Age and weak indivisibility

N. W. Sauer[1]
University of Calgary
Department of Mathematics and Statistics
and
Technische Hochschule Wien, Diskrete Mathematik
nsauer@math.ucalgary.ca

Abstract

Let H *be a countable infinite homogeneous relational structure. If* H *is indivisible then it is weakly indivisible and if it is weakly indivisible then it is age indivisible. There are examples of countable infinite homogeneous structures which are weakly indivisible but not indivisible. It is a natural question to ask for examples of countable age indivisible homogenous structures which are not weakly indivisible. I am not aware of any such. Could it possibly be true that age indivisibility for countable homogeneous relational structures implies weak indivisibility?*
Keywords: Homogeneous structures, partition theory, age indivisible, weakly indivisible, relational structures.

1. Introduction

For undefined notions and not explicitly stated results such as the Fraïssé theorems on amalgamation classes and homogeneous structures see [1] and [5].

We name relational structures using capital roman letters and the same letter in slanted format to indicated the corresponding base set. Let R be a relational structure and $S \subseteq R$. Then $\mathrm{R}\!\downarrow\! S$ denotes the restriction of R to S, that is the substructure of R induced by S. A *copy* of the relational structure R in the relational structure B is an induced substructure of B isomorphic to R. The relational structure R *can be embedded into the structure* B if it has a copy in B. The age of R consists of all finite structures which can be embedded into R. A countable set of finite relational structures having the same signature is an *age* if it is closed under isomorphism and induced substructures and is updirected, that is for any two structures in the age there is one into which both can be embedded. For every countable age $\mathbf{A}$ there is a countable structure R whose age is $\mathbf{A}$, see [1].

A *local isomorphism* of A is an isomorphism of a finite induced substructure of A to an induced substructure of A. A relational structure H is homogeneous if every local isomorphism of H has an extension to an automorphism of H.

Let R be a relational structure, then R is:

1. *Age indivisible* if for every structure A in the age of R there exists a structure B in the age of R so that for every partition (P_0, P_1) of B there exists $i \in 2$ and a copy of A in $\mathrm{B}\!\downarrow\! P_i$.

2. *Weakly indivisible* if for every partition (P_0, P_1) of R for which there exists an element A in the age of R which can not be embedded into $\mathrm{R}\!\downarrow\! P_0$, the structure R can be embedded into $\mathrm{R}\!\downarrow\! P_1$.

3. *Indivisible* if for every partition (P_0, P_1) of R there exists $i \in 2$ and a copy of R in $\mathrm{R}\!\downarrow\! P_i$.

A relational structure R is age indivisible if and only if for every partition of R into finitely many parts $(R_0, R_1, \dots, R_{n-1})$ there is an $i \in n$ so that every element of the age of R can be embedded into $\mathrm{R}\!\downarrow\! R_i$, for a proof see [2]. For some earlier results on indivisible ages and indivisible relations by M. Pouzet, see [3] and [4].

Note that if a relational structure is indivisible then it is weakly indivisible and it follows from the assertion above that if a relational structure is weakly indivisible it is age indivisible. At the end of Subsection 3.1 is an example of

[1]Supported by NSERC of Canada Grant # 691325

a countable infinite weakly indivisible homogeneous structure which is not indivisible. It is not difficult to generalize and produce infinitely many such examples.

There are two large classes of prominent countable infinite homogeneous structures. One of them is the class of homogeneous structures whose age has free amalgamation, see Subsection 3.1. According to Theorem 5 all of those homogeneous structures are weakly indivisible. Then there is the class of countable homogeneous metric spaces. The generic ones over a given distance set are known to be age indivisible, see [6]. With L. Nguyen Van Thé we proved recently that the generic countable homogeneous metric spaces with distance set $\{0, 1, 2, \dots, n\}$ for $n \in \omega$ are indivisible and the generic countable homogeneous metric space with distance set the non negative integers is weakly indivisible. Probably the best candidate for an age indivisible countable metric space which is not weakly indivsible is the rational Urysohn sphere. That is the generic countable homogeneous metric space with distance set the rationals in the interval $[0, 1]$. We can only prove that it is "approximately weakly indivisible". This rational Urysohn sphere has probably the best chance of being an example of an age indivisible structure which is not weakly indivisible. Corollary 1 settles the case of countable ultrahomogeneous spaces: A denumerable homogeneous ultrametric space is age indivisible if and only if it is weakly indivisible.

2. Ultrahomogeneous metric spaces

We need the following notions and Theorem 1 and Theorem 2 from [6].

Let f be a function of a set S to ω. Then $\mathrm{supp}(f) := \{s \in S : f(s) \neq 0\}$. Let λ be a countable chain. A function $\mathbf{a} : \lambda \to \omega$ with $2 \leq \mathbf{a}(\mu) \leq \omega$ for all $\mu \in \lambda$ will be called a *degree function for* λ. Let $\mathbf{a}$ be a degree function for λ, then

$$\omega^{[\mathbf{a}]} := \{f : \lambda \to \omega \mid \forall \mu \in \lambda\, (f(\mu) < \mathbf{a}(\mu)) \text{ and } \mathrm{supp}(f) \text{ is finite}\}.$$

If $\mathbf{a}(\mu) = \omega$ for every $\mu \in \lambda$, then the set $\omega^{[\mathbf{a}]}$ is usually denoted by $\omega^{[\lambda]}$. For $f, g \in \omega^{[\mathbf{a}]}$ let

$$\Delta(f, g) := \begin{cases} \infty, & \text{if } f = g; \\ \mu, & \text{for } \mu \text{ the smallest element in } \lambda \text{ with } f(\mu) \neq g(\mu). \end{cases}$$

For M a metric space and $a \in M$ let $\mathrm{Spec}(\mathrm{M}, a) := \{\mathrm{d}(a, x) \mid x \in M\}$ and $\mathrm{Spec}(\mathrm{M}) := \{\mathrm{d}(x, y) \mid x, y \in M\}$ and $B_a(s) := \{x \in M \mid d(a, x) \leq s\}$ and $\mathrm{Nerv}(M) := \{B_a(s) : a \in M, s \in \mathrm{Spec}(M, a)\}$. An *ultrametric space* M is a metric space with $\mathrm{d}(x, y) \leq \max\{\mathrm{d}(x, z), \mathrm{d}(y, z)\}$ for all $x, y, z \in M$.

Theorem 1. *Let λ be a countable chain and $\mathbf{a}$ a degree function for λ and $w : \lambda \cup \{\infty\} \to \mathbb{R}_+$ a strictly decreasing map with $w(\infty) = 0$ and image V. Let $\mathrm{d}_w := w \circ \Delta$. Then $\mathrm{M} = (\omega^{[\mathbf{a}]}, \mathrm{d}_w)$ is an ultrahomogeneous space with $\mathrm{Spec}(\mathrm{M}) = V$.*

Let M be a countable ultrametric space. Then the following properties are equivalent:

a. M *is isometric to some* $(\omega^{[\mathbf{a}]}, \mathrm{d}_w)$.

b. M *is homogeneous.*

c. M *is point-homogeneous. (That is, the automorphism group* $\mathrm{Iso}(\mathrm{M})$ *of* M *acts transitively on* M*.)*

The space $(\omega^{[\lambda]}, \mathrm{d}_w)$ is the countable homogeneous ultrametric space $\mathbb{U}lt_V$ associated with V.

Theorem 2. *Let M be a denumerable ultrametric space. Then the following properties are equivalent:*

i. M *is isometric to some $\mathbb{U}lt_V$, where V is dually well-ordered;*

ii. M *is point-homogeneous, the tree $(Nerv(M), \supseteq)$ is well founded and the degree of every non maximal element of it is infinite;*

iii. M *is homogeneous and indivisible;*

Let M be an ultrahomogeneous space and $B \in \mathrm{Nerv}(\mathrm{M})$ of diameter r. Then the relation $\mathrm{d}(x, y) < r$ on B is an equivalence relation. The equivalence classes of this relation are the *sons* of B.

Lemma 1. *Let λ be a countable chain and $\nu \in \lambda$ and $\mathbf{a}$ a degree function for λ with $\mathbf{a}(\nu) = n \in \omega$ and $w : \lambda \cup \{\infty\} \to \mathbb{R}_+$ a strictly decreasing map with $w(\infty) = 0$ and image V. Let $\mathrm{d}_w := w \circ \Delta$ and let $\mathrm{M} = (\omega^{[\mathbf{a}]}, \mathrm{d}_w)$ be the corresponding ultrahomogeneous space with $\mathrm{Spec}(\mathrm{M}) = V$.*

Then M is age divisible. (That is not age indivisible.)

Proof. The metric space A with $|A| = n$ and $\mathrm{d}(x, y) = w(\nu)$ for all $x, y \in A$ with $x \neq y$ is an element in the age of M. Let R be an element in the age of M. The balls in R of diameter $w(\nu)$ are disjoint and each contains at most $n \geq 2$ sons. Let P be the subset of R containing the elements of exactly one son from every ball of R which contains n elements. Then A can not be embedded into $\mathrm{R}\!\downarrow P$ and it also can not be embedded into $\mathrm{R}\!\downarrow (R \setminus P)$. □

Lemma 2. *Let λ be a countable chain and $w : \lambda \cup \{\infty\} \to \mathbb{R}_+$ a strictly decreasing map with $w(\infty) = 0$ and image V. Let $\mathrm{d}_w := w \circ \Delta$.*

Then the corresponding ultrahomogeneous space $\mathrm{M} = (\omega^{[\lambda]}, \mathrm{d}_w)$ is weakly indivisible.

Proof. Let $r \in V$. Observe that the balls in M of diameter r are pairwise disjoint and and cover M. If B is a ball of diameter r with $x, y \in B$ and if $z \in M$ but $z \notin B$ then $\mathrm{d}(x, z) = \mathrm{d}(y, z) > r \geq \mathrm{d}(x, y)$. This implies that if for every ball B of diameter r there is an embedding e_B of B into B then the union of those embeddings e_B is an embedding of M into M. Every ball B of diameter r has infinitely many sons. Hence if S is a set of points in M which contains points of only finitely many sons of every ball B of diameter r then there exists an embedding f of M into M with $f(x) \notin S$ for every point x of M.

Let A be an element in the age of M and $P \subseteq M$ so that A can not be embedded into $\mathrm{M}\!\downarrow P$. We will show by induction on $|A|$ that then M can be embedded into $\mathrm{M}\!\downarrow (M \setminus P)$. The Lemma obviously holds in case $|A| = 1$. Let r be the smallest distance between two different elements of A and K a ball in A of diameter r. Note that $|K| > 1$ and that any two different elements in K have distance r. Let $p, q \in K$ with $q \neq p$.

Let $\mathcal{F}$ be the set of isometries of $A \setminus \{p\}$ into P. Then $\mathcal{F} \neq \emptyset$, otherwise M could be embedded into $\mathrm{M}\!\downarrow (M \setminus P)$ because of the induction assumption. Let $\mathcal{B}$ be the set of balls B of diameter r for which there is an $f \in \mathcal{F}$ with $f(q) \in B$. It follows that $f''(K \setminus \{p\}) \subseteq B$ contains points from $|K| - 1$ sons of B. If P contains points of more than $|K| - 1$ sons of B then there is an element $x \in P \setminus f''(K \setminus \{p\})$ so that the space A can be embedded into $\mathrm{M}\!\downarrow (P \cup \{x\})$. It follows that $P \cap B$ contains points from only finitely many sons of B. If $S = \{f(q) \mid f \in \mathcal{F}\}$ then there is no isometry of $A \setminus \{p\}$ into $P \setminus S$. Note that $S \cap B$ contains points from only finitely many sons of B for every ball B of M of diameter r. Using induction, there is an embedding g of M into $\mathrm{M}\!\downarrow ((M \setminus P) \cup S)$.

Let M^* be the image of g. Then S contains for every ball B of M^* of diameter r points from only finitely many sons of B and hence there exists an embedding h of M^* into M^* with $h(x) \notin S$ for all points x of M^*. It follows that the composition $h \circ g$ is an embedding of M into $\mathrm{M}\!\downarrow (M \setminus P)$.

□

Theorem 3. *Let M be a denumerable ultrametric space. Then the following properties are equivalent:*

i. *M is isometric to some $\mathbb{U}lt_V$.*

ii. *M is point-homogeneous and the degree of every non maximal element of the tree $(Nerv(M), \supseteq)$ is infinite.*

iii. *M is homogeneous and weakly indivisible.*

Proof. Follows from Theorem 1 and Lemma 2. □

Corollary 1. *A denumerable homogeneous ultrametric space is age indivisible if and only if it is weakly indivisible.*

Proof. Follows from Theorem 3 and Lemma 1. □

3. A sufficient condition

For F a finite subset of elements of a homogeneous structure H and $a \in H \setminus F$ we denote by $\mathrm{orbit}(F, a)$ the set of elements $b \in H$ for which there exists an automorphism g of H with $g(a) = b$ and $g(x) = x$ for all $x \in F$. The *orbits* of H are all subsets of H of the form $\mathrm{orbit}(F, a)$ with $F \subseteq H$ finite and $a \in H \setminus F$. Note that if every orbit of H induces a substructure isomorphic to H then H is indivisible. (Enumerate H into an ω-sequence and observe that for every partition of H into two parts $(P, H \setminus P)$ we can find step by step an embedding of H into $\mathrm{H}\!\downarrow P$ or if we are obstructed from doing so, then one of the orbits is a subset of $H \setminus P$.)

Theorem 4. *Let* H *be a countable homogeneous structure so that for every finite subset* $F \subseteq H$ *and* $a \in H \setminus F$ *and element* V *in the age of* H *and element* $v \in V$ *there exists a copy* K *of* H *with* $K \subseteq H \setminus F$ *so that every embedding* e *of* $\mathrm{V}\!\downarrow(V \setminus \{v\})$ *into* K *has an extension* e^* *to an embedding of* V *into* H *with* $e^*(v) \in \mathrm{orbit}(F, a)$.

Then H *is weakly indivisible.*

Proof. Let V be an element in the age of H and let $(P, H \setminus P)$ be a partition of H so that there is no embedding of V into $\mathrm{H}\!\downarrow(H \setminus P)$. We will prove by induction on $|V|$ that there is an embedding of H into P. If $|V| = 1$ the assertion obviously holds. Let $v \in V$ and $\mathrm{U} = \mathrm{V}\!\downarrow(V \setminus \{v\})$.

Let $h_0, h_1, h_2, h_3, \dots$ be an ω-enumeration of H. If there is a sequence $p_0, p_1, p_2, p_3, \dots$ of elements in P so that the map $f : H \to P$ with $f(h_i) = p_i$ for all $i \in \omega$ is an embedding, we are done. Otherwise there exists a finite sequence $p_0, p_1, p_2, p_3, \dots, p_{n-1}$ of elements in P so that the function f mapping h_i to p_i for $i \in n$ is a local isomorphism which can not be extended to a local isomorphism mapping h_n into P. Because H has the mapping extension property there exists an element $a \in H$ so that the function f can be extended to a local isomorphism mapping h_n to an element $a \in H$. Let $F := \{p_i \mid i \in n\}$ and note that $\mathrm{orbit}(F, a) \cap P = \emptyset$ because f can not be extended to a function mapping h_n to an element in P.

Because K is an isomorphic copy of H and $(K \cap P, K \setminus P)$ is a partition of K it follows from the induction assumption that there is an embedding e of $\mathrm{V}\!\downarrow(V \setminus \{v\})$ into $\mathrm{K}\!\downarrow(K \setminus P)$ which according to the assumption of the Theorem has an extension e^* to an embedding of V into H with $e^*(v) \in \mathrm{orbit}(F, a)$.

We arrived at a contradiction. As observed earlier $\mathrm{orbit}(F, a) \cap P = \emptyset$ and hence $e^*(v) \notin P$ and hence in $H \setminus P$ and e^* is an embedding of V into $\mathrm{H}\!\downarrow(H \setminus P)$.

□

3.1. Free amalgamation

Let A be a relational structure. The elements x and y in A are *adjacent* if $x \neq y$ and there exists an $n \in \omega$ and an n-are relation R in the signature of A and an n-tuple $(z_0, z_1, \dots, z_{n-1})$ with entries which contain x and y and are elements of A and with $R(z_0, z_1, \dots, z_n)$. The relational structure A is *complete* if any two different of its elements are adjacent. Let A and B be two structures with the same signature. The *free amalgam of* A *and* B is the structure D with $D = A \cup B$ and $\mathrm{D}\!\downarrow A = \mathrm{A}$ and $\mathrm{D}\!\downarrow B = \mathrm{B}$ and if $x \in A \setminus B$ and $y \in B \setminus A$ then x and y are not adjacent in D. An age has free amalgamation if any two of its elements have free amalgamation. Note that if a countable age has free amalgamation then it has amalgamation and hence there exists a unique countable homogeneous structure with that age. An age has *free amalgamation over the empty set* if any two disjoint elements of the age have free amalgamation. An age has *free vertex amalgamation* if any two of its elements whose intersection is a singleton have free amalgamation.

The age $\mathbf{A}$ has free amalgamation if and only if there exists a set $\mathbf{B}$ of finite complete structures in the signature σ of the elements of $\mathbf{A}$ so that $\mathbf{A}$ is the class of all finite structures in signature σ which do not embed any of the elements in $\mathbf{B}$, see [5].

Lemma 3. *Let* H *be a countable homogeneous structure whose age has free vertex amalgamation. Then the age of* H *has free emptyset amalgamation.*

Proof. Let A and B be two elements in the age of H with $A \cap B = \emptyset$. Then there exists a copy A^* of A in H and because H is infinite an element $a^* \in H \setminus A^*$. It follows that A has an extension to an element A' in the age of H with $A' = A \cup \{a\}$ for some element $a \notin A$ so that $\mathrm{A}'\!\downarrow A = \mathrm{A}$. Similarly there exists an extension of B to an element B' in the age of H with $B' = B \cup \{b\}$ for some element $b \notin B$ so that $\mathrm{B}'\!\downarrow B = \mathrm{B}$. Let D be the free amalgam of A' with B' identifying a with b. The restriction of D to $A \cup B$ is then a free amalgam of A with B over the empty set.

□

Lemma 4. *Let* H *be a countable homogeneous structure whose age has free amalgamation over the empty set. Then there exists for every finite subset* F *of* H *a copy* K *of* H *in* H *with* $K \subseteq H \setminus F$ *and so that* $\mathrm{H}\!\downarrow(F \cup K)$ *is the free amalgam of* $\mathrm{H}\!\downarrow F$ *and* K.

Proof. Let H$\downarrow F :=$ F and D be a free amalgam of F with a copy, say C of H. Then the age of D is a subset of the age of H and hence there is an embedding f of D into H. There is an automorphism g of H with $g \circ f(x) = x$ for every element $x \in F$. Let $K := (g \circ f)''(C)$.

□

Theorem 5. *Let* H *be a countable homogeneous structure with free vertex amalgamation. Then* H *is weakly indivisible.*

Proof. Let F be a finite subset of H and $a \in H \setminus F$ and V in the age of H and $v \in V$. Let K be a copy of H in H with $K \subseteq H \setminus F$ and so that H$\downarrow (F \cup K)$ is the free amalgam of H$\downarrow F$ and K; which exists according to Lemma 3 and Lemma 4. Let $a \in H \setminus F$ and G := H$\downarrow (F \cup \{a\})$ and W a copy of V$\downarrow (V \setminus \{v\})$ in K with e an embedding V$\downarrow (V \setminus \{v\})$ with image W.

Let D be the free amalgam of G with V identifying a with v and let e' be the embedding of D$\downarrow (F \cup V \setminus \{v\})$ into H with $e'(x) = x$ for all $x \in F$ and $e'(x) = e(x)$ for all $x \in V \setminus \{v\}$. Then e' has an extension to an embedding e^* of D into H with $e^*(v) \in \text{orbit}(F, a)$. □

By graph, we mean simple graph, that is no loops or multiple edges.

Corollary 2. *Every countable infinite homogeneous graph* G *is weakly indivisible.*

Proof. Note that a graph is weakly indivisible if and only if its complement is weakly indivisible and that a graph is age indivisible if and only if the age of its complement is weakly indivisible. The K$_n$-free homogeneous graphs have free amalgamation and hence are weakly indivisible, they are actually indivisible according to [7]. The Rado graph has free amalgamation and it is easy to see that it is indivisible. (Every orbit of the Rado graph induces a subgraph isomorphic to the Rado graph.) Otherwise G is a union of complete graphs and hence weakly indivisible. The Corollary follows from the Lachlan and Woodrow classification of countable homogeneous graphs, see [8]. □

Let H be a countable homogeneous structure with binary signature. The orbits of H form a partial order under embedding. If H is indivisible then this partial order is a chain, see [9]. Let **A** be the class of all finite structures of the form $(V; E_0, E_1)$ where $(V; E_0)$ is a triangle free graph with vertex set V and edge set E_0 and $(V; E_1)$ is a triangle free graph with vertex set V and edge set E_1 and $E_0 \cap E_1 = \emptyset$. Then **A** is a countable age with free amalgamation. It follows from Theorem 5 that the countable homogeneous graph $\mathrm{H}_{\mathbf{A}}$ whose age is **A** is weakly indivisible. Let $x \in H$. The orbit of $\mathrm{H}_{\mathbf{A}}$ induced by all vertices adjacent to x by an edge in E_0 is the Rado graph in edge set E_1. The orbit of $\mathrm{H}_{\mathbf{A}}$ induced by all vertices adjacent to x by an edge in E_1 is the Rado graph in edge set E_0. Hence the partial order of orbits under embedding is not a chain. It follows that $\mathrm{H}_{\mathbf{A}}$ is not indivisible.

3.2. Oriented graphs

We will use Lachlan's classification of the countable homogeneous tournaments, see [10] and the classification of Cherlin of the countable homogeneous directed graphs, see [11].

Let L be a dense countable subset of the unit circle so that no two of its elements are opposite on the circle and so that L contains the point $(1, 0)$. The oriented graph L with vertex set L has an arc from x to y if $x \neq y$ and the counterclockwise angle from x to y is less than π. The graph L is the *local order*. The random Tournament, the order structure of the rationals and the local order are the three countable infinite homogeneous tournaments. The random tournament and the order structure of the rationals are easily seen to be divisible and hence weakly indivisible. Let L_0 be the set of elements in L whose angle with $(1, 0)$ is smaller than π and $L_1 = L \setminus L_0$. The partition of L into L_0 and L_1 shows that L is not age indivisible.

Note that the tournaments are the complete structures amongst the oriented graphs. Let $\mathcal{T}$ be a set of finite tournaments and $\mathbf{A}_{\mathcal{T}}$ the set of finite oriented graphs which do not embed any of the tournaments in $\mathcal{T}$ and $\mathrm{H}_{\mathcal{T}}$ the countable homogeneous oriented graph whose age is $\mathbf{A}_{\mathcal{T}}$. Then $\mathrm{H}_{\mathcal{T}}$ has free amalgamation. Hence all the "Henson graphs", see [12], are weakly indivisible according to Theorem 4. The countable infinite oriented graph with no edges is weakly indivisible. The random oriented graph has free amalgamation and hence is weakly indivisible.

Let C be a dense countable subset of the unit circle so that no two of its elements have a counterclockwise angle of $2\pi/3$ and so that C contains the point $(1, 0)$. The oriented graph C with vertex set C has an arc from x to y if $x \neq y$ and the counterclockwise angle from x to y is less than $2\pi/3$. The graph C is the *circular order*. Let C_0 be the set of elements in C whose angle with $(1,0)$ is smaller than $2\pi/3$ and C_1 the set of elements in C whose angle with $(1,0)$ is in between $2\pi/3$ and $4\pi/3$ and $C_2 := C \setminus (C_0 \cup C_1)$. The partition of C into C_0 and C_1 and C_2 shows that L is not age indivisible.

Let $n \in \omega$ and $\mathbf{A}_n$ the age of finite oriented graphs with no n-element independent set and H_n the corresponding countably infinite homogeneous oriented graph whose age is $\mathbf{A}_n$. Then H_n is weakly indivisible on account of Theorem 4 as follows: Let F be a finite subset of H_n and $a \in H_n \setminus F$ and $\mathrm{V} \in \mathbf{A}_n$ and $v \in V$. Let K be the set of all elements in $H_n \setminus F$ so that there is an arc from x to y for every element $x \in F$ and $y \in K$. Then $\mathrm{H}_n \downarrow K$ is a copy of H_n in H_n. It is easily seen that K satisfies the conditions of Theorem 4.

The random partial order is indivisible because each of its orbits is an isomorphic copy of it, hence it is weakly indivisible.

There are several more such countable homogeneous oriented graphs and a of course many other special cases of countable homogeneous structures. I could not detect a likely candidate which is age indivisible but not weakly indivisible. The best open case is the rational Urysohn sphere as mentioned in the introduction. In general metric spaces may provide such examples as all of the generic ones are known to be age indivisible.

References

[1] R. Fraïssé, Theory of relations, Studies in Logic and the Foundations of Mathematics, 145, North-Holland Publishing Co., Amsterdam, 2000.

[2] M. El-Zahar and N.W. Sauer, Ramsey-type properties of relational structures, Discrete Mathematics **94** (1991) 1-10.

[3] M. Pouzet, Relation minimale, Zeitschrift für Math. Logik, **25**, fasc. 4 (1979), p. 315-344.

[4] M. Pouzet, Relation Impartible, Dissertationnes, **103**, (1981), p. 1-48.

[5] N.W. Sauer, Appendix in: Theory of Relations, by R. Fraïssé, Revised Edition, in: *Studies in Logic and the Foundations of Mathematics*, **145**, North Holland 2000 ISBN 0-444-50542-3 CV 0.

[6] C. Delhomm and C, Laflamme and M. Pouzet and Norbert Sauer, Indivisible Ultrametric Spaces, to appear: Topology and its Applications, special issue Workshop on the Urysohn space, May 21-24, 2006, Ben-Gurion University of the Negev, Beer Sheva, Israel.

[7] M. El-Zahar and N.W. Sauer, The Indivisibility of the Homogeneous K_n-free graphs. Journal of Combinatorial Theory, Series B, **47**, No. 2, October 1989.

[8] A.H. Lachlan and R.E. Woodrow, Countable Ultrahomogeneous Undirected Graphs, Trans. Amer. Math. Soc. **262** (1980), 51-94.

[9] N.Sauer, Canonical vertex partitions, Combinatorics Probability and Computing, **12** (2003), issue 5-6, pp 671-704.

[10] A.H. Lachlan, Countable Homogeneous Tournaments, Trans. Amer. Math. Soc. **284** (1984), no. 2, 431-461.

[11] G. Cherlin, The classification of countable homogeneous directed graphs and countable homogeneous n-tournaments, Mem. Amer. Math. Soc. **131** (1998), no. 621, xiv+161pp.

[12] C.W. Henson, Countable homogeneous relational structures and $\aleph_0$-categorical theories, J. Symbolic Logic, **37**, (1972), 494-500.