

in particular, $g^{-1}A_\bullet g$ is a θ-stable maximal F-split torus. Let $\mathbf{U}_0$ be the unipotent radical of $\mathbf{Q}_0$. By [31, Lemma 2.4], $g^{-1}A_\bullet g$ is $(\mathbf{H} \cap \mathbf{U}_0)(F)$-conjugate to A_T. It follows that there exists $h' \in (\mathbf{H} \cap \mathbf{U}_0)(F)$ such that

$$A_T = h'^{-1}g^{-1}A_\bullet gh' = h'^{-1}h^{-1}A_\bullet hh' = (hh')^{-1}A_\bullet(hh'); \tag{5.5}$$

moreover, we have that $A_T = (hh')^{-1}A_\bullet(hh')$ is contained in $(hh')^{-1}\mathbf{P}(hh')$ and $\mathbf{P}$ is H-conjugate to a θ-stable A_T-semi-standard parabolic subgroup. □

Lemma 5.1.15.
1. *If n is odd, then there are no proper θ-elliptic A_T-semi-standard Levi subgroups of G.*
2. *If n is even, then $L = M_{(n/2,n/2)}$ is the only proper θ-elliptic A_T-semi-standard Levi subgroup of G, up to conjugacy by Weyl group elements $w \in W = W(G, A_T)$, such that $w^{-1}w_\ell w \in N_G(L) \setminus L$.*

Proof. We give only a sketch of the proof. We identify the Weyl group $W = W(G, A_T)$ of A_T in G with the subgroup of permutation matrices in G. We've already shown that if n even, then $L = M_{(n/2,n/2)}$ is θ-elliptic (*cf.* Lemma 5.1.9).

First, let $P = MN$ be a θ-stable maximal proper A_T-semi-standard parabolic subgroup of G with θ-stable A_T-semi-standard Levi subgroup M. It is well known that P is W-conjugate to a unique standard (block upper-triangular) maximal parabolic subgroup of G. In particular, $M = wM_{(n_1,n_2)}w^{-1}$ for some partition (n_1, n_2) of n and some $w \in W$. Moreover, M is θ-stable if and only if its F-split component $A_M = wA_{(n_1,n_2)}w^{-1}$ is contained in a torus $A_{(\underline{n})}$, for some balanced partition $(\underline{n})$ of n. Assume that M is θ-stable and let $(\underline{n})$ be the coarsest partition such that $wA_{(n_1,n_2)}w^{-1}$ is contained in $A_{(\underline{n})}$. One may regard the F-split component $A_M = wA_{(n_1,n_2)}w^{-1}$ of M as being obtained by a two-colouring of $(\underline{n})$. That is, regard $\mathrm{diag}(\underbrace{a, \ldots, a}_{n_1}, \underbrace{b, \ldots, b}_{n_2}) \mapsto A_{(\underline{n})}$ as a two-colouring of the partition $(\underline{n})$. It is straightforward to verify the following.

1. If n is odd, then, by considering (5.3), we see that $wA_{(n_1,n_2)}w^{-1}$ contains at least a rank-one F-split torus, non-central in G, consisting of θ-split elements. (Indeed, since n is odd, the central segment in (5.3) must appear.) In particular, M cannot be θ-elliptic by Lemma 1.6.3. Moreover,

by Lemma 1.6.6 M cannot contain a θ-elliptic Levi subgroup of G. It follows that there are no A_T-semi-standard θ-elliptic Levi subgroups when n is odd.

2. Suppose that n is even. In light of (5.3), we see that M is θ-elliptic if and only if $(\underline{n})$ is a refinement of the partition $(n/2, n/2)$ of n. In particular, it readily follows that $n_1 = n_2 = n/2$ and M is conjugate to L.

Observe that $\theta(w) = w_\ell^{-1} w w_\ell$, for any $w \in W$. It is straightforward to verify that $M = wLw^{-1}$ is θ-stable if and only if $w^{-1}w_\ell w \in N_G(L)$. It can be verified that a θ-stable conjugate $M = wLw^{-1}$ of L is θ-elliptic if and only if $w^{-1}w_\ell w \notin C_G(A_L) = L$.

Finally, we show that L does not contain any proper θ-elliptic Levi subgroups. We argue by contradiction. Suppose that $L' \subsetneq L$ is a θ-elliptic Levi subgroup of G. Notice that L' is also θ-elliptic in L. Since L' is proper in L, it follows from Lemma 1.6.6 that L' is contained in a θ-stable maximal proper Levi subgroup L'' of L. Without loss of generality, $L'' \cong M_{(k_1,k_2)} \times \mathbf{GL}_{n/2}(F)$. However, considering the action of θ on L described in (5.6), we observe that no such Levi subgroup L'' can be θ-stable. We conclude that L does not contain any proper θ-elliptic Levi subgroups. □

The θ-fixed points of $L = M_{(n/2,n/2)}$ and ${}^\gamma L$

Assume that n is even. The parabolic subgroup $Q = P_{(n/2,n/2)}$ is θ-stable and has θ-elliptic Levi subgroup $L = M_{(n/2,n/2)}$. Let U denote the unipotent radical $N_{(n/2,n/2)}$. We now determine the θ-fixed points of L. Let $l = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in L$, where $A, B \in \mathbf{GL}_{n/2}(E)$. Note that w_ℓ can be considered as a block-$(n/2 \times n/2)$ matrix

$$w_\ell = \begin{pmatrix} 0 & J_{n/2} \\ J_{n/2} & 0 \end{pmatrix},$$

and recall that $J_r^{-1} = J_r$ for any positive integer r. Then we compute that

$$\theta(l) = w_\ell^{-1}\left({}^t\sigma(l)^{-1}\right) w_\ell = \begin{pmatrix} J_{n/2}^{-1}\,{}^t\sigma(B)^{-1}J_{n/2} & 0 \\ 0 & J_{n/2}^{-1}\,{}^t\sigma(A)^{-1}J_{n/2} \end{pmatrix}. \tag{5.6}$$

It follows immediately from (5.6) that l is θ-fixed if and only if $B = \theta_{J_{n/2}}(A) = J_{n/2}^{-1}\left({}^t\sigma(A)^{-1}\right) J_{n/2}$. In particular, we have that

$$L^\theta = \left\{ \begin{pmatrix} A & 0 \\ 0 & \theta_{J_{n/2}}(A) \end{pmatrix} : A \in \mathbf{GL}_{n/2}(E) \right\} \cong \mathbf{GL}_{n/2}(E). \tag{5.7}$$

From (5.7), we immediately obtain a characterization of the θ-fixed points of the associate Levi subgroup $M = {}^\gamma L$ of the Δ_0-standard parabolic subgroup $P = {}^\gamma Q$.

Lemma 5.1.16. *The Levi subgroup $M = {}^\gamma L$ is the θ-stable Levi subgroup of a standard θ-split parabolic $P = MN = {}^\gamma P_{(n/2,n/2)}$. The θ-fixed points of M are isomorphic to a product of two copies of the unitary group $\mathbf{U}_{E/F,1_{n/2}} = \{x \in \mathbf{GL}_{n/2}(E) : x^{-1} = {}^t\sigma(x)\}$, where $1_{n/2}$ is the $n/2 \times n/2$ identity matrix.*

Proof. Let ${}^{\gamma}m \in M$, where $m \in L$. We have that

$$\theta({}^{\gamma}m) = w_\ell^{-1}\ {}^t\sigma(\gamma)^{-1}\ {}^t\sigma(m)^{-1}\ {}^t\sigma(\gamma)\ w_\ell = \gamma({}^t\sigma(\gamma)w_\ell\gamma)^{-1}\ {}^t\sigma(m)^{-1}\ ({}^t\sigma(\gamma)w_\ell\gamma)\gamma^{-1} = \gamma\ {}^t\sigma(m)^{-1}\gamma^{-1},$$

where the last equality holds since ${}^t\sigma(\gamma)w_\ell\gamma \in A_L$ centralizes $m \in L$. It follows that ${}^{\gamma}m = \theta({}^{\gamma}m)$ if and only if $m = {}^t\sigma(m)^{-1}$. Writing m as a block-diagonal matrix $m = \operatorname{diag}(A, B)$, we have $m = {}^t\sigma(m)^{-1}$ if and only if $A = {}^t\sigma(A)^{-1}$ and $B = {}^t\sigma(B)^{-1}$. It follows that

$$M^\theta = \left\{ \gamma \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \gamma^{-1} : A, B \in \mathbf{GL}_{n/2}(E), A = {}^t\sigma(A)^{-1}, B = {}^t\sigma(B)^{-1} \right\}, \tag{5.8}$$

and $M^\theta \cong \mathbf{U}_{E/F,1_{n/2}} \times \mathbf{U}_{E/F,1_{n/2}}$, as claimed. □

In Lemma 5.2.20, we will determine the θ-fixed points of the Levi subgroup of an arbitrary maximal θ-split parabolic subgroup with Levi subgroup associate to L. It is interesting to note that, even though we're interested in distinction by the quasi-split unitary group $\mathbf{U}_{E/F,w_\ell}$, we will need to consider distinction by (possibly non-quasi-split) unitary groups for Jacquet modules along θ-split parabolic subgroups (*cf.* §5.2.4).

5.2 Constructing RDS for $\mathbf{U}_{E/F}(F)\backslash\mathbf{GL}_n(E)$, n even

5.2.1 Characterization of inducing data

Definition 5.2.1. A representation π of G is *Galois invariant* if π is equivalent to its twist ${}^{\sigma}\pi$ by the Galois involution σ.

Recall that $X = \mathbf{X}(F)$ is the variety of Hermitian matrices in G and $\mathbf{H}^x = \mathbf{G}^{\theta_x}$ is a unitary group. Irreducible representations distinguished by unitary groups are characterized in the paper [20]. In particular, the authors show that an irreducible unitarizable representation π of G is H^x-distinguished if and only if π is Galois invariant. Although Feigon, Lapid and Offen prove much stronger results, we'll recall only what we need for our application. The following appears as [20, Corollary 13.5].

Theorem 5.2.2 (Feigon–Lapid–Offen)**.** *Let π be an irreducible admissible essentially square integrable representation of G. For any $x \in X$, the following conditions are equivalent:*

1. *the representation π is Galois invariant, that is $\pi \cong {}^{\sigma}\pi$.*
2. *the representation π is H^x-distinguished.*

In addition, $\dim \operatorname{Hom}_{H^x}(\pi, 1) \leq 1$.

The multiplicity-one statement appears as [20, Proposition 13.3]. It is known that multiplicity-one does not hold in general, see [20, Corollary 13.16] for instance, which gives a lower bound for the dimension of $\operatorname{Hom}_{H^x}(\pi, 1)$ for Galois invariant generic representations π. On the other hand, Feigon, Lapid and Offen are able to extend Theorem 5.2.2 to all *ladder representations* (*cf.* [20, Theorem 13.11]).

Remark 5.2.3. As in the Galois case, distinction by unitary groups has deep connections with quadratic base change. We refer the reader to the work of Feigon, Lapid and Offen [20] for a detailed discussion.

Proposition 5.2.4 (Hakim–Murnaghan)**.** *For $n \geq 2$, there exist infinitely many distinct equivalence classes of Galois invariant (respectively, non-Galois invariant) irreducible supercuspidal representations of $\mathbf{GL}_n(E)$.*

Proof. The proof is similar to that of Theorems 2.3.15 and 2.3.28. First, apply [56, Proposition 10.1] to obtain the existence of infinitely many pairwise inequivalent Galois invariant irreducible supercuspidal representations of $\mathbf{GL}_n(E)$. To complete the proof, apply [29, Theorem 1.1]. Alternatively, one may apply [28, Theorem 4.4]. □

Proposition 5.2.5. *Let ρ be an irreducible supercuspidal representation of $\mathbf{GL}_r(E)$, $r \geq 1$. For $k \geq 2$, the generalized Steinberg representation $\mathrm{St}(k,\rho)$ of $\mathbf{GL}_{kr}(E)$ is Galois invariant if and only if ρ is Galois invariant.*

Proof. Let $G = \mathbf{GL}_{kr}(E)$. First, we observe that the twisted representation ${}^\sigma\,\mathrm{St}(k,\rho)$ is equivalent to the generalized Steinberg representation $\mathrm{St}(k, {}^\sigma\rho)$. Indeed, ${}^\sigma\,\mathrm{St}(k,\rho)$ is the unique irreducible quotient of the induced representation ${}^\sigma\left(\iota_{P_{(\underline{r})}}^G \nu^{\frac{1-k}{2}}\rho \otimes \ldots \otimes \nu^{\frac{k-1}{2}}\rho\right)$; moreover, the map $f \mapsto {}^\sigma f = f \circ \sigma$ is an invertible intertwining operator from ${}^\sigma\left(\iota_{P_{(\underline{r})}}^G \nu^{\frac{1-k}{2}}\rho \otimes \ldots \otimes \nu^{\frac{k-1}{2}}\rho\right)$ to $\iota_{P_{(\underline{r})}}^G\, {}^\sigma\left(\nu^{\frac{1-k}{2}}\rho \otimes \ldots \otimes \nu^{\frac{k-1}{2}}\rho\right) = \iota_{P_{(\underline{r})}}^G \left(\nu^{\frac{1-k}{2}}\,{}^\sigma\rho \otimes \ldots \otimes \nu^{\frac{k-1}{2}}\,{}^\sigma\rho\right)$ (*cf.* Corollary 4.2.25). On the other hand, the induced representation $\iota_{P_{(\underline{r})}}^G \left(\nu^{\frac{1-k}{2}}\,{}^\sigma\rho \otimes \ldots \otimes \nu^{\frac{k-1}{2}}\,{}^\sigma\rho\right)$ has unique irreducible quotient $\mathrm{St}(k, {}^\sigma\rho)$. It follows that ${}^\sigma\,\mathrm{St}(k,\rho)$ is equivalent to $\mathrm{St}(k,{}^\sigma\rho)$. In particular, $\mathrm{St}(k,\rho)$ is Galois invariant if and only if $\mathrm{St}(k,\rho) \cong {}^\sigma\,\mathrm{St}(k,\rho) \cong \mathrm{St}(k,{}^\sigma\rho)$. The result now follows from [75, Theorem 9.7(b)], which gives us that $\mathrm{St}(k,\rho) \cong \mathrm{St}(k,{}^\sigma\rho)$ if and only if $\rho \cong {}^\sigma\rho$. □

We can realize the Steinberg representation St_n as $\mathrm{St}(n,1)$, where 1 is the trivial character.

Corollary 5.2.6. *For $n \geq 2$, the Steinberg representation St_n of $\mathbf{GL}_n(E)$ is Galois invariant.*

Corollary 5.2.7. *For $n \geq 2$, there exists a unitary twist of the Steinberg representation St_n of $\mathbf{GL}_n(E)$ that is not Galois invariant.*

Proof. Let $\chi : E^\times \to \mathbb{C}^\times$ be a (unitary) character of $E^\times$. By [75, Theorem 9.7(b)], $\chi\,\mathrm{St}_n \cong {}^\sigma(\chi\,\mathrm{St}_n)$ if and only if $\chi = {}^\sigma\chi$. A character $\chi : E^\times \to \mathbb{C}^\times$ is Galois invariant if and only if ${}^\sigma\chi = \chi$ if and only if χ is trivial on the kernel of the norm map $N_{E/F} : E^\times \to F^\times$ (this follows from Hilbert's Theorem 90 which allows us to identify $\ker N_{E/F}$ with $\{\frac{\sigma(a)}{a} : a \in E^\times\}$). Note that $\ker N_{E/F}$ is a non-trivial closed subgroup of $E^\times$. We can extend any non-trivial (unitary) character of $\ker N_{E/F}$ to $E^\times$ to obtain a unitary character χ of $E^\times$ such that ${}^\sigma\chi \neq \chi$. □

Remark 5.2.8. For the remainder of this chapter we assume that n is even. We set $Q = P_{(n/2,n/2)}$ with Levi factorization $L = M_{(n/2,n/2)}$ and $U = N_{(n/2,n/2)}$.

Note that Q is θ-stable with θ-elliptic Levi subgroup L. The group L is identified with $\mathbf{GL}_{n/2}(E) \times \mathbf{GL}_{n/2}(E)$ and any irreducible admissible representation of L is of the form $\pi_1 \otimes \pi_2$, where π_i is an irreducible admissible representation of $\mathbf{GL}_{n/2}(E)$, $i = 1, 2$. We characterize L^θ-distinction in the following proposition.

Proposition 5.2.9. *An irreducible admissible representation $\pi_1 \otimes \pi_2$ of L is L^θ-distinguished if and only if π_2 is equivalent to the Galois-twist of π_1, that is, if and only if $\pi_2 \cong {}^\sigma\pi_1$.*

We actually prove a slightly more general result from which Proposition 5.2.9 is a trivial corollary, by taking into account the description of L^θ given in (5.7).

Lemma 5.2.10. *Let $G' = \mathbf{GL}_m(E) \times \mathbf{GL}_m(E)$, $x \in \mathbf{GL}_m(E)$ a Hermitian matrix, and define*

$$H' = \left\{ \begin{pmatrix} A & 0 \\ 0 & \theta_x(A) \end{pmatrix} : A \in \mathbf{GL}_m(E) \right\}.$$

An irreducible admissible representation $\pi_1 \otimes \pi_2$ of G' is H'-distinguished if and only if π_2 is equivalent to the Galois-twist of π_1, i.e., $\pi_2 \cong {}^\sigma\pi_1$.

Proof. By Proposition 3.6.1 we have that $\pi_1 \otimes \pi_2$ is H'-distinguished if and only if π_2 is equivalent to ${}^{\theta_x}\widetilde{\pi}_1$, the θ_x-twist of the contragredient of π_1. It suffices to show that for any Hermitian matrix x in $\mathbf{GL}_m(E)$, and any irreducible admissible representation π of $\mathbf{GL}_m(E)$, the Galois-twisted representation ${}^\sigma\pi$ is equivalent to ${}^{\theta_x}\widetilde{\pi}$. By a result of Gel'fand and Kazhdan [25, Theorem 2], we have that $\widetilde{\pi}$ is equivalent to $\widehat{\pi}$, where the representation $\widehat{\pi}$, is defined by $\widehat{\pi}(g) = \pi({}^tg^{-1})$ acting on the space V of π. Applying the same result again, we have that $\widehat{(\widetilde{\pi})} \cong \widetilde{\widetilde{\pi}}$. Since π is admissible, by Proposition 2.1.2(3), we have $\widetilde{\widetilde{\pi}} \cong \pi$; thus, we see that $\widehat{(\widetilde{\pi})} \cong \pi$. On the other hand, the representation ${}^{\theta_x}\widetilde{\pi}$ on $\widetilde{V}$ is given by ${}^{\theta_x}\widetilde{\pi}(g) = \widetilde{\pi}(\theta_x(g))$. Using that x is Hermitian, i.e., ${}^t\sigma(x) = x$, for any $g \in \mathbf{GL}_m(E)$ we have that

$${}^{\theta_x}\widetilde{\pi}(g) = \widetilde{\pi}(x^{-1}{}^t\sigma(g)^{-1}x) = \widetilde{\pi}({}^t({}^tx\sigma(g){}^tx^{-1})^{-1}) = \widehat{(\widetilde{\pi})}({}^tx\sigma(g){}^tx^{-1}) = {}^\sigma\widehat{(\widetilde{\pi})}(xgx^{-1}) = {}^{\operatorname{Int} x^{-1}}({}^\sigma\widehat{(\widetilde{\pi})})(g).$$

We observe that ${}^{\theta_x}\widetilde{\pi}$ is equivalent to ${}^\sigma\widehat{(\widetilde{\pi})}$ since $\operatorname{Int} x^{-1}$ is an inner automorphism of $\mathbf{GL}_m(E)$. It is also clear that taking Galois twists commutes with taking π to $\widehat{\pi}$ (and even with twisting by $\operatorname{Int} x^{-1}$, since $x = {}^t\sigma(x)$ is Hermitian). Putting this all together, we have that

$${}^{\theta_x}\widetilde{\pi} \cong {}^{\operatorname{Int} x^{-1}}({}^\sigma\widehat{(\widetilde{\pi})}) \cong {}^\sigma\widehat{(\widetilde{\pi})} \cong {}^\sigma\pi,$$

which completes the proof of the proposition. □

5.2.2 H-distinction of an induced representation

Let τ' be an irreducible admissible representation of $\mathbf{GL}_{n/2}(E)$ and define $\tau = \tau' \otimes {}^\sigma\tau'$. By Proposition 5.2.9, the irreducible admissible representation τ of L is L^θ-distinguished. Let λ be a nonzero element of $\operatorname{Hom}_{L^\theta}(\tau, 1)$. As in Proposition 3.6.1, λ is defined using the pairing of τ' with its contragredient. By [49, Proposition 4.3.2], we have that $\delta_Q^{1/2}$ restricted to L^θ is equal to $\delta_{Q\cap H} = \delta_{Q^\theta}$. Applying Lemma 3.3.1, we obtain an injection of $\operatorname{Hom}_{L^\theta}(\tau, 1)$ into the space $\operatorname{Hom}_H(\pi, 1)$ of H-invariant linear forms on π. Let $\lambda^G \in \operatorname{Hom}_H(\pi, 1)$ be the image of λ under this injection, then λ^G is nonzero. In particular, we have the following result.

Proposition 5.2.11. *Let τ' be an irreducible admissible representation of $\mathbf{GL}_{n/2}(E)$. If $\tau = \tau' \otimes {}^\sigma\tau'$, then the induced representation $\pi = \iota_Q^G\tau$ is H-distinguished.*

Remark 5.2.12. Unless otherwise indicated, below π denotes an induced representation $\iota_Q^G\tau$, where $\tau = \tau' \otimes {}^\sigma\tau'$ and τ' is an irreducible admissible representation of $\mathbf{GL}_{n/2}(E)$.

5.2.3 Computation of exponents along maximal θ-split parabolic subgroups

We employ the same strategy as in Chapter 4 to compute the exponents of π along proper θ-split parabolic subgroups. Let $P = MN$ be a maximal θ-split parabolic subgroup of G. By Lemma 1.6.4, there exists $g \in (\mathbf{HT}_0)(F)$ such that $P = gP_\Theta g^{-1}$, where P_Θ is a Δ_0-standard maximal θ-split parabolic subgroup. By the Geometric Lemma 2.1.17 and Lemma 2.1.24, the exponents of $\pi = \iota_Q^G\tau$ along P are given by

$$\mathcal{E}xp_{A_M}(\pi_N) = \bigcup_{y\in M\backslash S(M,L)/L} \mathcal{E}xp_{A_M}(\mathcal{F}_N^y(\tau)),$$

and the exponents $\mathcal{E}xp_{A_M}(\mathcal{F}_N^y(\tau))$ are the central characters of the irreducible subquotients of the representations $\mathcal{F}_N^y(\tau)$. By Lemma 2.1.25, restriction of characters from A_M to the (θ, F)-split component S_M provides a surjection from $\mathcal{E}xp_{A_M}(\pi_N)$ to $\mathcal{E}xp_{S_M}(\pi_N)$.

There are two situations we need to consider:

Case (1): when $P \cap {}^yL = {}^yL$, and

Case (2): when $P \cap {}^yL \subsetneq {}^yL$ is a proper parabolic subgroup of ${}^yL$.

In *Case (2)*, we will reduce our application of the Relative Casselman's Criterion 2.2.18 to Casselman's Criterion 2.1.28 for the inducing representation τ of L. First, we prove the analog of the main technical result of Chapter 4.

A technical lemma on θ-split dominant cones

Lemma 5.2.13 is the analog of Lemma 4.2.19. In the present setting, we also have to consider non-standard θ-split parabolic subgroups in our analysis of the exponents of π. We will explain how to adapt Lemma 5.2.13 to handle the non-standard case.

The inducing parabolic subgroup Q is conjugate to the standard maximal θ-split parabolic P_Ω, where $\Omega = \Delta_0 \setminus \{{}^\gamma(\epsilon_{n/2} - \epsilon_{n/2+1})\}$. In particular, $Q = \gamma^{-1}P_\Omega\gamma$, $L = \gamma^{-1}M_\Omega\gamma$ and $U = \gamma^{-1}N_\Omega\gamma$. As usual, for a standard θ-split parabolic subgroup P_Θ of G, we use the representatives $[W_\Theta\backslash W_0/W_\Omega]$ for the double-coset space $P_\Theta\backslash G/P_\Omega$ (*cf.* Lemma 2.1.19). We have an isomorphism $P_\Theta\backslash G/P_\Omega \cong P_\Theta\backslash G/Q$ given by $w \mapsto w\gamma$.

Lemma 5.2.13. *Let P_Θ, given by $\Theta \subset \Delta_0$, be any maximal θ-split Δ_0-standard parabolic subgroup. Let $w \in [W_\Theta\backslash W_0/W_\Omega]$ such that $M_\Theta \cap {}^wM_\Omega = M_{\Theta\cap w\Omega}$ is a proper Levi subgroup of ${}^wM_\Omega = M_{w\Omega}$. We have the containment*

$$S_\Theta^- \setminus S_\Theta^1 S_{\Delta_0} \subset A_{\Theta\cap w\Omega}^{-w\Omega} \setminus A_{\Theta\cap w\Omega}^1 A_{w\Omega}. \tag{5.9}$$

See Definition 3.4.2, (2.14) and (1.8) for the notation used in (5.9). Recall that $S_\Theta^1 = S_\Theta(\mathcal{O}_F)$ and $A_{\Theta\cap w\Omega}^1 = A_{\Theta\cap w\Omega}(\mathcal{O}_F)$ (*cf.* Remark 2.2.19).

Proof. By assumption, P_Θ is maximal and by Proposition 5.1.8 we have that $\Theta = \Delta_0 \setminus \{{}^\gamma(\epsilon_{n_1} - \epsilon_{n_1+1})\}$, for some $1 \le n_1 \le n-1$. The parabolic subgroup P_Θ is the γ-conjugate the block-diagonal parabolic subgroup $P_{(n_1,n_2)}$, where $n_1 + n_2 = n$. In particular, $M_\Theta = {}^\gamma M_{(n_1,n_2)}$. The (θ, F)-split component S_Θ of M_Θ is equal to its F-split component A_Θ. Indeed, $A_\Theta = {}^\gamma A_{(n_1,n_2)}$ is contained in A_0 and A_0 is a maximal (θ, F)-split torus. By Lemma 5.1.4, $S_G = S_{\Delta_0}$ is equal to A_G. Explicitly, we have

$$S^1_\Theta = S_\Theta(\mathcal{O}_F) = \{{}^\gamma \operatorname{diag}(\underbrace{a,\ldots,a}_{n_1},\underbrace{b,\ldots,b}_{n_2}) : a,b \in \mathcal{O}_F^\times\},$$

and that

$$S^-_\Theta = \{{}^\gamma \operatorname{diag}(\underbrace{a,\ldots,a}_{n_1},\underbrace{b,\ldots,b}_{n_2}) : a,b \in F^\times,\ |ab^{-1}|_F \le 1\}.$$

We see immediately that

$$S^-_\Theta \setminus S^1_\Theta S_{\Delta_0} = \{{}^\gamma \operatorname{diag}(\underbrace{a,\ldots,a}_{n_1},\underbrace{b,\ldots,b}_{n_2}) : a,b \in F^\times, a \ne b, \text{ not both } a,b \in \mathcal{O}_F^\times,\ |ab^{-1}|_F \le 1\}.$$

On the other hand, we need to understand the dominant cone $A^{-w\Omega}_{\Theta\cap w\Omega} \setminus A^1_{\Theta\cap w\Omega}A_{w\Omega}$. Following the proof of Lemma 4.2.19, we have that $S_\Theta \subset A_{\Theta\cap w\Omega}$, $S^1_\Theta \subset A^1_{\Theta\cap w\Omega}$ and $S_G \subset A_{w\Omega}$. Moreover, we have that $S^1_\Theta S_G \subset S^-_\Theta \cap A^1_{\Theta\cap w\Omega}A_{w\Omega}$. To prove the desired result, it suffices to prove the opposite inclusion.

Suppose that $s = tz \in S^-_\Theta \cap A^1_{\Theta\cap w\Omega}A_{w\Omega}$, where $t \in A^1_{\Theta\cap w\Omega}$ and $z \in A_{w\Omega}$. Since $w \in [W_\Theta\backslash W_0/W_\Omega]$, we have that $w\Omega \subset \Phi^+_0$; moreover, since $M_{\Theta\cap w\Omega}$ is a proper Levi subgroup of $M_{w\Omega}$, we have that $\Theta \cap w\Omega \subsetneq w\Omega$ is a proper subset. By Lemma 3.4.1, $w\Omega$ cannot be contained in Φ^+_Θ. It follows that there exists $\alpha \in w\Omega \setminus (\Theta \cap w\Omega)$ such that $\alpha \in \Phi^+_0$ and $\alpha \notin \Phi^+_\Theta$. There is a unique expression

$$\alpha = \sum_{j=1}^{n-1} c_j {}^\gamma(\epsilon_j - \epsilon_{j+1}) \qquad\qquad c_j \in \mathbb{Z}, c_j \ge 0,$$

such that $c_{n_1} \ne 0$, because $\alpha \notin \Phi^+_\Theta$ and $\Theta = \Delta_0 \setminus \{{}^\gamma(\epsilon_{n_1} - \epsilon_{n_1+1})\}$. First, observe that

$$\alpha(s) = \alpha(t)\alpha(z) = \alpha(t) \in \mathcal{O}_F^\times,$$

since $z \in A_{w\Omega}$, so $z \in \ker\alpha$, and $t \in A_{\Theta\cap w\Omega}(\mathcal{O}_F)$. Note that $A_{\Theta\cap w\Omega} \supset A_{w\Omega}$ and both are F-split subtori of the (θ, F)-split torus A_0. In particular, both t and z are θ-split, i.e., $\theta(t) = t^{-1}$ and $\theta(z) = z^{-1}$. In fact, since A_0 is (θ, F)-split, we have that $S_{\Theta\cap w\Omega} = A_{\Theta\cap w\Omega}$ and $S_{w\Omega} = A_{w\Omega}$. Working explicitly again, we have that $s = {}^\gamma \operatorname{diag}(\underbrace{a,\ldots,a}_{n_1},\underbrace{b,\ldots,b}_{n_2})$ and

$$\alpha(s) = \left(\sum_{j=1}^{n-1} c_j {}^\gamma(\epsilon_j - \epsilon_{j+1})\right)(s) = \prod_{j=1}^{n-1} {}^\gamma(\epsilon_j - \epsilon_{j+1})(s)^{c_j} = {}^\gamma(\epsilon_{n_1} - \epsilon_{n_1+1})(s)^{c_{n_1}} = (ab^{-1})^{c_{n_1}},$$

where c_{n_1} is a positive integer. On the other hand, we saw above that, $\alpha(s) = \alpha(t) \in \mathcal{O}_F^\times$; it follows that $|ab^{-1}|_F^{c_{n_1}} = 1$. Moreover, since c_{n_1} is a positive integer, $|ab^{-1}|_F$ is a rational root of unity and $|ab^{-1}|_F = 1$, that is, $ab^{-1} \in \mathcal{O}_F^\times$. Our goal is to realize $s = tz$ as an element of the product $S^1_\Theta S_G$. We can write s as the product

$$s = {}^\gamma \operatorname{diag}(\underbrace{a,\ldots,a}_{n_1},\underbrace{b,\ldots,b}_{n_2}) = {}^\gamma \operatorname{diag}(\underbrace{ab^{-1},\ldots,ab^{-1}}_{n_1},\underbrace{1,\ldots,1}_{n_2}) {}^\gamma \operatorname{diag}(\underbrace{b,\ldots,b}_{n_1+n_2}),$$

where, since $ab^{-1} \in \mathcal{O}_F^\times$,

$$^{\gamma}\operatorname{diag}(\underbrace{ab^{-1},\dots,ab^{-1}}_{n_1},\underbrace{1,\dots,1}_{n_2})$$

is an element of $S^1_\Theta = A_\Theta(\mathcal{O}_F)$, and $^{\gamma}\operatorname{diag}(b,\dots,b)$ lies in S_G. This shows that $S^-_\Theta \cap A^1_{\Theta\cap w\Omega}A_{w\Omega} \subset S^1_\Theta S_G$ and completes the proof of the lemma. $\square$

Next, we generalize the previous lemma to *all* (non-standard) maximal θ-split parabolic subgroups. Let $P = MN$ be a maximal θ-split parabolic subgroup of G. By Lemma 1.6.4, there exists $g \in (\mathbf{HT}_0)(F)$ such that $P = gP_\Theta g^{-1}$, where P_Θ is a Δ_0-standard maximal θ-split parabolic subgroup. We may take $S_M = gS_\Theta g^{-1}$ and then we have $S^-_M = gS^-_\Theta g^{-1}$. Let $y \in P\backslash G/Q$, given by $y = gw\gamma$, where $w \in [W_\Theta\backslash W_0/W_\Omega]$. We observe that $^yL = g(M_{w\Omega})g^{-1}$. In particular, $M \cap {}^yL = g(M_{\Theta\cap w\Omega})g^{-1}$ and $A_{M\cap {}^yL} = g(A_{\Theta\cap w\Omega})g^{-1}$. The dominant part of the torus $A_{M\cap {}^yL}$ will be denoted by $A^{-{}^yL}_{M\cap {}^yL}$ and is determined by the simple roots $gw\Omega$ of the maximal (θ, F)-split torus $^gA_0$ in ^yL.

Lemma 5.2.14. *Let $P = MN$ be any maximal θ-split parabolic subgroup of G with θ-stable Levi M and unipotent radical N. Choose a maximal subset Θ of Δ_0 and an element $g \in (\mathbf{HT}_0)(F)$ such that $P = gP_\Theta g^{-1}$. Let $y = gw\gamma \in P\backslash G/Q$, where $w \in [W_\Theta\backslash W_0/W_\Omega]$, such that $M \cap {}^yL$ is a proper Levi subgroup of ^yL. Then we have the containment*

$$S^-_M \setminus S^1_M S_G \subset A^{-{}^yL}_{M\cap {}^yL} \setminus A^1_{M\cap {}^yL}A_{{}^yL}. \tag{5.10}$$

Proof. The (θ, F)-split component S_Θ of M_Θ is equal to its F-split component A_Θ, so we have that $S_M = A_M$ (*cf.* proof of Lemma 5.2.13). We also have that $S^-_M = gS^-_\Theta g^{-1}$ and $S^1_M = gS^1_\Theta g^{-1}$; moreover, since $S_G = S_{\Delta_0}$ is central in G we obtain

$$S^-_M \setminus S_G S^1_M = g(S^-_\Theta)g^{-1} \setminus S_G g(S^1_\Theta)g^{-1}. \tag{5.11}$$

By Lemma 5.2.13, we have that

$$S^-_\Theta \setminus S^1_\Theta S_{\Delta_0} \subset A^{-w\Omega}_{\Theta\cap w\Omega} \setminus A^1_{\Theta\cap w\Omega}A_{w\Omega}.$$

By the equality in (5.11), it suffices to show that

$$A^{-{}^yL}_{M\cap {}^yL} \setminus A^1_{M\cap {}^yL}A_{{}^yL} = g\left(A^{-w\Omega}_{\Theta\cap w\Omega}\right)g^{-1} \setminus g\left(A^1_{\Theta\cap w\Omega}\right)g^{-1}\ g\left(A_{w\Omega}\right)g^{-1}. \tag{5.12}$$

Indeed, if (5.12) holds, then we have

$$\begin{aligned} S^-_M \setminus S_G S^1_M &= gS^-_\Theta g^{-1} \setminus S_G g S^1_\Theta g^{-1} \\ &\subset g\left(A^{-w\Omega}_{\Theta\cap w\Omega}\right)g^{-1} \setminus g\left(A^1_{\Theta\cap w\Omega}\right)g^{-1}\ g\left(A_{w\Omega}\right)g^{-1} && \text{(Lemma 5.2.13)} \\ &= A^{-{}^yL}_{M\cap {}^yL} \setminus A^1_{M\cap {}^yL}A_{{}^yL}, \end{aligned}$$

as claimed. The truth of (5.12) immediately follows from how we determine the dominant part of $A_{M\cap {}^yL}$. As above, we have that $M \cap {}^yL = g(M_{\Theta\cap w\Omega})g^{-1}$ and $A_{M\cap {}^yL} = g(A_{\Theta\cap w\Omega})g^{-1}$. Moreover, $A^1_{M\cap {}^yL} = g(A^1_{\Theta\cap w\Omega})g^{-1}$. Given a root $\alpha \in \Phi_0$ we obtain a root $g\alpha$ of $^gA_0$ in G by setting $g\alpha = \alpha \circ \operatorname{Int} g^{-1}$, as usual. Explicitly, we have that

$$A^{-^yL}_{M\cap {}^yL} = \{a \in A_{M\cap {}^yL} : |g\beta(a)| \leq 1,\ \beta \in w\Omega \setminus \Theta \cap w\Omega\}. \tag{5.13}$$

In fact, we have that $M \cap {}^yL$ is determined (as Levi subgroup of ${}^yL$) by the simple roots $g(\Theta \cap w\Omega) \subset gw\Omega$ of ${}^gA_0$ in ${}^yL = gwM_\Omega$. It is immediate that

$$A^{-^yL}_{M\cap {}^yL} = g\left(A^{-w\Omega}_{\Theta\cap w\Omega}\right) g^{-1}, \tag{5.14}$$

from which (5.12) follows, completing the proof of the lemma. □

Jacquet modules and exponents in Case (1)

Let $P = MN$ be any maximal θ-split parabolic subgroup of G with θ-stable Levi M and unipotent radical N, as above. Let $y = gw\gamma \in P\backslash G/Q$, where $w \in [W_\Theta\backslash W_0/W_\Omega]$. Assume that we are in *Case (1)*, then $P \cap {}^yL = {}^yL$ and in particular, $M \cap {}^yL = {}^yL$. That is, M and ${}^yL$ are associate (maximal) Levi subgroups. Observe that, ${}^\gamma P_{(n/2,n/2)}$ is the only Δ_0-standard θ-split maximal parabolic subgroup with a Levi subgroup associate to $L = M_{(n/2,n/2)}$. The only parabolic subgroups we must consider in this case are of the form $P = gP_\bullet g^{-1}$, where $P_\bullet = {}^\gamma P_{(n/2,n/2)}$ and $g \in (\mathbf{T}_0\mathbf{H})(F)$. In *Case (1)*, we show that the exponents of $\pi = \iota_Q^G\tau$ along P are unitary when τ is unitary.

Lemma 5.2.15. *There are exactly two elements y in the double coset space $P\backslash G/Q$ such that $M\cap {}^yL = {}^yL$. We take representatives $g\gamma$ and $g\gamma w_L$, where $w_L = \begin{pmatrix} 0 & 1_{n/2} \\ 1_{n/2} & 0 \end{pmatrix} \in N_G(L)$ and $1_{n/2}$ is the $n/2 \times n/2$ identity matrix.*

Proof. To see this use the definition of $[W_\Omega\backslash W/W_\Omega]$, where $\Omega = \Delta\setminus\{\epsilon_{n/2}-\epsilon_{n/2+1}\}$. The set $[W_\Omega\backslash W/W_\Omega]$ has size $n/2$ consisting of elements of the form

$$\begin{pmatrix} 1_k & & & \\ & & 1_m & \\ & 1_m & & \\ & & & 1_k \end{pmatrix},$$

where $n = 2k + 2m$, and only 1_n and w_L normalize L. There are bijections

$$\begin{gathered} Q\backslash G/Q \to P_\bullet\backslash G/Q \to P\backslash G/Q \\ QxQ \mapsto P_\bullet\gamma xQ \mapsto Pg\gamma xQ. \end{gathered}$$

Indeed, $QxQ = Q\gamma^{-1}\gamma xQ$, then left-multiply by γ to obtain $P_\bullet\gamma xQ$. Similarly, we have $P_\bullet g^{-1}g\gamma xQ$ and we left-multiply by g to obtain $Pg\gamma xQ$. Note that the parabolic $w_LQw_L^{-1} = Q^{op}$ is the opposite of Q. □

Observe that, for a representation τ' of $\mathbf{GL}_{n/2}(E)$, we have ${}^{w_L}(\tau' \otimes {}^\sigma\tau') \cong {}^\sigma\tau' \otimes \tau'$. We obtain the following.

Corollary 5.2.16. *Let $\tau = \tau' \otimes {}^\sigma\tau'$ be an irreducible admissible representation of L. Let $P = {}^gP_\Omega$, $g \in (\mathbf{T_0H})(F)$, be a maximal θ-split parabolic subgroup with Levi associate to L. Let $\pi = \iota_Q^G\tau$.*

1. *If* $y = g\gamma$, *then* $\mathcal{F}_N^y(\tau) = {}^{g\gamma}\tau = {}^{g\gamma}(\tau' \otimes {}^\sigma\tau')$.
2. *If* $y = g\gamma w_L$, *then* $\mathcal{F}_N^y(\tau) = {}^{g\gamma w_L}\tau = {}^{g\gamma}({}^\sigma\tau' \otimes \tau')$

Proof. Indeed, for $y = g\gamma x$, where $x \in Q\backslash G/Q$ and x normalizes L, we have

$$M \cap {}^yQ = M \cap g\gamma Q\gamma^{-1}g^{-1} = M \cap P = M$$

and

$$P \cap {}^yL = P \cap g\gamma L\gamma^{-1}g^{-1} = P \cap M = M = {}^yL,$$

so we have that

$$\mathcal{F}_N^y(\tau) = \iota_M^M\left(({}^y\tau)_{\{e\}}\right) = {}^y\tau.$$

□

If τ is an irreducible unitary (e.g., a discrete series) representation of L, then the two subquotients $\mathcal{F}_N^{g\gamma}(\tau) = {}^{g\gamma}(\tau' \otimes {}^\sigma\tau')$ and $\mathcal{F}_N^{g\gamma w_L}(\tau) = {}^{g\gamma}({}^\sigma\tau' \otimes \tau')$ of π_N are irreducible and unitary.

Jacquet modules and exponents in Case (2)

Let $P = MN$ be any maximal θ-split parabolic subgroup of G with θ-stable Levi M and unipotent radical N, as above. Let $y = gw\gamma \in P\backslash G/Q$, where $w \in [W_\Theta\backslash W_0/W_\Omega]$. Assume that we are in *Case (2)*, then $P \cap {}^yL \subsetneq {}^yL$ is a proper parabolic subgroup of ${}^yL$ and, in particular, $M \cap {}^yL \subsetneq {}^yL$. If τ is a discrete series representation of L, then τ satisfies Casselman's Criterion 2.1.28. The argument that the exponents of $\pi = \iota_Q^G\tau$ along P then satisfy the Relative Casselman's Criterion 2.2.18 follows exactly as in Chapter 4 (*cf.* Proposition 4.2.20). In particular, we may apply Lemmas 4.2.15 and 4.2.16 along both standard and non-standard θ-split parabolic subgroups. We also have the necessary technical Lemmas 5.2.13 and 5.2.14 in place. We record this result as a proposition, which is the direct analog of Proposition 4.2.20.

Proposition 5.2.17. *Let P be a maximal θ-split parabolic subgroup of G and $y \in P\backslash G/Q$ such that $P \cap {}^yL$ is a proper parabolic subgroup of ${}^yL$. Let τ be a discrete series representation of L. The exponents of $\pi = \iota_Q^G\tau$ along P contributed by the subquotient $\mathcal{F}_N^y(\tau) = \iota_{M\cap {}^yQ}^M({}^y\tau)_{N\cap {}^yL}$ of π_N satisfy the condition in* (2.14).

5.2.4 Regularity and distinction of Jacquet modules

Let $P = gP_\bullet g^{-1}$ be a θ-split parabolic with Levi subgroup associate to L, where $P_\bullet = {}^\gamma P_{(n/2,n/2)}$ and $g \in (\mathbf{HT}_0)(F)$. Assume that $\tau = \tau' \otimes {}^\sigma\tau'$ is a regular representation of L. Note that τ is regular if and only if $\tau' \ncong {}^\sigma\tau'$. Under this regularity assumption, we argue that neither of the irreducible subquotients $\mathcal{F}_N^y(\tau)$ of π_N (*cf.* Corollary 5.2.16), can be M^θ-distinguished. Once we identify the subgroup M^θ of θ-fixed points in M, this will follow from the regularity assumption and Theorem 5.2.2.

Remark 5.2.18. There is a right-action of G on the variety X of Hermitian matrices, given by $h\cdot g = {}^t\sigma(g)hg$, where $h \in X$ and $g \in G$. This action is compatible with the right action of G on the set of involutions $\theta_h = \operatorname{Int} h^{-1} \circ {}^t\sigma(\ )^{-1}$, where $h \in X$ That is, we have $g \cdot \theta_h = \theta_{h\cdot g}$. Notice that since ${}^t\sigma(\gamma)w_\ell\gamma \in A_L$, we have that $\gamma\cdot\theta$ restricted to L is equal to θ_e restricted to L.

The Levi subgroup $M = gM_\bullet g^{-1}$ is θ-stable, therefore $g^{-1}\theta(g) \in N_G(M_\bullet)$. Using that P is θ-split, i.e., $\theta(P) = P^{op}$, we have

$$gP_\bullet^{op}g^{-1} = P^{op} = \theta(P) = \theta(g)\theta(P_\bullet)\theta(g^{-1}) = \theta(g)P_\bullet^{op}\theta(g)^{-1},$$

where $\theta(P_\bullet) = P_\bullet^{op}$ holds because $P_\bullet$ is also θ-split. It follows that $g^{-1}\theta(g) \in N_G(P_\bullet^{op})$. By [72, Corollary 6.4.10], $N_G(P_\bullet^{op}) = P_\bullet^{op}$ and so we conclude that $g^{-1}\theta(g) \in N_G(M_\bullet) \cap P_\bullet^{op}$. We have that $g^{-1}\theta(g) \in N_G(M_\bullet) \cap P_\bullet^{op}$ and we have $N_G(M_\bullet) = \gamma N_G(L)\gamma^{-1}$. Define the element

$$x_L = \gamma^{-1}g^{-1}\theta(g)\gamma. \tag{5.15}$$

We have that $x_L \in N_G(L) \cap Q^{op}$. Observe that x_L is an element of L. Indeed, we have the following lemma.

Lemma 5.2.19. *The intersection $N_G(L) \cap Q^{op}$ of the normalizer of L in G with the opposite parabolic Q^{op} of Q is equal to L.*

Proof. Suppose that $q = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \in N_G(L) \cap Q^{op}$, and let $l = \mathrm{diag}(l_1, l_2)$ be an arbitrary element of L. We have that

$$qlq^{-1} = \begin{pmatrix} Al_1 & 0 \\ Bl_1 & Cl_2 \end{pmatrix}\begin{pmatrix} A^{-1} & 0 \\ -C^{-1}BA^{-1} & C^{-1} \end{pmatrix} = \begin{pmatrix} Al_1A^{-1} & 0 \\ Bl_1A^{-1} - Cl_2C^{-1}BA^{-1} & Cl_2C^{-1} \end{pmatrix}. \tag{5.16}$$

Since the element (5.16) must lie in L, we must have that $Bl_1A^{-1} - Cl_2C^{-1}BA^{-1} = 0$, for all $l_1, l_2 \in \mathbf{GL}_{n/2}(E)$. This occurs if and only if $B = Cl_2C^{-1}Bl_1^{-1}$ for all $l_1, l_2 \in \mathbf{GL}_{n/2}(E)$. If we take $l_2 = 1_{n/2}$ to be the identity, then we require that $Bl_1 = B$ for any $l_1 \in \mathbf{GL}_{n/2}(E)$, which occurs if and only if $B(l_1 - 1_{n/2}) = 0$ for any $l_1 \in \mathbf{GL}_{n/2}(E)$. In particular, since there exists $l_1 \in \mathbf{GL}_{n/2}(E)$ such that $l_1 - 1_{n/2}$ is invertible, we must have that $B = 0$. It follows that $N_G(L) \cap Q^{op} = L$, as claimed. □

In particular, by Lemma 5.2.19, we have that $x_L = \gamma^{-1}g^{-1}\theta(g)\gamma = \mathrm{diag}(x_1, x_2) \in L$. Moreover, we can show that x_L is Hermitian. Indeed, since ${}^t\gamma = \gamma = \sigma(\gamma)$ and $w_\ell = {}^tw_\ell = \sigma(w_\ell)$, we have

$$\begin{aligned}
{}^t\sigma(x_L) &= {}^t\sigma(\gamma^{-1}g^{-1}\theta(g)\gamma) \\
&= {}^t\sigma(\gamma^{-1}g^{-1}w_\ell^{-1}({}^t\sigma(g)^{-1})w_\ell\gamma) \\
&= \sigma({}^t\gamma)\sigma({}^tw_\ell)g^{-1}\sigma({}^tw_\ell)^{-1}\sigma({}^tg)^{-1}\sigma({}^t\gamma)^{-1} \\
&= \gamma w_\ell g^{-1}w_\ell^{-1t}\sigma(g)^{-1}\gamma^{-1} \\
&= \gamma w_\ell g^{-1}w_\ell^{-1t}\sigma(g)^{-1}w_\ell w_\ell^{-1}\gamma^{-1} \\
&= \gamma w_\ell g^{-1}\theta(g)w_\ell^{-1}\gamma^{-1} \\
&= z\gamma^{-1}g^{-1}\theta(g)\gamma z^{-1} && \text{where } z = {}^t\sigma(\gamma)w_\ell\gamma = \gamma w_\ell\gamma \in A_L, \\
&= zx_Lz^{-1} \\
&= x_L.
\end{aligned}$$

In fact, we have that x_1 and x_2 are Hermitian elements of $\mathbf{GL}_{n/2}(E)$. Note that, upon restriction to L, $g\gamma \cdot \theta = x_L \cdot \theta_e = \mathrm{Int}\, x_L^{-1} \circ {}^t\sigma(\ )^{-1}$. Note also that $x_L \cdot \theta_e = \theta_{e \cdot x_L} = \theta_{x_L}$. In particular, $l \in L$ is $g\gamma \cdot \theta$-fixed if and only if l is θ_{x_L}-fixed. This can be viewed as a special case of Lemma 2.2.6. Explicitly, $l \in L$ is θ_{x_L}-fixed if and only if $l = x^{-1t}\sigma(l)^{-1}x_L$. Since x_L is Hermitian, we have that $L^{g\gamma\cdot\theta} = L^{\theta_{x_L}}$ is equal to the product $\mathbf{U}_{E/F,x_1} \times \mathbf{U}_{E/F,x_2}$ of unitary groups. Now, observe that $M^\theta = g\gamma L^{g\gamma\cdot\theta}(g\gamma)^{-1}$. Indeed, if $m = g\gamma l\gamma^{-1}g^{-1} \in M$, where $l \in L$, then m is θ-fixed if and only if $l = (g\gamma\cdot\theta)(l)$ is $(g\gamma\cdot\theta)$-fixed.

We record the results so far as a lemma.

Lemma 5.2.20. *Let $M = g\gamma M_{(n/2,n/2)}\gamma^{-1}g^{-1}$, where $g \in (\mathbf{HT}_0)(F)$.*

1. *The subgroup M^θ of θ-fixed points in M is the $g\gamma$-conjugate of $L^{g\gamma\cdot\theta}$.*
2. *We have that $L^{g\gamma\cdot\theta} = L^{\theta_{x_L}}$ is equal to the product $\mathbf{U}_{E/F,x_1} \times \mathbf{U}_{E/F,x_2}$ of unitary groups.*
3. *Explicitly, $M^\theta = g\gamma\left(\mathbf{U}_{E/F,x_1} \times \mathbf{U}_{E/F,x_2}\right)\gamma^{-1}g^{-1}$ is isomorphic to a product of unitary groups.*
4. *Let τ be an irreducible admissible representation of L. Then ${}^{g\gamma}\tau$ is M^θ-distinguished if and only if τ is $\mathbf{U}_{E/F,x_1} \times \mathbf{U}_{E/F,x_2}$-distinguished.*

From Corollary 5.2.16, it follows that M^θ-distinction of ${}^{g\gamma}\tau$ (respectively, ${}^{g\gamma w_L}\tau$) is equivalent to $\mathbf{U}_{E/F,x_1}$-distinction of τ' and $\mathbf{U}_{E/F,x_2}$-distinction of ${}^\sigma\tau'$ (respectively, $\mathbf{U}_{E/F,x_1}$-distinction of ${}^\sigma\tau'$ and $\mathbf{U}_{E/F,x_2}$-distinction of τ'). If $\tau = \tau' \otimes {}^\sigma\tau'$ is a regular discrete series representation, then $\tau' \ncong {}^\sigma\tau'$. It follows from Theorem 5.2.2 that neither τ' nor ${}^\sigma\tau'$ can be distinguished by any unitary group. If $P = MN$ is any θ-split parabolic such that M is associate to L, then by Lemma 5.2.20, neither of the irreducible subquotients of π_N, described in Corollary 5.2.16, can be M^θ-distinguished. Therefore, we have the following result.

Corollary 5.2.21. *Let $\pi = \iota_Q^G\tau$, where $\tau = \tau' \otimes {}^\sigma\tau'$ is a discrete series representation such that $\tau' \ncong {}^\sigma\tau'$. Let $P = {}^gP_\Omega$, where $g \in (\mathbf{HT}_0)(F)$, be any maximal θ-split parabolic subgroup with θ-stable Levi $M = P \cap \theta(P)$ associate to L. Neither of the two irreducible unitary subquotients of π_N, twists of $\tau' \otimes {}^\sigma\tau'$ and ${}^\sigma\tau' \otimes \tau'$ (cf. Lemma 5.2.16), can be M^θ-distinguished.*

5.2.5 RDS for $\mathbf{GL}_n(E)/\mathbf{U}_{E/F}(F)$ for n even, inducing from $P_{(n/2,n/2)}$

Finally, we have the following theorem. The proof of Theorem 5.2.22 follows that of Theorem 4.2.1.

Theorem 5.2.22. *Let $n \geq 4$ be an even integer. Let $Q = P_{(n/2,n/2)}$ be the upper-triangular parabolic subgroup of G with standard Levi factor $L = M_{(n/2,n/2)}$ and unipotent radical $U = N_{(n/2,n/2)}$. Let $\pi = \iota_Q^G\tau$, where $\tau = \tau' \otimes {}^\sigma\tau'$, and τ' is a discrete series representation of $\mathbf{GL}_{n/2}(E)$ such that τ' is not Galois invariant, i.e., $\tau' \ncong {}^\sigma\tau'$. The representation π is a relative discrete series representation for $H\backslash G$ that does not occur in the discrete series of G.*

Proof. By Theorem 2.1.22, the unitary representation π is irreducible since τ is unitary and regular. In addition, π is H-distinguished by Proposition 5.2.11. The representation π does not occur in the discrete series of G by Zelevinsky's classification [75]. Let λ denote a fixed H-invariant linear form on π (note that we do not have multiplicity-one for π [20]). It suffices to show that π satisfies the Relative Casselman's Criterion 2.2.18. Let $P = MN$ be a proper θ-split parabolic subgroup of G. The exponents of π along P are the central characters of the irreducible subquotients of the representations $\mathcal{F}_N^y(\tau)$ given by the Geometric Lemma 2.1.17 (see §5.2.3). By Proposition 5.2.17, the condition (2.14) is satisfied when $P \cap {}^yL$ is a proper parabolic subgroup of ${}^yL$. As in Lemma 4.2.16, the only unitary exponents of π along P occur when $P \cap {}^yL = {}^yL$. By Corollary 5.2.21, π_N cannot be M^θ-distinguished when $P \cap {}^yL = {}^yL$. In the latter case, since π_N is not M^θ-distinguished, Proposition 3.5.1 guarantees that the unitary exponents of π along P do not contribute to $\mathcal{E}xp_{S_M}(\pi_N, r_P\lambda)$. Therefore, (2.14) is satisfied for every proper θ-split parabolic subgroup of G. Finally, by Theorem 2.2.18 the representation π appears in the discrete spectrum of $H\backslash G$. In particular, π is (H,λ)-relatively square integrable for all $\lambda \in \mathrm{Hom}_H(\pi, 1)$. □

We conclude this chapter with the following existence results.

Proposition 5.2.23. *Let $n \geq 4$ be an integer.*

1. *There exist infinitely many equivalence classes of discrete series representations ρ of $\mathbf{GL}_n(E)$ that are not Galois invariant.*
2. *If n is not a prime, then there exist infinitely many equivalence classes of non-supercuspidal discrete series representations τ of $\mathbf{GL}_n(E)$ that are not Galois invariant.*
3. *If n is prime, then there exists a twist of the Steinberg representation St_n of $\mathbf{GL}_n(E)$ that is not Galois invariant.*

Proof. The first statement follows directly from Proposition 5.2.4. To prove the second statement, combine Proposition 5.2.4 and Proposition 5.2.5. The last statement is Corollary 5.2.7. □

Corollary 5.2.24. *Let $n \geq 8$ be an even integer such that n is not twice an odd prime. There exist infinitely many equivalence classes of RDS representations of the form constructed in Theorem 5.2.22 and such that the discrete series τ is not supercuspidal.*

Proof. Apply Proposition 5.2.23 and [75, Theorem 9.7(b)]. □

Remark 5.2.25. The RDS representations constructed in Theorem 5.2.22 are irreducibly induced from discrete series and are thus tempered generic representations of G [41].

Remark 5.2.26. Theorem 5.2.22 is a direct analog of Theorem 4.2.1. Indeed, the inducing representation $\tau = \tau' \otimes {}^\sigma\tau'$ is exactly an L^θ-distinguished discrete series of the θ-elliptic Levi subgroup L and the requirement that $\tau' \not\cong {}^\sigma\tau'$ is exactly the condition required to ensure that τ is regular.

Remark 5.2.27. As in Remark 4.2.3, if we instead assume that τ' is supercuspidal in Theorem 5.2.22, then π is a non-supercuspidal relatively supercuspidal representation of G. A proof of this result can be obtained by a slight modification of our argument. Indeed, when $P \cap {}^yL$ is proper in ${}^yL$, the subquotients $\mathcal{F}_N^y(\tau)$ of the Jacquet module vanish since τ is supercuspidal. Otherwise, $\mathcal{F}_N^y(\tau)$ cannot be M^θ-distinguished. By Proposition 3.5.1, $r_P\lambda = 0$, for every proper θ-split parabolic subgroup P of G and any $\lambda \in \mathrm{Hom}_H(\pi, 1)$. Finally, by Theorem 2.2.16 (Kato–Takano), π is relatively supercuspidal. Moreover, since π is parabolically induced, π is not supercuspidal. Again, note that this modification of Theorem 5.2.22 can be obtained by more direct methods; see, for instance, the work of Murnaghan [57] for such results in a more general setting.

5.2.6 Exhaustion of the discrete spectrum

Again, we do not know if our construction of RDS via Theorem 5.2.22 exhausts all non-discrete relative discrete series in the unitary case. We refer the reader to the discussion of this issue in §4.2.6.

Chapter 6

Representations that are not relatively supercuspidal

Let G be the F-points of an arbitrary connected reductive group defined over F and let H be the subgroup of fixed points of an involution θ. In this chapter, we study the support of relative matrix coefficients defined with respect to invariant forms λ^G produced via Proposition 3.3.1. Recall from Definition 2.2.9 that a smooth representation (π, V) of G is (H, λ^G)-relatively supercuspidal if and only if all of the λ^G-relative matrix coefficients are compactly supported modulo $Z_G H$. We aim to determine if the RDS representations produced in Theorem 4.2.1 and Theorem 5.2.22 are non-(H, λ^G)-relatively supercuspidal. For convenience in applying the results of [44], in this chapter, we work the opposite convention for relative matrix coefficients (*cf.* §2.2.1) – as functions on $G/Z_G H$ given by $g \mapsto \langle \lambda^G, \pi(g^{-1})v \rangle$.

6.1 Support of functions on $G/Z_G H$

Let $Q = LU$ be a parabolic subgroup of G with θ-elliptic Levi subgroup L and unipotent radical U. By Proposition 1.6.8, the subgroups Q, U and U^{op} are θ-stable. Moreover, Q^θ is a parabolic subgroup of H with Levi factorization $Q^\theta = L^\theta U^\theta$ [31].

Remark 6.1.1. In what follows, we make the following two assumptions:

(A1) the restriction of the quasi-character $\delta_Q^{1/2}$ to L^θ is equal to the modular character δ_{Q^θ} of Q^θ.

(A2) L contains a non-trivial (θ, F)-split torus that is not central in L.

The assumption (A1) on the modular function of Q is not strictly necessary, but it simplifies the consideration of the relevant relative matrix coefficients for representations of L.

Remark 6.1.2. The assumptions (A1) and (A2) are satisfied by the inducing subgroups in Theorems 4.2.1 and 5.2.22. In particular, under the additional assumption of Proposition 6.1.3, the results of this chapter apply to the RDS representations produced via Theorem 4.2.1 and Theorem 5.2.22. However, it is unknown if the inducing discrete series representations τ are relatively supercuspidal in the linear, Galois and unitary cases.

Let τ be an irreducible L^θ-distinguished representation of L and let $\lambda \in \mathrm{Hom}_{L^\theta}(\tau, 1)$ be nonzero. Define $\pi = \iota_Q^G \tau$ and construct λ^G from λ using Lemma 3.3.1. The goal of this chapter is to prove the following.

Proposition 6.1.3. *Let $Q = LU$ be a θ-stable parabolic subgroup of G with θ-elliptic Levi subgroup L and unipotent radical U. Assume that (A1) and (A2) hold. Let τ be an irreducible L^θ-distinguished representation of L. Let $\lambda^G \in \mathrm{Hom}_H(\iota_Q^G \tau, 1)$, constructed from a nonzero element λ of $\mathrm{Hom}_{L^\theta}(\tau, 1)$ via Lemma 3.3.1. If τ is not (L^θ, λ)-relatively supercuspidal, then the induced representation $\pi = \iota_Q^G \tau$ is not (H, λ^G)-relatively supercuspidal.*

Corollary 6.1.4. *Let $\pi = \iota_Q^G \tau$ be an irreducible (H, λ^G)-relatively square integrable (RDS) representation constructed as in Theorem 4.2.1 and Theorem 5.2.22. If the inducing discrete series representation τ is not (L^θ, λ)-relatively supercuspidal, then π is not (H, λ^G)-relatively supercuspidal.*

The representation π admits a central character ω_π since π is induced from an irreducible representation. By Lemma 2.2.2, since π is H-distinguished, the character ω_π must be trivial on $Z_G \cap H$. Moreover, since Z_G is the almost direct product $S_G(Z_G \cap H)^\circ$ [33], we have that a smooth right H-invariant function ϕ on G is compactly supported modulo $Z_G H$ if and only if ϕ has compact support modulo $S_G H$. It turns out that we can understand the support of such a function by considering how the support behaves as we "approach infinity" along non-central (θ, F)-split tori.

Fix a maximal (θ, F)-split torus S_0 of G and a θ-base Δ_0 of the root system $\Phi_0 = \Phi(G, A_0)$ of G, where A_0 is a θ-stable maximal F-split torus containing S_0 (*cf.* §1.5). Define S_0^+ to be the set

$$S_0^+ = \{s \in S_0 : s^{-1} \in S_0^-\}, \tag{6.1}$$

where we recall that $S_0^- = \{s \in S_0 : |\alpha(s)|_F \leq 1, \text{ for all } \alpha \in \Delta_0\}$. Given a subset $\Theta \subset \Delta_0$, the sets $S_\Theta^+(\epsilon)$, $0 < \epsilon \leq 1$, are defined analogously (*cf.* equation (1.8)),

$$S_\Theta^+(\epsilon) = \{s \in S_\Theta : s^{-1} \in S_\Theta^-(\epsilon)\}.$$

Definition 6.1.5. Let $\Theta \subset \Delta_0$ be a θ-split subset and let S_Θ be the associated standard (θ, F)-split torus (*cf.* §1.5.1). The sequence $\{s_j\}_{j\in\mathbb{N}}$ approaches S_Θ-*infinity* if the sequences $\{|\alpha(s_j)|_F\}_{j\in\mathbb{N}}$ diverge to infinity for all $\alpha \in \Delta_0 \setminus \Theta$.

For example, if we take $s \in S_0^+ \setminus S_0^1$ such that $|\alpha(s)| > 1$ for all $\alpha \in \Delta_0 \setminus \Delta_0^\theta$, then the sequence $\{s^n\}_{n\in\mathbb{N}}$ approaches S_0-infinity. For a non-standard (θ, F)-split torus S, we can extend Definition 6.1.5 to S using that $S = gS_\Theta g^{-1}$, for some $g \in (\mathbf{HM}_0)(F)$ and some standard (θ, F)-split torus S_Θ (Lemma 1.6.4). Precisely, the sequence $\{gs_jg^{-1}\}_{j\in\mathbb{N}} \subset S$ approaches S-infinity if and only if the sequence $\{s_j\}_{j\in\mathbb{N}} \subset S_\Theta$ approaches S_Θ-infinity.

To prove the Proposition 6.1.3, we apply the contrapositive of [44, Lemma 6.7], which forms part of Kato and Takano's characterization of relatively supercuspidal representations. There is an analog of the Cartan decomposition for p-adic symmetric spaces due to Delorme and Sécherre [18]. In particular, there exists a compact subset $\mathfrak{C}$ of G and a finite subset $\mathfrak{X}$ of $(\mathbf{HM}_0)(F)$ such that $G = \mathfrak{C}S_0^+\mathfrak{X}^{-1}H$, where $\mathfrak{X}^{-1} = \{x^{-1} : x \in \mathfrak{X}\}$.

Lemma 6.1.6 ([44, Lemma 6.7])**.** *Let $\lambda \in \mathrm{Hom}_H(\pi, 1)$ be a nonzero H-invariant form on an admissible H-distinguished representation (π, V) of G. Let $P = gP_\Theta g^{-1}$ be a θ-split parabolic subgroup of G, for* *some θ-split subset $\Theta \subset \Delta_0$ and $g \in (\mathbf{HM}_0)(F)$. If $r_P(\lambda) = 0$, then for all $v \in V$ there exists a positive real number $\epsilon \leq 1$, depending on Θ and g, such that the relative matrix coefficient $\varphi_{\lambda,v}$ vanishes identically on $\mathfrak{C}S_\Theta^+(\epsilon)g^{-1}H$.*

The contrapositive of Lemma 6.1.6 is the following.

Lemma 6.1.7 (Contrapositive of Lemma 6.1.6)**.** *Let $\lambda \in \mathrm{Hom}_H(\pi, 1)$ be a nonzero H-invariant form on an admissible H-distinguished representation (π, V) of G. Let $P = gP_\Theta g^{-1}$ be a θ-split parabolic subgroup of G, for some θ-split subset $\Theta \subset \Delta_0$ and $g \in (\mathbf{HM}_0)(F)$. If there exists $v \in V$ such that, for all $0 < \epsilon \leq 1$, the function $\varphi_{\lambda,v}$ is non-zero on some element of $\mathfrak{C}S_\Theta^+(\epsilon)g^{-1}H$, then $r_P(\lambda) \neq 0$.*

Remark 6.1.8. Taking a net $\{s_\epsilon\}_{\epsilon>0}$, where $s_\epsilon \in S_\Theta^+(\epsilon)$ and letting ϵ go to zero, is equivalent to having $\{s_\epsilon\}_{\epsilon>0}$ approach S_Θ-infinity as $\epsilon \to 0$.

If $0 < \epsilon_1 \leq \epsilon_2 \leq 1$, then $S_\Theta^-(\epsilon_1) \subset S_\Theta^-(\epsilon_2) \subset S_\Theta^-$. By definition, $S_\Theta^+(\epsilon) = \{s \in S_\Theta : s^{-1} \in S_\Theta^-(\epsilon)\}$; therefore $S_\Theta^+(\epsilon_1) \subset S_\Theta^+(\epsilon_2) \subset S_\Theta^+$. In particular, as ϵ goes to zero the cosets $\mathfrak{C}S_\Theta^+(\epsilon)g^{-1}H$ shrink, and $\mathfrak{C}S_\Theta^+(\epsilon_1)g^{-1}H \subset \mathfrak{C}S_\Theta^+(\epsilon_2)g^{-1}H$, if $0 < \epsilon_1 \leq \epsilon_2 \leq 1$. We have the following.

Corollary 6.1.9. *Let $\lambda \in \mathrm{Hom}_H(\pi, 1)$ be a nonzero H-invariant form on an admissible H-distinguished representation (π, V) of G. Let S be the (θ, F)-split component of a proper θ-split parabolic P of G. If there exists $v \in V$ such that the function $\varphi_{\lambda,v}$ is non-zero on a sequence $\{s_n\}_{n\in\mathbb{N}}$ approaching S-infinity, then π is not (H, λ)-relatively supercuspidal.*

Proof. Apply Lemma 6.1.7 to show that $r_P(\lambda) \neq 0$, then the result follows from Theorem 2.2.16. □

6.1.1 Proof of Proposition 6.1.3

In this subsection, we work under the assumptions of Proposition 6.1.3.

Let S be a maximal non-central, by (A2), (θ, F)-split torus of L. By assumption τ is not (L^θ, λ)-relatively supercuspidal; therefore, there is some relative matrix coefficient $\varphi_{\lambda,v}$ of τ that is not compactly supported modulo $S_L L^\theta$. By the analog of the Cartan decomposition [18] applied to L, there is a compact subset $\mathfrak{C}_L$ of L and a finite subset $\mathfrak{X}_L$ of $(\mathbf{L}^\theta C_{\mathbf{L}}(S))(F)$ such that $L = \mathfrak{C}_L S^+ \mathfrak{X}_L^{-1} L^\theta$. The support of $\varphi_{\lambda,v}$ is not contained in any subset of L of the form $CS_L L^\theta$, where C is compact. In fact, since $\mathfrak{X}_L$ is finite and $\varphi_{\lambda,v}$ is right L^θ-invariant, there exists a sequence $\{\ell_n\}_{n\in\mathbb{N}} \subset L$ such that $\varphi_{\lambda,v}(\ell_n) \neq 0$, and where $\ell_n = o_n s_n x^{-1}$, where $o_n \in \mathfrak{C}_L$, $x \in \mathfrak{X}_L$ (fixed), and $\{s_n\} \subset S^+$ approaches S-infinity. Replacing S with the maximal (θ, F)-split torus $S' = xSx^{-1}$ of L, we have that $\varphi_{\lambda,v}$ is nonzero on the sequence $\{\ell_n = o_n' s_n'\}_{n\in\mathbb{N}}$, with $o_n' = o_n x^{-1}$, $s_n' = x s_n x^{-1}$, and where $\{s_n'\}$ approaches S'-infinity. The sequence $\{\ell_n\}$ has non-compact image modulo $S_L L^\theta$ and modulo $S_G H$ (also recall that L is θ-elliptic and $S_G = S_L$).

The idea of the proof of Proposition 6.1.3 is to understand the support of the λ^G-relative matrix coefficients of π in terms of the support of the λ-relative matrix coefficients of τ. For simplicity, we assume that we can find a vector $v \in V_\tau$ such that $\varphi_{\lambda,v}$ is not compactly supported on S/S_L. If this is not the case, we may still produce $\varphi_{\lambda,v}$ and a sequence $\{\ell_n : \varphi_{\lambda,v}(\ell_n) \neq 0\}$ with non-compact image modulo $S_L L^\theta$ (and modulo $S_G H$), as above. The argument below to produce a non-compactly supported λ^G-relative matrix coefficient φ_{λ^G, f_v} of π still goes through with only a slight adjustment. In particular, one need only apply Lemma 6.1.7 instead of Corollary 6.1.9.

Proof of Proposition 6.1.3. We show that if τ admits a non-compactly supported (modulo $Z_L L^\theta$) λ-relative matrix coefficient, then π admits a non-compactly supported (modulo $Z_G H$) λ^G-relative matrix coefficient. Let V_π denote the space of π and V_τ denote the space of the inducing representation τ. We want to construct a vector $f \in V_\pi$ such that the λ^G-relative matrix coefficient $\varphi_{\lambda^G,f}$ is nonzero on a sequence of (θ, F)-split elements in S approaching S-infinity. To that end, for an arbitrary $f \in V_\pi$, we consider the restriction of the λ^G-relative matrix coefficient $\varphi_{\lambda^G,f}$ to S. Using the definition of λ^G given in (3.1), and that $S \subset L$, we see that

$$\langle \lambda^G, \pi(s)^{-1} f\rangle = \int_{Q^\theta\backslash H} \langle \lambda, f(hs^{-1})\rangle \, d\mu(h) = \int_{Q^\theta\backslash H} \langle \lambda, \delta_Q^{1/2}(s^{-1})\tau(s^{-1})f(shs^{-1})\rangle \, d\mu(h).$$

Replacing s by an arbitrary element ℓ of L, we have

$$\langle \lambda^G, \pi(\ell)^{-1} f\rangle = \int_{Q^\theta\backslash H} \langle \lambda, \delta_Q^{1/2}(\ell^{-1})\tau(\ell^{-1})f(\ell h\ell^{-1})\rangle \, d\mu(h). \tag{6.2}$$

If we can appropriately control the elements $f(\ell h \ell^{-1}) \in V_\tau$, then the integrand in (6.2) is essentially a λ-relative matrix coefficient of τ. This gives us a way to link the support of the λ-relative matrix coefficients of τ with the support of the λ^G-relative matrix coefficients of π, via the integral (6.2).

Let K be a compact open subgroup of G that has Iwahori factorization with respect to $Q = LU$. Then the product map (in any order)

$$(L \cap K) \times (U \cap K) \times (U^{op} \cap K) \to K,$$

is bijective. Take a vector $v \in V_\tau$ such that $s \mapsto \langle \lambda, \delta_Q^{1/2}(s^{-1})\tau(s^{-1})v\rangle = \delta_Q^{1/2}(s^{-1})\varphi_{\lambda,v}(s)$ is not compactly supported on S/S_L. Define $f_v \in V_\pi$ to be zero off of $QK = Q(U^{op} \cap K)$ and such that $f_v(\bar{u}) = v$ for $\bar{u} \in U^{op} \cap K$. Since we require $f_v \in V_\pi$ this completely determines the function $f_v : G \to V_\tau$. Now consider the possible values of $f_v(\ell h \ell^{-1})$. By construction, f_v is zero unless $\ell h \ell^{-1} \in Q(U^{op} \cap K)$. Suppose that $\ell h \ell^{-1} \in Q(U^{op} \cap K)$. This occurs if and only if h lies in

$$(\ell^{-1} Q \ell)\ell^{-1}(U^{op} \cap K)\ell = Q(U^{op} \cap \ell^{-1} K \ell), \tag{6.3}$$

where the equality in (6.3) holds since $\ell \in L$ normalizes U and U^{op}. Write $h = q\bar{u}$ where $q \in Q$ and $\bar{u} \in U^{op} \cap \ell^{-1} K \ell$. Since h is θ-fixed,

$$q\bar{u} = h = \theta(h) = \theta(q)\theta(\bar{u}),$$

where $\theta(q) \in Q$ and $\theta(\bar{u}) \in U^{op}$ since both subgroups are θ-stable. The product map $(q', \bar{u}') \mapsto q'\bar{u}'$ on

$$L \times U \times U^{op} = Q \times U^{op} \to G$$

is one-to-one; therefore, $\theta(q) = q$ and $\theta(\bar{u}) = \bar{u}$. It follows that h is equivalent to $\bar{u}$ in $Q^\theta\backslash H$; moreover, $\ell \bar{u} \ell^{-1} \in U^{op} \cap K$ since $\bar{u} \in (U^{op} \cap \ell^{-1} K \ell)^\theta$ and so $f_v(\ell \bar{u} \ell^{-1}) = v$.

The compact open subgroup $\ell^{-1} K \ell$ also has Iwahori factorization with respect to Q. In particular, the image of $(\ell^{-1} K \ell)^\theta$ in $Q^\theta\backslash H$ is equal to the image of $(U^{op} \cap \ell^{-1} K \ell)^\theta$ which is open and thus has positive measure. From (6.2), integrating over the image of $(U^{op} \cap \ell^{-1} K \ell)^\theta$ in $Q^\theta\backslash H$, we obtain

$$\langle \lambda^G, \pi(\ell^{-1}) f_v\rangle = c_\ell \langle \lambda, \delta_Q^{1/2}(\ell^{-1})\tau(\ell^{-1})v\rangle = c_\ell \cdot \delta_Q^{1/2}(\ell^{-1})\varphi_{\lambda,v}(\ell), \tag{6.4}$$

where $c_\ell = \mu((U^{op} \cap \ell^{-1} K \ell)^\theta) > 0$. It is immediate that if the support of $\varphi_{\lambda,v}$ on S/S_L is non-compact then the same holds for φ_{λ^G, f_v}. In particular, by the assumption on $\varphi_{\lambda,v}$, there exists a sequence $\{s_n\}_{n\in\mathbb{N}}$ approaching S-infinity such that $\varphi_{\lambda,v}(s_n) \neq 0$, for all $n \in \mathbb{N}$. For this same sequence, we have that $\varphi_{\lambda^G, f_v}(s_n) \neq 0$, for every $n \in \mathbb{N}$. By Corollary 6.1.9, π is not (H, λ^G)-relatively supercuspidal. $\square$

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Appendix A

RDS for $\mathbf{GL}_{n-1}(F) \times \mathbf{GL}_1(F) \backslash \mathbf{GL}_n(F)$

We determine (nearly) all relative discrete series representations for $\mathbf{GL}_{n-1}(F) \times \mathbf{GL}_1(F) \backslash \mathbf{GL}_n(F)$. We classify the RDS up to showing that certain representations induced from principal series are *not* RDS. An explicit classification of the irreducible $\mathbf{GL}_{n-1}(F) \times \mathbf{GL}_1(F)$-distinguished representations of $\mathbf{GL}_n(F)$ was given by Venketasubramanian [74]. We exhibit the RDS for $\mathbf{GL}_{n-1}(F) \times \mathbf{GL}_1(F) \backslash \mathbf{GL}_n(F)$ as parabolically induced representations from a θ-elliptic Levi subgroup L. In contrast to Theorems 4.2.1 and 5.2.22, the L^θ-distinguished inducing representation is *not* a discrete series representation of L.

Here, we freely use Bernstein and Zelevinsky's notation [7, 75] for parabolically induced representations. We often write GL_m for $\mathbf{GL}_m(F)$ and we let I_m denote the $m \times m$ identity matrix.

A.1 $\mathbf{GL}_{n-1}(F)$-distinction for representations of $\mathbf{GL}_n(F)$

We begin be recalling some results of Venketasubramanian [74], which is based largely on the work of Prasad [60]. Let $G = \mathbf{GL}_n(F)$, and let $\underline{\mathrm{H}}$ be the image of $\mathbf{GL}_{n-1}(F)$ embedded into G by

$$h \mapsto \begin{pmatrix} h & \\ & 1 \end{pmatrix}.$$

Let π be an irreducible admissible representation of G. Let ν denote the character $|\det(\cdot)|_F$ of GL_m, with m understood from context. We use 1_m to denote the trivial character of GL_m. First, we recall the following result of Prasad [60].

Lemma A.1.1 (Prasad)**.** *Let π be an irreducible admissible infinite dimensional representation of GL_2 and let χ be a character of the diagonal torus T that restricts to the central character of π on the centre, then $\mathrm{Hom}_T(\pi, \chi)$ is nonzero.*

Theorem A.1.2 (Venketasubramanian)**.** *For $n \geq 3$, π is $\underline{\mathrm{H}}$-distinguished if and only if the Langlands parameter $\mathcal{L}(\pi)$, associated to π by the Local Langlands Correspondence (LLC), has a subrepresentation $\mathcal{L}(1_{n-2})$ of dimension $n-2$ corresponding to the trivial representation of GL_{n-2} and such that $\mathcal{L}(\pi)/\mathcal{L}(1_{n-2})$ corresponds under the LLC to either an infinite dimensional representation or one of the characters $\nu^{\pm\left(\frac{n-2}{2}\right)}$ of GL_2.*

For our purposes, we are interested in the following result which appears as [74, Corollary 6.15].

Theorem A.1.3 (Venketasubramanian)**.** *The following is a list of all irreducible admissible representations of G that are $\underline{\mathrm{H}}$-distinguished:*

1. *the trivial representation,*
2. $\iota_{P_{(n-1,1)}}^G(\nu^{-\frac{1}{2}}\otimes\chi)$ *where* $\chi\neq\nu^{\frac{n-1}{2}},\nu^{-(\frac{n+1}{2})}$ *and its contragredient,*
3. $\iota_{P_{(n-2,2)}}^G(1_{n-2}\otimes\tau)$ *where τ is an irreducible supercuspidal representation of* GL_2*, a twist of the Steinberg representation or the irreducible principal series $\chi_1\times\chi_2$, where $\chi_1,\chi_2\neq\nu^{\pm\frac{n-1}{2}}$,*
4. *the representation L_n and its contragredient $L_n^\vee$ (defined below).*

For $n\geq 2$, Venketasubramanian defines L_n to be the unique irreducible quotient of the representation $\iota_{P_{(n-1,1)}}^G(\nu^{\frac{1}{2}}\otimes\nu^{\frac{n+1}{2}})$. In particular, note that $L_2=\mathrm{St}_2\,\nu$. The representation $\iota_{P_{(n-1,1)}}^G(\nu^{\frac{1}{2}}\otimes\nu^{\frac{n+1}{2}})$ has length two, and the sequence

$$0\to\nu\to\iota_{P_{(n-1,1)}}^G(\nu^{\frac{1}{2}}\otimes\nu^{\frac{n+1}{2}})\to L_n\to 0,$$

is exact.

A.2 Distinction by $\mathbf{GL}_{n-1}(F)$ versus $\mathbf{GL}_{n-1}(F)\times\mathbf{GL}_1(F)$

Let $\theta=\mathrm{Int}\,w$ be the inner automorphism given by conjugation by the matrix

$$w=\begin{pmatrix} & & 1\\ & I_{n-2} & \\ 1 & & \end{pmatrix},$$

note that w is diagonalizable; in particular, there is $x\in G$ such that $xwx^{-1}=\mathrm{diag}(I_{n-1},-1)$. In fact, we have that θ is G-equivalent to $\theta'=\mathrm{Int}\,\mathrm{diag}(I_{n-1},-1)$. It follows that the group $\mathbf{H}=\mathbf{G}^\theta$ of θ-fixed points is isomorphic to $\mathbf{GL}_{n-1}\times\mathbf{GL}_1$. Write $H=\mathbf{H}(F)$ for the F-points of $\mathbf{H}$. The subgroup H of G is conjugate to the subgroup

$$H'=\left\{\begin{pmatrix} X & 0\\ 0 & a\end{pmatrix}: X\in\mathrm{GL}_{n-1}, a\in F^\times\right\},$$

which has $\underline{\mathrm{H}}$ as a subgroup. It is straightforward to check that H'-distinction is equivalent to H-distinction (*cf.* Lemma 2.2.6); moreover, we have the following.

Lemma A.2.1. *An irreducible admissible representation π of G is H'-distinguished if and only if π is $\underline{\mathrm{H}}$-distinguished and has trivial central character.*

Proof. If π is H'-distinguished then it is also distinguished by the subgroup $\underline{\mathrm{H}}$ of H'. To prove the converse, let $\lambda\in\mathrm{Hom}_{\underline{\mathrm{H}}}(\pi,1_{n-1})$ and denote the central character of π by ω_π. We can show that $\lambda\in\mathrm{Hom}_{H'}(\pi,1\otimes\omega_\pi)$, where we are regarding ω_π as a character of $F^\times\cong Z_G$. Let $v\in V_\pi$ and let

$$h=\begin{pmatrix} X & 0\\ 0 & a\end{pmatrix}\in H'.$$

We compute that

$$\lambda(\pi(h)v) = \lambda\left(\pi\begin{pmatrix} a^{-1}X & 0 \\ 0 & 1\end{pmatrix}\pi\begin{pmatrix} a & & \\ & \ddots & \\ & & a\end{pmatrix}v\right) = \lambda(\omega_\pi(a)v) = \omega_\pi(a)\lambda(v);$$

therefore $\lambda \in \mathrm{Hom}_{H'}(\pi, 1 \otimes \omega_\pi)$. If ω_π is trivial, then we have that $\lambda \in \mathrm{Hom}_{H'}(\pi, 1)$, and since $\lambda \neq 0$ we have the desired result. $\square$

Combining Lemma A.2.1 with Theorem A.1.3 we have:

Corollary A.2.2. *An irreducible admissible representation π is H-distinguished if and only if*

1. *π is the trivial representation*
2. *$\pi = \iota^G_{P_{(n-2,2)}}(1_{n-2} \otimes \tau)$ where τ is an irreducible admissible infinite dimensional representation of GL_2 with trivial central character.*

Proof. By Lemma A.2.1 it suffices to show that the remaining representations in Theorem A.1.3 do not have trivial central character. Write $\omega_\bullet$ for the central character of the irreducible representation $\pi_\bullet$ and recall that $\widetilde{\pi}_\bullet$ has central character $\omega_\bullet^{-1}$ (*cf.* Lemma 2.1.6). Let $z = \mathrm{diag}(a, \dots, a) \in Z_G$.

1. Consider $\pi_1 = \iota^G_{P_{(n-1,1)}}(\nu^{-\frac{1}{2}} \otimes \chi)$, where $\chi \neq \nu^{\frac{n-1}{2}}, \nu^{-\left(\frac{n+1}{2}\right)}$ and its contragredient. We have that

$$\begin{aligned} \omega_1(z) &= \delta^{1/2}_{P_{(n-1,1)}}(z)(\nu^{-\frac{1}{2}} \otimes \chi)(z) \\ &= \nu^{-\frac{1}{2}}\,\mathrm{diag}_{n-1}(a, \dots, a)\chi(a) && (\delta^{1/2}_{P_{(n-1,1)}}|_{Z_G} \equiv 1) \\ &= |a|^{-(n-1)/2}\chi(a) \end{aligned}$$

   which is non-trivial for some $a \in F^\times$ since $\chi \neq \nu^{\frac{n-1}{2}}$. In particular ω_1 and ω_1^{-1} are non-trivial; therefore π_1 and $\widetilde{\pi}_1$ are not H-distinguished.

2. Next, consider $\pi_2 = \iota^G_{P_{(n-2,2)}}(1_{n-2} \otimes \sigma)$, where σ as non-trivial central character. We have that

$$\omega_2(z) = \delta^{1/2}_{P_{(n-2,2)}}(z)(1_{n-2} \otimes \sigma)(z) = \omega_\sigma(\mathrm{diag}(a, a)) \qquad (\text{since } \delta^{1/2}_{P_{(n-2,2)}}|_{Z_G} \equiv 1),$$

   which is non-trivial by the assumption on σ. Therefore, π_2 is not H-distinguished.

3. Finally, we consider L_n and $\widetilde{L_n}$. Let $\pi_3 = \iota^G_{P_{(n-1,1)}}(\nu^{\frac{1}{2}} \otimes \nu^{\frac{n+1}{2}})$. The representation π_3 has central character $\omega_3 = \nu$. Indeed,

$$\pi_3(z) = \delta^{1/2}_{P_{(n-2,2)}}(z)\nu^{\frac{1}{2}} \otimes \nu^{\frac{n+1}{2}}(z) = |a|^{(n-1)/2}|a|^{(n+1)/2} = |a|^n = \nu(z).$$

   Since ω_3-representations form an abelian category, the quotient L_n also has central character $\nu \neq 1$; therefore, neither L_n nor $\widetilde{L}_n$ are H-distinguished.

$\square$

Remark A.2.3. An irreducible admissible infinite dimensional representation of GL_2 is either supercuspidal, a twist of the Steinberg representation, or an irreducible principal series [11].

Remark A.2.4. For a twist $\tau = \eta \otimes \mathrm{St}$ of the Steinberg representation of GL_2 to have trivial central character, the character η must be square-trivial, i.e., η is trivial on the set $(F^\times)^2$ of squares in $F^\times$. In particular, a square-trivial character η is effectively a character of the finite group $F^\times/(F^\times)^2$ and must be unitary. Similarly, for an irreducible principal series representation $\chi_1 \times \chi_2$ to have trivial central character, we require that $\chi_2 = \chi_1^{-1}$. We are still assuming, as in Theorem A.1.3, that $\chi_1 \neq \nu^{\pm\frac{n-1}{2}}$ and since we want $\chi_1 \times \chi_1^{-1}$ to be irreducible, we require that $\chi_1 \neq \nu^{\pm 1/2}$.

Corollary A.2.5. *An irreducible representation π in the discrete spectrum of $H\backslash G$ is equivalent to a representation of the form $\iota_{P_{(n-2,2)}}^G(1_{n-2} \otimes \tau)$, where τ is an irreducible admissible infinite dimensional representation of* GL_2 *with trivial central character.*

A.3 Irreducible principal series representations of $\mathbf{GL}_2(F)$

For a very nice discussion of the principal series representations of $\mathrm{GL}_2 = \mathbf{GL}_2(F)$, we refer to [11]. Let χ_1 be a quasi-character of $F^\times$ such that $\chi_1 \neq \nu^{\pm 1/2}$. The principal series $\tau = \chi_1 \times \chi_1^{-1}$ is a self-contragredient irreducible admissible representation of GL_2 with trivial central character. In fact, $\chi_1 \times \chi_1^{-1} = \iota_{B_2}^{\mathrm{GL}_2}(\chi_1 \otimes \chi_1^{-1})$ is the parabolically induced representation from the standard upper-triangular Borel subgroup $B_2 = T_2N_2$ of GL_2 and the character $\chi_1 \otimes \chi_1^{-1}$ of the diagonal torus T_2. If $\chi_1 \neq \chi_1^{-1}$ then the Jacquet module τ_{N_2} of τ along the standard Borel subgroup B_2 of GL_2 is completely reducible and

$$\tau_{N_2} = \delta_B^{1/2}\left(\chi_1 \otimes \chi_1^{-1}\right) \oplus \delta_B^{1/2}\left(\chi_1^{-1} \otimes \chi_1\right), \qquad (\chi_1 \neq \chi_1^{-1}), \tag{A.1}$$

as a representation of the F-split torus T_2. Both of the irreducible subquotients of (A.1) are trivial on the centre of GL_2. On the other hand if $\chi_1 = \chi_1^{-1}$ then the Jacquet module τ_{N_2} is the two-dimensional T_2-representation given by

$$\mathrm{diag}(t_1, t_2) \mapsto \delta_B^{1/2}\left(\mathrm{diag}(t_1,t_2)\right)\ \chi_1(t_1)\chi_1^{-1}(t_2)\ \begin{pmatrix} 1 & \mathrm{val}_F(t_1/t_2) \\ 0 & 1 \end{pmatrix}, \qquad (\chi_1 = \chi_1^{-1}) \tag{A.2}$$

where $\mathrm{val}_F : F \to \mathbb{Z}$ denotes the additive valuation on F. In the case that $\chi_1 = \chi_1^{-1}$, the Jacquet module τ_{N_2} (A.2) has two irreducible subquotients isomorphic to the character $\delta_B^{1/2}\chi_1 \otimes \chi_1^{-1}$ of T_2, which is trivial on the centre of GL_2.

A.4 RDS by induction from θ-elliptic Levi subgroups

We now describe how to obtain the representations in Corollary A.2.2 via parabolic induction from θ-stable parabolic subgroups of G with θ-elliptic Levi subgroups. We then apply the Relative Casselman's Criterion 2.2.18 to these representations and determine which are relative discrete series.

A.4.1 Tori, roots and θ-split parabolic subgroups

Let A_0 be the diagonal maximal F-split torus of G. Observe that A_0 is θ-stable. Indeed, if $a = \mathrm{diag}(a_1, \ldots, a_n) \in A_0$, then we have $\theta(a) = \mathrm{diag}(a_n, a_2, \ldots, a_{n-1}, a_1)$. The F-split torus

$$S_0 = \{\mathrm{diag}(a, 1, \ldots, 1, a^{-1}) : a \in F^\times\},$$

is a maximal (θ, F)-split torus of G contained in A_0. Let $W \cong N_G(A_0)/A_0$ be the Weyl group of G and let $\Phi = \Phi(G, A_0)$ be the root system of A_0 in G. Recall that $\Phi = \{\epsilon_i - \epsilon_j : 1 \leq i \neq j \leq n\}$, where ϵ_i denotes the i^{th} F-rational coordinate character of A_0. The following lemma is readily verified.

Lemma A.4.1. *The standard base $\Delta_0 = \{\epsilon_i - \epsilon_{i+1} : 1 \leq i \leq n-1\}$ of Φ is a θ-base for Φ and the subset of θ-fixed simple roots is $\Delta_0^\theta = \{\epsilon_i - \epsilon_{i+1} : 2 \leq i \leq n-2\}$.*

In this setting, $P = P_{\Delta_0^\theta}$ is the only proper Δ_0-standard θ-split parabolic subgroup. It can be shown, as in Chapter 4, that P is the unique proper θ-split parabolic subgroup of G up to H-conjugacy. Explicitly, P is equal to the standard block upper-triangular parabolic subgroup $P_{(1,n-2,1)}$. Let $P = MN$ be the usual Levi factorization of P with $M = M_{(1,n-2,1)}$ and $N = N_{(1,n-2,1)}$.

A.4.2 Inducing data

Let $Q = LU$ be the parabolic subgroup of G with Levi subgroup $\mathrm{GL}_{n-2} \times \mathrm{GL}_2$ of the form

$$L = \left\{ \begin{pmatrix} a & & b \\ & X & \\ c & & d \end{pmatrix} : X \in \mathrm{GL}_{n-2}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2 \right\},$$

and unipotent radical

$$U = \left\{ \begin{pmatrix} 1 & & 0 \\ u & I_{n-2} & v \\ 0 & & 1 \end{pmatrix} : u, v \in F^{n-2} \right\}.$$

It is straightforward to verify that L is θ-elliptic. By Proposition 1.6.8 Q and its opposite Q^{op} are θ-stable parabolic subgroups. Let τ be an irreducible admissible infinite dimensional self-contragredient representation of GL_2 with trivial central character. Define $\pi = \iota_Q^G(1_{n-2} \otimes \tau)$ to be the representation parabolically induced from the representation $1_{n-2} \otimes \tau$ of L.

To apply Lemma 3.3.1 and prove that π is H-distinguished, we need to show that there exists a linear form λ in the space $\mathrm{Hom}_{L^\theta}((1_{n-2} \otimes \tau)\delta_Q^{1/2}, \delta_{Q^\theta})$. First, observe that

$$Q^\theta : \left\{ \begin{pmatrix} a & & b \\ u & X & u \\ b & & a \end{pmatrix} : X \in \mathrm{GL}_{n-2}, a^2 - b^2 \neq 0, u \in F^{n-2} \right\}.$$

The modular characters of Q and Q^θ are given by

$$\delta_Q \begin{pmatrix} a & & b \\ & X & \\ c & & d \end{pmatrix} = |\det X|^2 \left| \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right|^{-(n-2)}$$

and

$$\delta_Q^\theta \begin{pmatrix} a & & b \\ & X & \\ b & & a \end{pmatrix} = |\det X||a+b|^{-(n-2)},$$

respectively. We will produce the required invariant form λ from an element λ' of the space $\mathrm{Hom}_A(\tau\nu^{-\frac{(n-2)}{2}}, \chi)$, where χ denotes the character

$$\chi \begin{pmatrix} a & b \\ b & a \end{pmatrix} = |a+b|^{-(n-2)},$$

of the F-split torus

$$A = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a^2 - b^2 \neq 0 \right\},$$

which is conjugate to the diagonal torus T_2 of GL_2. Observe that χ is the restriction of δ_{Q^θ} to the copy of A in the "outer" GL_2-block of L; similarly $\nu^{-(n-2)/2}$ is the restriction of $\delta_Q^{1/2}$ to the GL_2 block of L. By assumption, τ has trivial central character. It is straightforward to check that $\nu^{-(n-2)/2}$ and χ agree on the centre of GL_2. By Lemma A.1.1, and since the diagonal torus of GL_2 is conjugate to A, we have that $\mathrm{Hom}_A(\tau\nu^{-(n-2)/2}, \chi)$ is nonzero. Let $\lambda' \in \mathrm{Hom}_A(\tau\nu^{-(n-2)/2}, \chi)$ be nonzero. We can identify the space W_τ of τ and the space of $(1 \otimes \tau)$. By doing so, we can regard λ' as a linear form λ on the space of $(1_{n-2} \otimes \tau)\delta_Q^{1/2}$. It follows immediately that λ is a nonzero element of $\mathrm{Hom}_{L^\theta}((1_{n-2} \otimes \tau)\delta_Q^{1/2}, \delta_{Q^\theta})$. We may now apply Lemma 3.3.1 and let $\lambda^G \in \mathrm{Hom}_H(\pi, 1)$ be the nonzero image of λ.

A.4.3 Computation of Jacquet modules and exponents

The Geometric Lemma 2.1.17 gives us that there is a filtration of π_N such that the associated graded object is given by

$$\mathrm{gr}(\pi_N) = \bigoplus_{y \in M\backslash S(M,L)/L} \mathcal{F}_N^y(1_{n-2} \otimes \tau), \tag{A.3}$$

where

$$\mathcal{F}_N^y(1_{n-2} \otimes \tau) = \iota_{M\cap {}^yQ}^M {}^y(1_{n-2} \otimes \tau)_{N \cap {}^yL} \tag{A.4}$$

By Lemma 2.1.24, we may utilize (A.3) and (A.4) to compute the exponents π along P. First, we will choose nice representatives for the double cosets $P\backslash G/Q$. Recall that $P = P_{(1,n-2,1)}$ is a standard parabolic of G and Q is conjugate to the standard parabolic $P_{(n-2,2)}$. More precisely, let x_Q be the permutation matrix representing the cycle $(1\ 2\ 3\ \ldots\ n-2\ n-1)$, then $Q = x_Q P_{(n-2,2)} x_Q^{-1}$. Observe that the map $PyP_{(n-2,2)} \mapsto Pyx_Q^{-1}Q$ gives an isomorphism between $P\backslash G/P_{(n-2,2)}$ and $P\backslash G/Q$. For our choice of Δ_0, a root $\epsilon_i - \epsilon_j$ is positive if and only if $i < j$, in this case we write $\epsilon_i - \epsilon_j > 0$. We also recall that for $w \in W$ the action of w on $\epsilon_i - \epsilon_j$ is given by

$$\begin{aligned} w(\epsilon_i - \epsilon_j) &= (\epsilon_i - \epsilon_j) \circ \operatorname{Ad} w^{-1} \\ &= \epsilon_{w(i)} - \epsilon_{w(j)}, \end{aligned}$$

where we regard w and element of the permutation group S_n on $\{1, \ldots, n\}$. For consistency with the notation in Chapters 4 and 5, write $\Theta = \Delta_0^\theta = \{\epsilon_i - \epsilon_{i+1} : 2 \le i \le n-2\}$, and set $\Omega = \Delta_0 \setminus \{\epsilon_{n-2} - \epsilon_{n-1}\}$. Then we have $P = P_\Theta$ and $P_{(n-2,2)} = P_\Omega$ is conjugated to Q by x_Q.

Lemma A.4.2. *For Θ and Ω above, $[W_\Theta\backslash W/W_\Omega]$ is equal to*

$$\{e, (1\ 2\ \ldots\ n-2\ n-1), (n-2\ n\ n-1), (1\ 2\ \ldots\ n-3\ n-2\ n\ n-1)\},$$

where the non-trivial elements of $W \cong \mathfrak{S}_n$ are written in cycle notation.

Proof. The claim is straightforward to verify using the definition of $[W_\Theta\backslash W/W_\Omega]$. We omit the details. □

Corollary A.4.3. *A set of representatives for $P\backslash G/Q$ is given by $[W_\Theta\backslash W/W_\Omega]x_Q^{-1}$, which is equal to the set $\{e, (n-1\ n-2\ \ldots\ 3\ 2\ 1), (n-2\ n-3\ \ldots\ 3\ 2\ 1)(n-1\ n), (n-1\ n)\}$, where the permutation matrices are written in cycle notation.*

We are now in a position to compute the subquotients $\mathcal{F}_N^y(1 \otimes \tau)$ of π_N given in (A.4). Here $1 = 1_{n-2}$ is the trivial character of GL_{n-2}. Recall that $B_2 = T_2N_2$ is the upper triangular Borel subgroup of GL_2. We work as explicitly as possible.

Case 1: $y = e$

If $y = e$, then $P \cap L \cong \mathrm{GL}_{n-2} \times B_2$ is a (maximal) parabolic subgroup of L with Levi subgroup $M \cap L = M$, and unipotent radical $N \cap L \cong F$. Notice also that since $M \subset L$ we have $Q \cap M = M$. It follows that

$$\mathcal{F}_N^e(1 \otimes \tau) = (1_{n-2} \otimes \tau)_{N \cap L} = 1_{n-2} \otimes \tau_{N_2}$$

where 1_{n-2} is the trivial character on the GL_{n-2} block of $M = F^\times \times \mathrm{GL}_{n-2} \times F^\times$, and τ_{N_2} is the Jacquet module of τ along the Borel B_2 of GL_2. Explicitly,

$$\mathcal{F}_N^e(1 \otimes \tau) \begin{pmatrix} a & & \\ & A & \\ & & b \end{pmatrix} = 1_{n-2}(A) \otimes \tau_{N_2} \begin{pmatrix} a & \\ & b \end{pmatrix}. \tag{A.5}$$

If we take $\tau = \eta \otimes \mathrm{St}$ to be a twist of the Steinberg representation of GL_2, then it is well known that $(\eta \otimes \mathrm{St})_{N_2} = \delta_{B_2}^{1/2}(\eta \otimes \eta)$ and we have that

$$\mathcal{F}_N^e(1 \otimes \tau) = |\cdot|^{\frac{1}{2}}\eta \otimes 1_{n-2} \otimes |\cdot|^{-\frac{1}{2}}\eta \tag{A.6}$$

is a quasi-character of M. While if τ is supercuspidal, then $\mathcal{F}_N^e(1 \otimes \tau)$ vanishes since $\tau_{N_2} = 0$. We'll deal with the case that τ is an irreducible principal series below.

Case 2: $y = (n-1\ n-2\ \dots\ 3\ 2\ 1) = x_Q^{-1}$

If $y = x_Q^{-1}$, then we have that ${}^yQ = P_{(n-2,2)}$ and ${}^yL = M_{(n-2,2)}$. The parabolic $P \cap {}^yL$ of $M_{(n-2,2)}$ is equal to $P_{(1,n-3)} \times B_2$, with Levi subgroup consisting of block-diagonal matrices $M \cap {}^yL = \mathrm{GL}_1 \times \mathrm{GL}_{n-3} \times \mathrm{GL}_1 \times \mathrm{GL}_1$. The parabolic subgroup $M \cap {}^yQ$ of M is equal to $\mathrm{GL}_1 \times P_{(n-3,1)} \times \mathrm{GL}_1$. We have that ${}^y(1_{n-2} \otimes \tau)$ is a representation of ${}^yL$; restricting to $P \cap {}^yL$ observe that

$$ {}^y(1_{n-2} \otimes \tau) \begin{pmatrix} a & \vec{n} & 0 & 0 \\ 0 & X & 0 & 0 \\ 0 & 0 & b & * \\ 0 & 0 & 0 & c \end{pmatrix} = (1_{n-2} \otimes \tau) \begin{pmatrix} b & 0 & 0 & * \\ 0 & a & \vec{n} & 0 \\ 0 & 0 & X & 0 \\ 0 & 0 & 0 & c \end{pmatrix} = 1_{n-2} \begin{pmatrix} a & \vec{n} \\ 0 & X \end{pmatrix} \otimes \tau \begin{pmatrix} b & * \\ 0 & c \end{pmatrix}, $$

where $a, b, c \in F^\times, X \in \mathrm{GL}_{n-3}, \vec{n} \in F^{n-3}$ and $* \in F$. This observation lets us compute the Jacquet module ${}^y(1_{n-2} \otimes \tau)_{N \cap {}^yL}$. We see that

$$ \begin{aligned} {}^y(1_{n-2} \otimes \tau)_{N \cap {}^yL} \begin{pmatrix} a & & & \\ & X & & \\ & & b & \\ & & & c \end{pmatrix} &= (1_{n-2})_{N_{(1,n-3)}} \begin{pmatrix} a & \\ & X \end{pmatrix} \otimes \tau_{N_2} \begin{pmatrix} b & \\ & c \end{pmatrix} \\ &= \delta_{P_{(1,n-3)}}^{-1/2} \begin{pmatrix} a & \\ & X \end{pmatrix} \otimes \tau_{N_2} \begin{pmatrix} b & \\ & c \end{pmatrix} \\ &= |a|^{-\frac{(n-3)}{2}} |\det X|^{\frac{1}{2}} \otimes \tau_{N_2} \begin{pmatrix} b & \\ & c \end{pmatrix}. \end{aligned} $$

If we take $\tau = \eta \otimes \mathrm{St}$ to be a twist of the Steinberg representation, then we have

$$ {}^y(1_{n-2} \otimes \tau)_{N \cap {}^yL} \begin{pmatrix} a & & & \\ & X & & \\ & & b & \\ & & & c \end{pmatrix} = |a|^{-\frac{(n-3)}{2}} |\det X|^{\frac{1}{2}} |b|^{\frac{1}{2}} \eta(b) |c|^{-\frac{1}{2}} \eta(c). $$

It follows that

$$ \begin{aligned} \mathcal{F}_N^y(1 \otimes \tau) &= \iota_{M \cap {}^yQ}^M \, {}^y(1_{n-2} \otimes \tau)_{N \cap {}^yL} \\ &= \iota_{\mathrm{GL}_1 \times P_{(n-3,1)} \times \mathrm{GL}_1}^M \left(|\cdot|^{-\frac{(n-3)}{2}} \otimes |\det(\cdot)|^{\frac{1}{2}} \otimes \eta |\cdot|^{\frac{1}{2}} \otimes \eta |\cdot|^{-\frac{1}{2}} \right) \\ &= |\cdot|^{-\frac{(n-3)}{2}} \otimes \iota_{P_{(n-3,1)}}^{\mathrm{GL}_{n-2}} \left(|\det(\cdot)|^{\frac{1}{2}} \otimes \eta |\cdot|^{\frac{1}{2}} \right) \otimes \eta |\cdot|^{-\frac{1}{2}}. \end{aligned} $$

Again, if τ is supercuspidal, then τ_{N_2} is the zero T_2-module and ${}^y(1_{n-2} \otimes \tau)_{N \cap {}^yL} = 0$; moreover, $\mathcal{F}_N^y(1 \otimes \tau) = 0$, in this case.

Case 3: $y = (n-1\ n)$

Suppose that $y = (n-1\ n)$. First we describe ${}^yL$. Let

$$l = \begin{pmatrix} a & & b \\ & x_{i,j} & \\ c & & d \end{pmatrix}, \qquad\qquad 1 \leq i, j \leq n-2$$

be an arbitrary element of L. Then we have

$$ {}^yl = yly^{-1} = \begin{pmatrix} a & 0 & \dots & 0 & b & 0 \\ 0 & x_{1,1} & \dots & x_{1,n-3} & 0 & x_{1,n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & x_{n-3,1} & \dots & x_{n-3,n-3} & 0 & x_{n-3,n-2} \\ c & 0 & \dots & 0 & d & 0 \\ 0 & x_{n-2,1} & \dots & x_{n-2,n-3} & 0 & x_{n-2,n-2} \end{pmatrix}.$$

We now observe that $M \cap {}^yL = M_{(1,n-3,1,1)} \cong \mathrm{GL}_1 \times \mathrm{GL}_{n-3} \times \mathrm{GL}_1 \times \mathrm{GL}_1$ consists of block-diagonal matrices, and that the parabolic $P \cap {}^yL$ is isomorphic to $P_{(n-3,1)} \times B_2$, with unipotent radical

$$N \cap {}^yL = \left\{ \begin{pmatrix} 1 & 0 & \dots & 0 & * & 0 \\ 0 & 1 & 0 & 0 & 0 & * \\ \vdots & 0 & \ddots & 0 & \vdots & \vdots \\ \vdots & \vdots & 0 & 1 & 0 & * \\ \vdots & 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix} \right\}.$$

On the other hand, ${}^yQ \cap M = ({}^yL \cap M)({}^yU \cap M)$, so we just need to understand ${}^yU$. One readily observes that

$${}^yU = \left\{ \begin{pmatrix} 1 & 0 & \dots & \dots & 0 & 0 \\ * & 1 & 0 & 0 & * & \vdots \\ \vdots & 0 & \ddots & 0 & \vdots & \vdots \\ * & \vdots & 0 & 1 & * & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \\ * & 0 & \dots & 0 & * & 1 \end{pmatrix} \right\};$$

therefore, $M \cap {}^yQ = \mathrm{GL}_1 \times P_{(n-3,1)} \times \mathrm{GL}_1$ is a parabolic subgroup of M. One computes that the $M \cap {}^yL$-representation ${}^y(1_{n-2} \otimes \tau)_{N \cap {}^yL}$ is given by

$$\begin{aligned} {}^y(1_{n-2} \otimes \tau)_{N \cap {}^yL} \begin{pmatrix} a & & & \\ & X & & \\ & & b & \\ & & & c \end{pmatrix} &= (1_{n-2})_{N_{(n-3,1)}} \begin{pmatrix} X & \\ & c \end{pmatrix} \otimes \tau_{N_2} \begin{pmatrix} a & \\ & b \end{pmatrix} \\ &= \delta_{P_{(n-3,1)}}^{-1/2} \begin{pmatrix} X & \\ & c \end{pmatrix} \otimes \tau_{N_2} \begin{pmatrix} a & \\ & b \end{pmatrix}. \end{aligned}$$

If we take $\tau = \eta \otimes \mathrm{St}$ to be a twist of the Steinberg representation, then we have

$${}^y(1_{n-2} \otimes \tau)_{N \cap {}^yL} \begin{pmatrix} a & & & \\ & X & & \\ & & b & \\ & & & c \end{pmatrix} = |a|^{\frac{1}{2}} \eta(a)\ |\det(X)|^{-\frac{1}{2}}\ |b|^{-\frac{1}{2}} \eta(b)\ |c|^{\frac{n-3}{2}}.$$

It follows that, for $\tau = \eta \otimes \mathrm{St}$ we have

$$\begin{aligned} \mathcal{F}_N^y(1 \otimes \tau) &= \iota_{\mathrm{GL}_1 \times P_{(n-3,1)} \times \mathrm{GL}_1}^M \left(|\cdot|^{\frac{1}{2}} \eta \otimes |\det(\cdot)|^{-\frac{1}{2}} \otimes |\cdot|^{-\frac{1}{2}} \eta \otimes |\cdot|^{\frac{n-3}{2}} \right) \\ &= |\cdot|^{\frac{1}{2}} \eta \otimes \iota_{P_{(n-3,1)}}^{\mathrm{GL}_{n-2}} \left(|\det(\cdot)|^{-\frac{1}{2}} \otimes |\cdot|^{-\frac{1}{2}} \eta \right) \otimes |\cdot|^{\frac{n-3}{2}}. \end{aligned}$$

As above, if τ is supercuspidal, then $\mathcal{F}_N^y(1 \otimes \tau) = 0$, since $\tau_{N_2} = 0$ and ${}^y(1_{n-2} \otimes \tau)_{N \cap {}^yL} = 0$.

Case 4: $y = (n-2\ n-3\ \ldots\ 3\ 2\ 1)(n-1\ n)$

First we note that since $L = x_Q M_{(n-2,2)} x_Q^{-1}$ we have that ${}^yL = {}^{yx_Q}M_{(n-2,2)}$; moreover, yx_Q is the cycle $(n-2\ n\ n-1)$. It follows that ${}^yL$ is the subgroup of matrices of the form

$$\begin{pmatrix} x_{1,1} & \cdots & x_{1,n-3} & 0 & 0 & x_{1,n-2} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ x_{n-3,1} & \cdots & x_{n-3,n-3} & 0 & 0 & x_{n-3,n-2} \\ 0 & \cdots & 0 & a & b & 0 \\ 0 & \cdots & 0 & c & d & 0 \\ x_{n-2,1} & \cdots & x_{n-1,n-2} & 0 & 0 & x_{n-2,n-2} \end{pmatrix},$$

where $(x_{i,j}) \in \mathrm{GL}_{n-2}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2$. Moreover, we have that ${}^yQ = (yx_Q)P_{(n-2,2)}(yx_Q)^{-1}$ and we observe that ${}^yQ$ consists of matrices of the form

$$\left\{ \begin{pmatrix} x_{1,1} & \dots & x_{1,n-3} & u_1 & v_1 & x_{1,n-2} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ x_{n-3,1} & \dots & x_{n-3,n-3} & u_{n-3} & v_{n-3} & x_{n-3,n-2} \\ 0 & \dots & 0 & a & b & 0 \\ 0 & \dots & 0 & c & d & 0 \\ x_{n-2,1} & \dots & x_{n-1,n-2} & u_{n-2} & v_{n-2} & x_{n-2,n-2} \end{pmatrix} \right\},$$

where $(x_{i,j}) \in \mathrm{GL}_{n-2}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2$ and $u_i, v_i \in F$, $1 \le i \le n-2$. It follows that $M \cap {}^yL = M_{(1,n-4,2,1)}$ and the parabolic subgroup $P \cap {}^yL$ is isomorphic to the subgroup $P_{(1,n-4,1)} \times \mathrm{GL}_2$ of $\mathrm{GL}_{n-2} \times \mathrm{GL}_2$. We also see that $M \cap {}^yQ$ is equal to the parabolic subgroup $\mathrm{GL}_1 \times P_{(n-4,2)} \times \mathrm{GL}_1$ of M. Let $m = \mathrm{diag}(a, X, A, b) \in M \cap {}^yL$ where $a, b \in F^\times$, $X \in \mathrm{GL}_{n-4}$ and $A \in \mathrm{GL}_2$. We see that

$$\begin{aligned} {}^y(1_{n-2} \otimes \tau)_{N \cap {}^yL} \,\mathrm{diag}(a, X, A, b) &= (1_{n-2})_{N_{(1,n-4,1)}}(\mathrm{diag}(a, X, b)) \otimes \tau(A) \\ &= \delta_{P_{(1,n-4,1)}}^{-1/2}(\mathrm{diag}(a, X, b)) \otimes \tau(A) \\ &= |a|^{-\frac{(n-3)}{2}} |b|^{\frac{(n-3)}{2}} \otimes \tau(A) \end{aligned}$$

We can write

$${}^y(1_{n-2} \otimes \tau)_{N \cap {}^yL} = |\cdot|^{-\frac{(n-3)}{2}} \otimes 1_{n-4} \otimes \tau \otimes |\cdot|^{\frac{(n-3)}{2}}$$

as a representation of $M \cap {}^yL = M_{(1,n-4,2,1)}$. It follows that

$$\begin{aligned} \mathscr{F}_N^y(1 \otimes \tau) &= \iota_{\mathrm{GL}_1 \times P_{(n-4,2)} \times \mathrm{GL}_1}^M \left(|\cdot|^{-\frac{(n-3)}{2}} \otimes 1_{n-4} \otimes \tau \otimes |\cdot|^{\frac{n-3}{2}} \right) \\ &= |\cdot|^{-\frac{(n-3)}{2}} \otimes \iota_{P_{(n-4,2)}}^{\mathrm{GL}_{n-2}} (1_{n-4} \otimes \tau) \otimes |\cdot|^{\frac{n-3}{2}}. \end{aligned}$$

A.4.4 Distinguished subquotients of Jacquet modules

By Proposition 3.5.1, to apply the Relative Casselman's Criterion 2.2.18, it suffices to consider the exponents coming from M^θ-distinguished (irreducible) subquotients of π_N. Here we focus on the case that $\tau = \eta \otimes \mathrm{St}$ is a self-contragredient twist of the Steinberg representation of GL_2 (with trivial central character) and the case that τ is an irreducible principal series as in Theorem A.1.3. We need only consider the possibility that τ is supercuspidal in *Case 4*.

Case 1: $y = e$

If τ is either a twist of the Steinberg or an irreducible principal series, then the representation $\mathscr{F}_N^e(1 \otimes \tau)$ of M is M^θ-distinguished since M^θ acts trivially via $\mathscr{F}_N^e(1 \otimes \tau)$. Indeed,

$$\mathscr{F}_N^e(1 \otimes \tau)\begin{pmatrix} a & & \\ & A & \\ & & a \end{pmatrix} = 1_{n-2}(A) \otimes \tau_{N_2}\begin{pmatrix} a & \\ & a \end{pmatrix}$$

and $\tau_{N_2}(\mathrm{diag}(a,a))$ is trivial since τ was assumed to have trivial central character. In the case that τ is an irreducible principal series, both of the irreducible subquotients of $\mathscr{F}_N^e(1 \otimes \tau)$ are M^θ-distinguished; moreover, both irreducible subquotients have trivial central character.

Case 2: $y = (n-1\ n-2\ \ldots\ 3\ 2\ 1) = x_Q^{-1}$

If $\tau = \eta \otimes \mathrm{St}$ is a twist of the Steinberg representation, then $\mathscr{F}_N^y(1_{n-2} \otimes \tau)$ is not M^θ-distinguished. Indeed, the central character χ_N^y of $\mathscr{F}_N^y(1_{n-2} \otimes \tau)$ is non-trivial on A_M^θ which is impossible if $\mathscr{F}_N^y(1_{n-2} \otimes \tau)$ is distinguished (*cf.* Lemma 2.2.2). Let $\mathrm{diag}(a,b,\ldots,b,a) \in A_M^\theta$, then since $\delta_{P_{(n-3,1)}}$ is trivial on $\mathrm{diag}(b,\ldots,b) \in \mathrm{GL}_{n-2}$ we have

$$\begin{aligned}
\chi_N^y(\mathrm{diag}(a,b,\ldots,b,a) &= |a|^{-\frac{(n-3)}{2}} |\det(\mathrm{diag}(b,\ldots,b))|^{\frac{1}{2}} \eta(b)|b|^{\frac{1}{2}} \eta(a)|a|^{-\frac{1}{2}} \\
&= |a|^{-\frac{(n-3)}{2}} |a|^{-\frac{1}{2}} |b|^{\frac{(n-3)}{2}} |b|^{\frac{1}{2}} \eta(b)\eta(a) \\
&= |a|^{-\frac{(n-2)}{2}} |b|^{\frac{(n-2)}{2}} \eta(a)\eta(b),
\end{aligned}$$

which is non-trivial for some $a, b \in F^\times$, since η is unitary.

If $\tau = \chi_1 \times \chi_1^{-1}$ is an irreducible principal series, by Lemma 2.1.26 the central characters of the irreducible subquotients of $\mathscr{F}_N^y(1_{n-2} \otimes \tau)$ are the restrictions to A_M of the irreducible subquotients of ${}^y(1_{n-2} \otimes \tau)_{N \cap {}^yL}$. Let $z = (a,b,\ldots,b,a) \in A_M^\theta$, then the action of z via ${}^y(1_{n-2} \otimes \tau)_{N \cap {}^yL}$ is given by

$$z \mapsto |a|^{-\frac{(n-3)}{2}} |\det(bI_{n-3})|^{1/2} \tau_{N_2}\left(\mathrm{diag}(b,a)\right);$$

therefore, z acts on an irreducible subquotient of ${}^y(1_{n-2} \otimes \tau)_{N \cap {}^yL}$ via a character of the form

$$z \mapsto |a|^{-\frac{(n-3)}{2}} |b|^{\frac{(n-3)}{2}} \delta_{B_2}^{1/2}\left(\mathrm{diag}(b,a)\right) \mu(b)\mu^{-1}(a) = |a|^{-\frac{(n-2)}{2}} |b|^{\frac{(n-2)}{2}} \mu(b)\mu^{-1}(a),$$

where $\mu = \chi_1^{\pm 1}$. In particular, since $\chi_1 \neq \nu^{\pm\frac{(n-2)}{2}}$, the central characters of the irreducible subquotients of $\mathscr{F}_N^y(1_{n-2} \otimes \tau)$ are non-trivial on A_M^θ. In particular, the irreducible subquotients of $\mathscr{F}_N^y(1_{n-2} \otimes \tau)$ cannot be M^θ-distinguished by Lemma 2.2.2.

Case 3: $y = (n-1\ n)$

We show that, if $\tau = \eta \otimes \mathrm{St}$ is a twist of the Steinberg representation, then $\mathscr{F}_N^y(1_{n-2} \otimes \eta \otimes \mathrm{St})$ is not M^θ-distinguished, again by applying Lemma 2.2.2. Indeed, the central character χ_N^y of $\mathscr{F}_N^y(1_{n-2} \otimes \eta \otimes \mathrm{St})$ is non-trivial on A_M^θ. Let $\mathrm{diag}(a,b,\ldots,b,a) \in A_M^\theta$, then since $\delta_{P_{(n-3,1)}}$ is trivial on $\mathrm{diag}(b,\ldots,b) \in \mathrm{GL}_{n-2}$

we have

$$\begin{aligned}\chi_N^y(\operatorname{diag}(a,b,\ldots,b,a) &= |a|^{\frac{1}{2}}\eta(a)|\det(\operatorname{diag}(b,\ldots,b))|^{-\frac{1}{2}}\eta(b)|b|^{-\frac{1}{2}}|a|^{\frac{n-3}{2}} \\ &= |a|^{\frac{n-2}{2}}|b|^{-\frac{n-2}{2}}\eta(a)\eta(b)\end{aligned}$$

which is non-trivial for some $a, b \in F^\times$, since η is unitary.

If $\tau = \chi_1 \times \chi_1^{-1}$ is an irreducible principal series, by Lemma 2.1.26 the central characters of the irreducible subquotients of $\mathcal{F}_N^y(1_{n-2} \otimes \tau)$ are the restrictions to A_M of the irreducible subquotients of ${}^y(1_{n-2} \otimes \tau)_{N \cap {}^yL}$. Let $z = (a, b, \ldots, b, a) \in A_M^\theta$, then the action of z via ${}^y(1_{n-2} \otimes \tau)_{N \cap {}^yL}$ is given by

$$z \mapsto |\det(bI_{n-3})|^{-1/2}|a|^{\frac{(n-3)}{2}}\tau_{N_2}(\operatorname{diag}(a,b))\,;$$

therefore, z acts on an irreducible subquotient of ${}^y(1_{n-2} \otimes \tau)_{N \cap {}^yL}$ via a character of the form

$$z \mapsto |a|^{\frac{(n-3)}{2}}|b|^{-\frac{(n-3)}{2}}\delta_{B_2}^{1/2}(\operatorname{diag}(a,b))\,\mu(a)\mu^{-1}(b) = |a|^{\frac{(n-2)}{2}}|b|^{-\frac{(n-2)}{2}}\mu(a)\mu^{-1}(b),$$

where $\mu = \chi_1^{\pm 1}$. In particular, since $\chi_1 \neq \nu^{\pm\frac{(n-2)}{2}}$ the central characters of the irreducible subquotients of $\mathcal{F}_N^y(1_{n-2} \otimes \tau)$ are non-trivial on A_M^θ. It follows from Lemma 2.2.2 that, the irreducible subquotients of $\mathcal{F}_N^y(1_{n-2} \otimes \tau)$ cannot be M^θ-distinguished.

Case 4: $y = (n-2\ n-3\ \ldots\ 3\ 2\ 1)(n-1\ n)$

We show that $\mathcal{F}_N^y(1_{n-2} \otimes \tau)$ is not M^θ-distinguished. In this case, the central character of $\mathcal{F}_N^y(1_{n-2} \otimes \tau)$ is trivial on $M^\theta \cap A_M$, so we need a more sophisticated argument than above. Let (σ, W) be the representation $\iota_{P_{(n-4,2)}}^{\mathrm{GL}_{n-2}}(1_{n-4} \otimes \tau)$ of GL_{n-2}. From above,

$$\mathcal{F}_N^y(1_{n-2} \otimes \tau) = |\cdot|^{-\frac{(n-3)}{2}} \otimes \sigma \otimes |\cdot|^{\frac{n-3}{2}}.$$

Assume that $\mathcal{F}_N^y(1_{n-2} \otimes \tau)$ is M^θ-distinguished; we'll reach a contradiction. We identify the space of $\mathcal{F}_N^y(1_{n-2} \otimes \tau)$ with the space W of σ. Let $\mu \in W^*$ be a nonzero M^θ-invariant linear form on $\mathcal{F}_N^y(1_{n-2} \otimes \tau)$. If we restrict to the subgroup GL_{n-2} of M^θ, we have that (σ, W) is GL_{n-2}-distinguished. In particular, $\mu \in \operatorname{Hom}_{\mathrm{GL}_{n-2}}(\sigma, 1)$ is nonzero; however, this is a contradiction to Lemma A.4.4. Since (σ, W) cannot be GL_{n-2}-distinguished, it follows that $\mathcal{F}_N^y(1_{n-2} \otimes \tau)$ is not M^θ-distinguished.

Lemma A.4.4. *Let P be a proper parabolic subgroup of a reductive p-adic group G, with Levi subgroup M and unipotent radical N. Let (σ, W) be an irreducible admissible representation of M with trivial central character. Then the parabolically induced representation $\iota_P^G\sigma$ is not G-distinguished. That is, there are no nonzero G-invariant linear forms on the space V of $\iota_P^G\sigma$.*

Proof. Argue by contradiction and suppose that $\lambda \in \operatorname{Hom}_G(\iota_P^G\sigma, 1_G)$. By Bernstein's Second Adjoint Theorem (see [62], for instance), the functor $(\cdot)_{N^{op}}$ of Jacquet restriction along P^{op} is right adjoint to ι_P^G, again here P^{op} is the opposite parabolic to P. In particular,

$$\operatorname{Hom}_G(\iota_P^G\sigma, 1_G) \cong \operatorname{Hom}_M(\sigma, (1_G)_{N^{op}});$$

note that $(1_G)_{N^{op}} = \delta_{P^{op}}^{-1/2} = \delta_P^{1/2}$. Since λ is nonzero, the above isomorphism gives us a nonzero element

$$\lambda' \in \mathrm{Hom}_M(\sigma, \delta_P^{1/2}).$$

Let $w \in W$ such that $\langle \lambda', w\rangle \neq 0$. Then for any $z \in A_M$, we have

$$\delta_P^{1/2}(z)\langle \lambda', w\rangle = \langle \lambda', \sigma(z)w\rangle = \langle \lambda', w\rangle,$$

where the last equality holds since σ has trivial central character. It follows that

$$\left(\delta_P^{1/2}(z) - 1\right)\langle \lambda', w\rangle = 0$$

and since $\langle \lambda', w\rangle \neq 0$, we have $\delta_P^{1/2}(z) = 1$, for all $z \in A_M$; however, this is a contradiction, since δ_P is non-trivial on A_M when P is a proper parabolic subgroup. We conclude that $\lambda = 0$ and thus $\mathrm{Hom}_G(\iota_P^G\sigma, 1_G) = \{0\}$, completing the proof. □

To summarize, in *Case 4*, the irreducible subquotient $\mathscr{F}_N^y(1_{n-1} \otimes \tau)$ of π_N is not M^θ-distinguished for any τ as in Corollary A.2.2(2).

A.4.5 The discrete spectrum of $\mathbf{GL}_{n-1}(F) \times \mathbf{GL}_1(F)\backslash\mathbf{GL}_n(F)$

The exponents of π along P are the central A_M-quasi-characters of $\mathscr{F}_N^y(1 \otimes \tau)$. As noted above, we need only consider the exponents contributed by the M^θ-distinguished subquotients of π_N.

Theorem A.4.5. *Let τ be an irreducible admissible infinite dimensional representation of GL_2 with trivial central character. Let π be the irreducible H-distinguished representation $\pi = \iota_Q^G(1_{n-2} \otimes \tau)$ of G parabolically induced from $Q = LU$.*

1. *if τ is supercuspidal, then π is relatively supercuspidal,*
2. *if $\tau = \eta \otimes \mathrm{St}$ is a twist of St by a square-trivial character η of $F^\times$, then π lies in the discrete series of $H\backslash G$,*
3. *if $\tau = \chi_1 \times \chi_1^{-1}$ is an irreducible principal series ($\chi_1 \neq \nu^{\pm 1/2}$, and $\chi_1 \neq \nu^{\pm \frac{n-1}{2}}$ as in Theorem A.1.3), and if $r_P\lambda^G \neq 0$, then π does not lie in the discrete spectrum of $H\backslash G$.*

Proof. Recall that P is the only proper θ-split parabolic subgroup of G up to H-conjugacy. By Lemma 3.2.3, it is sufficient to consider the exponents of π along P in applying the Relative Casselman's Criterion 2.1.28. Moreover, in this situation, we need only consider Jacquet modules of π along P when applying Theorem 2.2.16 (*cf.* [44, Remark 6.10]).

1. If τ is supercuspidal, then we saw above that either $\mathscr{F}_N^y(\tau)$ vanishes or is not M^θ-distinguished. It follows that $r_P\lambda^G$ vanishes for any $\lambda^G \in \mathrm{Hom}_H(\pi, 1)$. We conclude that π is relatively supercuspidal.
2. Now, assume that $\tau = \eta \otimes \mathrm{St}$ is a twist of the Steinberg representation St of GL_2. By Proposition 3.5.1, it is enough to consider the exponents of π_N corresponding to M^θ-distinguished irreducible

subquotients of π_N. We have that $S_M = S_0$; moreover

$$S_0^1 = S_0(\mathcal{O}_F) = \{\mathrm{diag}(u, 1, \dots, 1, u^{-1}) : u \in \mathcal{O}_F^\times\},$$
$$S_0^- = \{\mathrm{diag}(a, 1, \dots, 1, a^{-1}) : a \in F^\times, |a|_F \leq 1\},$$

and $S_G = \{e\}$. When $\tau = \eta \otimes \mathrm{St}$, the character $\chi_e = \mathcal{F}_N^e(1 \otimes \tau)$ is an exponent of π along P and is the only M^θ-distinguished (irreducible) subquotient of π_N. Observe that

$$|\chi_e(\mathrm{diag}(a, 1, \dots, 1, a^{-1})| = |a|^{\frac{1}{2}}|\eta(a)||a^{-1}|^{-\frac{1}{2}}|\eta(a^{-1})| = |a| < 1, \tag{A.7}$$

for all $s = \mathrm{diag}(a, 1, \dots, 1, a^{-1}) \in S_0^- \setminus S_G S_0^1$. Therefore, by Theorem 2.2.18, we have that π is a RDS for $H\backslash G$.

3. Finally, if $\tau = \chi_1 \times \chi_1^{-1}$ is an irreducible principal series, we saw above that $\mathcal{F}_N^e(1 \otimes \tau)$ is the only M^θ-distinguished subquotient of π_N. By Proposition 3.5.1, it is enough to consider the exponents of π_N corresponding to the irreducible subquotients of $\mathcal{F}_N^e(1 \otimes \tau)$. The irreducible subquotients of $\mathcal{F}_N^e(1 \otimes \tau)$ are both M^θ-distinguished and the central characters are trivial. Since the central characters of the only two irreducible M^θ-distinguished subquotients of π_N are trivial, the generalized eigenspace $(V_N)_{1,\infty}$ must support $r_P\lambda^G \neq 0$ (*cf.* Proposition 3.5.1). In addition, $|1(s)| = 1$ for all $s \in S_0^- \setminus S_G S_0^1$ and $1 \in \mathcal{E}xp_{S_M}(\mathcal{F}_N^e(1 \otimes \tau)) \subset \mathcal{E}xp_{S_M}(\pi_N, r_P\lambda^G)$. In particular, π fails the condition (2.14) and π is not a (H, λ^G)-relatively square integrable by Theorem 2.2.18.

□

Remark A.4.6. 1. Theorem A.4.5(1) appears as [44, Proposition 8.2.3].

2. For $n = 3$ and $\tau = \mathrm{St}$, Theorem A.4.5(2) appears in [45, §5.1].

3. In light of Corollary A.2.5, we fall short of giving a complete classification of RDS representations for $H\backslash G$ only by the additional assumption on $r_P\lambda^G$ in Theorem A.4.5(3). We do not expect π to be RDS when τ is a principal series representation. However, at the moment, we do not have sufficient information about the M^θ-invariant linear forms $r_P\lambda^G$ to give a proof that $r_P\lambda^G \neq 0$.

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