Sphere packings in 3-space

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Abstract

In this paper we survey results on packings of congruent spheres in 3-dimensional spaces of constant curvature. The topics discussed are as follows:
- Hadwiger numbers of convex bodies and kissing numbers of spheres;
- Touching numbers of convex bodies;
- Newton numbers of convex bodies;
- One-sided Hadwiger and kissing numbers;
- Contact graphs of finite packings and the combinatorial Kepler problem;
- Isoperimetric problems for Voronoi cells and the strong dodecahedral conjecture;
- The strong Kepler conjecture.

Each topic is discussed in details along with some open problems. Four topics from the above list are treated in spaces of dimension different from three as well.

1 Introduction

A family of (not necessarily infinitely many) non-overlapping congruent balls in 3-dimensional space of constant curvature is called a packing of congruent balls in the given 3-space which is either Euclidean ($\mathbb{E}^3$) or spherical ($\mathbb{S}^3$) or hyperbolic ($\mathbb{H}^3$). The goal of this paper is to survey several of the most recent results on 3-dimensional sphere packings. On the one hand, this area seems to be one of the most active research areas of Discrete Geometry on the other hand, it is one of the oldest research areas of mathematics ever studied. The topics discussed in separate sections of this paper are the following ones:
- Hadwiger numbers of convex bodies and kissing numbers of spheres;
- Touching numbers of convex bodies;

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- Newton numbers of convex bodies;
- One-sided Hadwiger and kissing numbers;
- Contact graphs of finite packings and the combinatorial Kepler problem;
- Isoperimetric problems for Voronoi cells and the strong dodecahedral conjecture;
- The strong Kepler conjecture.

Each section outlines the state of the art of relevant research along with some open research problems. Also, we feel important to mention that although in this paper emphases are on the 3-dimensional case, four topics from the above list are treated in all dimensions. Last but not least the paper intends to complement the very recent papers of Casselman [17] and of Pfender and Ziegler [49] on similar topics.

2 Hadwiger numbers of convex bodies and kissing numbers of spheres

Let $\mathbf{K}$ be a convex body (i.e. a compact convex set with nonempty interior) in d-dimensional Euclidean space $\mathbb{E}^d, d \geq 2$. Then the Hadwiger number $H(\mathbf{K})$ of $\mathbf{K}$ is the largest number of non-overlapping translates of $\mathbf{K}$ that can all touch $\mathbf{K}$. An elegant observation of Hadwiger [25] is the following.

Theorem 2.1 *For every d-dimensional convex body $\mathbf{K}$,*

$$H(\mathbf{K}) \leq 3^d - 1,$$

where equality holds if and only if $\mathbf{K}$ is an affine d-cube.

On the other hand, in another elegent paper Swinnerton-Dyer [54] proved the following lower bound for Hadwiger numbers of convex bodies.

Theorem 2.2 *For every d-dimensional $(d \geq 2)$convex body $\mathbf{K}$,*

$$d2 + d \leq H(\mathbf{K}).$$

Actually, finding a better lower bound for Hadwiger numbers of d-dimensional convex bodies is a highly challanging open problem for all $d \geq 4$. (It is not hard to see that the above theorem of Swinnerton-Dyer is sharp for dimensions 2 and 3.) The best lower bound known in dimensions $d \geq 4$ is due to Talata [56], who applying Dvoretzky's theorem on spherical sections of centrally symmetric convex bodies succeded to show the following inequality.

Theorem 2.3 *There exists an absolute constant $c > 0$ such that*

$$2^{cd} \leq H(\mathbf{K})$$

holds for every positive integer d and for every d-dimensional convex body $\mathbf{K}$.

Now, if we look at convex bodies different from a Euclidean ball in dimensions larger than 2, then our understanding of their Hadwiger numbers is very limited. Namely, we know the Hadwiger numbers of the following convex bodies different from a ball. The result for tetrahedra is due to Talata [57] and the rest was proved by Larman and Zong [36].

Theorem 2.4 *The Hadwiger numbers of tetrahedra, octahedra and rhombic dodecahedra are all equal to 18.*

In order to gain some more insight on Hadwiger numbers it is natural to pose the following question.

Problem 2.5 *For what integers k with $12 \leq k \leq 26$ does there exist a 3-dimensional convex body with Hadwiger number k? What is the Hadwiger number of a $d-$dimensional simplex (resp., crosspolytope) for $d \geq 4$?*

The second main problem in this section is fondly known as the kissing number problem. The kissing number τ_d is the maximum number of nonoverlapping d-dimensional balls of equal size that can touch a congruent one in $\mathbb{E}^d$. In three dimension this question was the subject of a famous discussion between Isaac Newton and David Gregory in 1964. So, it is not surprising that the literature on the kissing number problem is "huge". Perhaps the best source of information on this problem is the book [18] of Conway and Sloane. In what follows we give a short description of the present status of this problem.

$\tau_2 = 6$ is trivial. However, determining the value of τ_3 is not a trivial issue. Actually the first complete and correct proof of $\tau_3 = 12$ was given by Schütte and van der Waerden [53] in 1953. The subsequent (two pages) often cited proof of Leech [37], which is impressively short, contrary to the common belief does contain some gaps. It can be completed though, see for example, [42]. Further more recent proofs can be found in [14], [1] and in [47]. None of these are short proofs either and one may wonder whether there exists a proof of $\tau_3 = 12$ in THE BOOK at all. (For more information on this see the very visual paper [17].) Thus, we have the following theorem.

Theorem 2.6 $\tau_2 = 6$ *and* $\tau_3 = 12$.

The race for finding out the kissing numbers of Euclidean balls of dimension larger than 3 was always and is even today one of the most visible research projects of mathematics. Following the chronological ordering, here are the major inputs. Coxeter [19] conjectured and Böröczky [13] proved the following theorem, where $F_d(\alpha) = \frac{2^d U}{d!\omega_d}$ is the Schläfli function with U standing for the spherical volume of a regular spherical $(d-1)$-dimensional simplex of dihedral angle 2α and with ω_d denoting the surface volume of the d-dimensional unit ball.

Theorem 2.7 $\tau_d \leq \frac{2F_{d-1}(\beta)}{F_d(\beta)}$, *where* $\beta = \frac{1}{2}\operatorname{arcsec} d$.

It was another breakthrough when Delsarte's linear programming method (for details see for example [49]) was applied to the kissing number problem and so, when Kabatiansky and Levenshtein [35] succeded to improve the upper bound of the previous theorem for large d as follows. The lower bound mentioned below was found by Wyner [58] several years earlier.

Theorem 2.8 $2^{0.2075d(1+o(1))} \leq \tau_d \leq 2^{0.401d(1+o(1))}$.

As the gap between the lower and upper bounds is exponential it was a great surprise when Levenshtein [37] and Odlyzko and Sloane [48] independently found the following exact values for τ_d.

Theorem 2.9 $\tau_8 = 240$ *and* $\tau_{24} = 196560$.

In addition, Bannai and Sloane [2] were able to prove the following.

Theorem 2.10 *There is a unique way (up to isometry) of arranging* 240 *(resp.,* 196560*) nonoverlapping unit spheres in* 8*-dimensional (resp.,* 24*-dimensional) Euclidean space such that they touch another unit sphere.*

The latest surprise came when Musin [45], [46] extending Delsarte's method found the kissing number of 4-dimensional Euclidean balls. Thus, we have

Theorem 2.11 $\tau_4 = 24$.

In connection with Musin's result we believe in the following conjecture.

Conjecture 2.12 *There is a unique way (up to isometry) of arranging* 24 *nonoverlapping unit spheres in* 4*-dimensional Euclidean space such that they touch another unit sphere.*

3 Touching numbers of convex bodies

The touching number $t(\mathbf{K})$ of a convex body $\mathbf{K}$ in d-dimensional Euclidean space $\mathbb{E}^d$ is the largest possible number of mutually touching translates of $\mathbf{K}$ lying in $\mathbb{E}^d$. In a very elegant paper Danzer and Grünbaum [20] have shown the following fundamental inequality.

Theorem 3.1 *For an arbitrary convex body* $\mathbf{K}$ *of* $\mathbb{E}^d$

$$t(\mathbf{K}) \leq 2^d$$

with equality if and only if $\mathbf{K}$ *is an affine d-cube.*

It is natural to ask for a non-trivial lower bound for $t(\mathbf{K})$. Brass [15] as an application of Dvoretzky's well-known theorem gave a partial answer for the existence of such a lower bound.

Theorem 3.2 *For each* k *there exists a* $d(k)$ *such that for any convex body* $\mathbf{K}$ *of* $\mathbb{E}^d$ *with* $d \geq d(k)$

$$k \leq t(\mathbf{K}).$$

It is remarkable that the natural sounding conjecture of Petty [55] stated next is still open for all $d \geq 4$.

Conjecture 3.3 *For each convex body* $\mathbf{K}$ *of* $\mathbb{E}^d, d \geq 4$

$$d + 1 \leq t(\mathbf{K}).$$

A generalization of the concept of touching numbers was introduced by K. Bezdek, M. Naszódi and B. Visy [7] as follows. The mth touching number (or the mth Petty number) $t(m, \mathbf{K})$ of a convex body $\mathbf{K}$ of $\mathbb{E}^d$ is the largest cardinality of (possible overlapping) translates of $\mathbf{K}$ in $\mathbb{E}^d$ such that among any m translates always there are two touching ones. Note that $t(2, \mathbf{K}) = t(\mathbf{K})$. The following theorem proved by K. Bezdek, M. Naszódi and B. Visy [7] states some upper bounds for $t(m, \mathbf{K})$.

Theorem 3.4 *Let* $t(\mathbf{K})$ *be an arbitrary convex body in* $\mathbb{E}^d$*. Then*

$$t(m, \mathbf{K}) \leq \min\left\{(m-1)4^d, \binom{2^d + m - 1}{2^d}\right\}$$

holds for all $m \geq 2, d \geq 2$*. Also, we have the inequalities*

$$t(3, \mathbf{K}) \leq 2 \cdot 3^d,\ \ t(m, \mathbf{K}) \leq (m-1)[(m-1)3^d - (m-2)]$$

for all $m \geq 4, d \geq 2$. Moreover, if $\mathbf{B^d}$ (resp., $\mathbf{C^d}$) denotes a d-dimensional ball (resp., d-dimensional affine cube) of $\mathbb{E}^d$, then

$$t(2, \mathbf{B^d}) = d + 1,\ t(m, \mathbf{B^d}) \leq (m-1)3^d,\ t(m, \mathbf{C^d}) = (m-1)2^d$$

hold for all $m \geq 2, d \geq 2$.

We cannot resist on raising the following question (for more details see [7]).

Problem 3.5 *Prove or disprove that if $\mathbf{K}$ is an arbitrary convex body in $\mathbb{E}^d$ with $d \geq 2$ and $m > 2$, then*

$$(m-1)(d+1) \leq t(m, \mathbf{K}) \leq (m-1)2^d.$$

4 Newton numbers of convex bodies

According to L. Fejes Tóth [23] the Newton number $N(\mathbf{K})$ of a convex body $\mathbf{K}$ in $\mathbb{E}^d$ is defined as the largest number of congruent copies of $\mathbf{K}$ that can touch $\mathbf{K}$ without having interior points in common. (Note that unlike in the case of Hadwiger numbers here it is not necessary at all to use translated copies of the given convex body in fact, often it is better to use rotated or reflected ones.) For the special case when $\mathbf{K}$ is a ball we refer the reader to Section 2 of this paper. Here we focus on the case when $\mathbf{K}$ is different from a ball. Somewhat surprisingly, in this case only planar results are known. Namely, Linhart [41] and Böröczky [12] determined the Newton numbers of regular convex polygons.

Theorem 4.1 *If $N(n)$ denotes the Newton number of a regular convex n-gon in $\mathbb{E}^2$, then*

$$N(3) = 12, N(4) = 8 \text{ and } N(n) = 6 \text{ for all } n \geq 5.$$

L. Fejes Tóth [22] proved the following - in some cases quite sharp - upper bound for the Newton numbers of convex domains (i.e. compact convex sets with nonempty interior) in $\mathbb{E}^2$.

Theorem 4.2 *A convex domain with diameter D and minimum width W cannot be touched by more than*

$$\left[(4 + 2\pi)\frac{D}{W} + 2 + \frac{W}{D}\right]$$

non-overlapping congruent copies of it.

Applying this theorem to convex domains of constant width we get the upper bound 8 for their Newton numbers. This result was improved by Schopp [52] as follows.

Theorem 4.3 *The Newton number of any convex domain of constant width in* $\mathbb{E}^2$ *is at most 7 and the Newton number of a Reuleaux triangle is exactly 7.*

We close this section with a rather natural question, which to the best of our knowledge has not been yet studied.

Problem 4.4 *Prove or disprove that the Newton number of a d-dimensional* ($d \geq 3$) *Euclidean cube is* $3^d - 1$.

5 One-sided Hadwiger and kissing numbers

K. Bezdek and P. Brass [8] assigned to each convex body $\mathbf{K}$ in $\mathbb{E}^d$ a specific positive integer called the one-sided Hadwiger number $h(\mathbf{K})$ as follows: $h(\mathbf{K})$ is the largest number of non-overlapping translates of $\mathbf{K}$ that touch $\mathbf{K}$ and that all lie in a closed supporting half-space of $\mathbf{K}$. In [8], using the Brunn-Minkowski inequality, K. Bezdek and P. Brass proved the following sharp upper bound for the one-sided Hadwiger numbers of convex bodies.

Theorem 5.1 *If* $\mathbf{K}$ *is an arbitrary convex body in* $\mathbb{E}^d$, *then*

$$h(\mathbf{K}) \leq 2 \cdot 3^{d-1} - 1.$$

Moreover, equality is attained if and only if $\mathbf{K}$ *is a d-dimensional affine cube.*

The notion of one-sided Hadwiger numbers was introduced to study the (discrete) geometry of the so-called k^+-neighbour packings, which are packings of translates of a given convex body in $\mathbb{E}^d$ with the property that each packing element is touched by at least k others from the packing, where k is a given positive integer. As this area of discrete geometry has a rather large literature we refer the interested reader to [8] for a brief survey on the relevant results. Here, we emphasize the following corollary of the previous theorem proved also in [8].

Theorem 5.2 *If* $\mathbf{K}$ *is an arbitrary convex body in* $\mathbb{E}^d$, *then any* k^+*-neighbour packing by translates of* $\mathbf{K}$ *with* $k \geq 2 \cdot 3^{d-1}$ *must have a positive density in* $\mathbb{E}^d$. *Moreover, there is a* $(2 \cdot 3^{d-1} - 1)^+$*-neighbour packing by translates of a d-dimensional affine cube with density* 0 *in* $\mathbb{E}^d$.

It is obvious that the one-sided Hadwiger number of any circular disk in $\mathbb{E}^2$ is 4. However, the 3-dimensional analogue statement is harder to get. As it turns out the one-sided Hadwiger number of the 3-dimensional Euclidean ball is 9. One of the shortest proofs of this fact was found by A. Bezdek and K. Bezdek [3]. Since here we are studying Euclidean balls their one-sided Hadwiger numbers we simply call one-sided kissing numbers.

Theorem 5.3 *The one-sided kissing number of the* 3*-dimensional Euclidean ball is* 9*.*

As we have mentioned before Musin [46] has just announced a proof of the long-standing conjecture that the kissing number of the 4-dimensional Euclidean ball is 24. Based on that K. Bezdek [10] gave a proof of the following.

Theorem 5.4 *The one-sided kissing number of the* 4*-dimensional Euclidean ball is either* 18 *or* 19*.*

The proof of the above theorem supports the following conjecture.

Conjecture 5.5 *The one-sided kissing number of the* 4*-dimensional Euclidean ball is* 18*.*

6 Contact graphs of finite packings and the combinatorial Kepler problem

Let $\mathbf{K}$ be an arbitrary convex body in $\mathbb{E}^d$. Then the contact graph of an arbitrary finite packing by non-overlapping translates of $\mathbf{K}$ in $\mathbb{E}^d$ is the (simple) graph whose vertices correspond to the packing elements and whose two vertices are connected by an edge if and only if the corresponding two packing elements touch each other. One of the most basic questions on contact graphs is to find out the maximum number of edges that a contact graph of n translates of the given convex body $\mathbf{K}$ can have in $\mathbb{E}^d$. Harborth [31] proved the following remarkable result on the contact graphs of congruent circular disk packings in $\mathbb{E}^2$.

Theorem 6.1 *The maximum number of touching pairs in a packing of n congruent circular disks in $\mathbb{E}^2$ is precisely*

$$\lfloor 3n - \sqrt{12n-3} \rfloor.$$

In a very recent paper [16] Brass extended the above result to the "unit circular disk packings" of normed planes as follows.

Theorem 6.2 *The maximum number of touching pairs in a packing of n translates of a convex domain $\mathbf{K}$ in $\mathbb{E}^2$ is $\lfloor 3n - \sqrt{12n-3} \rfloor$, if $\mathbf{K}$ is not a parallelogram, and $\lfloor 4n - \sqrt{28n-12} \rfloor$, if $\mathbf{K}$ is a parallelogram.*

The analogue question in the hyperbolic plane has been studied by Bowen in [11]. We prefer to quote his result in the following geometric way.

Theorem 6.3 *Consider circle packings in the hyperbolic plane, by finitely many congruent circles, which maximize the number of touching pairs for the given number of congruent circles. Then such a packing must have all of its centers located on the vertices of a triangulation of the hyperbolic plane by congruent equilateral triangles, provided the diameter* D *of the circles is such that an equilateral triangle in the hyperbolic plane of side length* D *has each of its angles is equal to* $\frac{2\pi}{N}$ *for some* $N > 6$.

It is not hard to see that one can extend the above result to $\mathbb{S}^2$ exactly in the way as the above phrasing suggests. However, we get a more general approach if we do the following: Take n non-overlapping unit diameter balls in a convex position in $\mathbb{E}^3$, that is assume that there exists a 3-dimensional convex polyhedron whose vertices are center points moreover, each center point belongs to the boundary of that convex polyhedron, where $n \geq 4$ is a given integer. Obviously, the shortest distance among the center points is at least one. Then count the unit distances showing up between pairs of center points but, count only those pairs that generate a unit line segment on the boundary of the given 3-dimensional convex polyhedron. Finally, maximize this number for the given n and label this maximum by $c(n)$. The following theorem was found by D. Bezdek [4] who also pointed out its interesting relation to protein folding as well as to Dürer's unsolved geometric problem on edge-unfolding of convex polyhedra. He calls the convex polyhedra showing up in the theorem below "higher order deltahedra" mainly because they form an extension of "deltahedra" classified earlier by Freudenthal and van der Waerden in [24].

Theorem 6.4 $c(n) \leq 3n - 6$, *where equality is attained for infinitely many* n *namely, for those for which there exists a* 3*-dimensional convex polyhedron whose each face is an edge-to-edge union of some regular triangles of side length one such that the total number of generating regular triangles on the boundary of the convex polyhedron is precisely* $2n-4$ *with a total number of* $3n-6$ *sides of length one and with a total number of* n *vertices.*

Now, we are ready to phrase the Combinatorial Kepler Problem. As its name suggests this problem is strongly related to the Kepler Conjecture on the densest unit sphere packings in $\mathbb{E}^3$ (for more details see Section 7 of this paper).

Problem 6.5 *For a given* n *find the largest number* $K(n)$ *of touching pairs in a packing of* n *congruent balls in* $\mathbb{E}^3$.

This problem is quite open. The first part of the following theorem was proved by D. Bezdek [4] the second part by K. Bezdek [10].

Theorem 6.6

(i) $K(4) = 6, K(5) = 9, K(6) = 12$ *and* $K(7) = 15$.

(ii) $K(n) < 6n - 0.59n^{\frac{2}{3}}$ *for all* $n \geq 4$.

We close this section with two upper bounds for the number of touching pairs in an arbitrary finite packing of translates of a convex body, proved by K. Bezdek in [6] . In order to state these theorems in a short way we need a bit of notation. Let $\mathbf{K}$ be an arbitrary convex body in $\mathbb{E}^d, d \geq 3$. Then let $\delta(\mathbf{K})$ denote the density of a densest packing of translates of the convex body $\mathbf{K}$ in $\mathbb{E}^d, d \geq 3$. Moreover, let $\mathrm{Iq}(\mathbf{K}) = \frac{(\mathrm{Svol}_{d-1}(\mathrm{bd}\mathbf{K}))^d}{(\mathrm{Vol}_d(\mathbf{K}))^{d-1}}$ be the isoperimetric quotient of the convex body $\mathbf{K}$, where $\mathrm{Svol}_{d-1}(\mathrm{bd}\mathbf{K})$ denotes the $(d-1)$-dimensional surface volume of the boundary $\mathrm{bd}\mathbf{K}$ of $\mathbf{K}$ and $\mathrm{Vol}_d(\mathbf{K})$ denotes the d-dimensional volume of $\mathbf{K}$. Moreover, let $\mathbf{B}$ denote the closed d-dimensional ball of radius 1 centered at the origin in $\mathbb{E}^d$. Finally, let $\mathbf{K}_0 = \frac{1}{2}(\mathbf{K} + (-\mathbf{K}))$ be the normalized (centrally symmetric) difference body assigned to $\mathbf{K}$ with $H(\mathbf{K}_0)$ (resp., $h(\mathbf{K}_0)$) standing for the Hadwiger number (resp., one-sided Hadwiger number) of $\mathbf{K}_0$.

Theorem 6.7 *The number of touching pairs in an arbitrary packing of* $n > 1$ *translates of the convex body* $\mathbf{K}$ *in* $\mathbb{E}^d, d \geq 3$ *is at most*

$$\frac{H(\mathbf{K}_0)}{2} \cdot n - \frac{1}{2^d \cdot (\delta(\mathbf{K}_0))^{\frac{(d-1)}{d}}} \cdot \Big(\frac{Iq(\mathbf{B})}{Iq(\mathbf{K}_0)}\Big)^{\frac{1}{d}} \cdot n^{\frac{(d-1)}{d}} - (H(\mathbf{K}_0) - h(\mathbf{K}_0) - 1).$$

Theorem 6.8 *The number of touching pairs in an arbitrary packing of* $n > 1$ *translates of the convex body* $\mathbf{K}$ *in* $\mathbb{E}^d, d \geq 3$ *is at most*

$$\frac{3^d - 1}{2} \cdot n - \frac{\omega_d^{\frac{1}{d}}}{2^{d+1}} \cdot n^{\frac{(d-1)}{d}},$$

where $\omega_d = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$ *is the volume of a* d*-dimensional ball of radius* 1 *in* $\mathbb{E}^d$.

7 Isoperimetric problems for Voronoi cells - the strong dodecahedral conjecture and the truncated octahedral conjecture

Recall that a family of non-overlapping 3-dimensional balls of radii 1 in Euclidean 3-space, $\mathbb{E}^3$ is called a unit ball packing in $\mathbb{E}^3$. The density of the packing is the proportion of space covered by these unit balls. The sphere packing problem asks for the densest packing of unit balls in $\mathbb{E}^3$. The conjecture that the density of any unit ball packing in $\mathbb{E}^3$ is at most $\frac{\pi}{\sqrt{18}} = 0.74078\ldots$ is often attributed to Kepler that he stated in 1611. The problem of proving the Kepler conjecture appears as part of Hilbert's 18th problem [32]. Using an ingenious argument which works in any dimension, Rogers [51] obtained the upper bound $0.77963\ldots$ for the density of unit ball packings in $\mathbb{E}^3$. This bound has been improved by Lindsey [40], and Muder [43], [44] to $0.773055\ldots$. Hsiang [33], [34] proposed an elaborate line of attack (along the ideas of L. Fejes Tóth suggested 40 years earlier), but his claim that he settled Kepler's conjecture seems exaggerated. However, so far no one has found any serious gap in the approach of Hales [26], [27], [28], [29], although no one has been able to fully verify it either. This is not too surprising, given that the detailed argument is described in several papers and relies on long computer aided calculations of more than 5000 subproblems.

Theorem 7.1 *The densest packing of unit balls in $\mathbb{E}^3$ has density $\frac{\pi}{\sqrt{18}}$, which is attained by the "cannonball packing".*

For several of the above mentioned papers Voronoi cells of unit ball packings play a central role. Recall that the Voronoi cell of a unit ball in a packing of unit balls in $\mathbb{E}^3$ is the set of points that are not farther away from the center of the given ball than from any other ball's center. As it is well-known, the Voronoi cells of a unit ball packing in $\mathbb{E}^3$ form a tiling of $\mathbb{E}^3$. One of the most attractive problems on Voronoi cells is the Dodecahedral Conjecture first phrased by L. Fejes Tóth in [21]. According to this the volume of any Voronoi cell in a packing of unit balls in $\mathbb{E}^3$ is at least as large as the volume of a regular dodecahedron with inradius 1. Very recently Hales and McLaughlin [30] announced a solution to this problem. Thus, we have the following theorem.

Theorem 7.2 *The volume of any Voronoi cell in a packing of unit balls in $\mathbb{E}^3$ is at least as large as the volume of a regular dodecahedron with inradius* 1.

Now, take the following stronger version of the Dodecahedral Conjecture often called the Strong Dodecahedral Conjecture. It was first phrased in [5].

Conjecture 7.3 *The surface area of any Voronoi cell in a packing with unit balls in $\mathbb{E}^3$ is at least as large as* $16.6508\ldots$ *the surface area of a regular dodecahedron of inradius* 1.

It is easy to see that if true, then it implies the Dodecahedral Conjecture. The best known result in connection with the Strong Dodecahedral Conjecture has been proved by K. Bezdek and E. Daróczy-Kiss in [9]. In order to phrase it properly we introduce a bit of terminology. A face cone of a Voronoi cell in a packing with unit balls in $\mathbb{E}^3$ is the convex hull of the face chosen and the center of the unit ball sitting in the given Voronoi cell. The surface area density of a unit ball in a face cone is simply the spherical area of the region of the unit sphere (centered at the apex of the face cone) that belongs to the face cone divided by the Euclidean area of the face. It should be clear from these definitions that if we have an upper bound for the surface area density in face cones of Voronoi cells, then the reciprocal of this upper bound times 4π (the surface area of a unit ball) is a lower bound for the surface area of Voronoi cells. Now, we are ready to state the main theorem of [9].

Theorem 7.4 *The surface area density of a unit ball in any face cone of a Voronoi cell in an arbitrary packing of unit balls of* $\mathbb{E}^3$ *is at most*

$$\frac{-9\pi + 30 \arccos\left(\frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{5}\right)\right)}{5\tan\left(\frac{\pi}{5}\right)} = 0.77836\ldots,$$

and so the surface area of any Voronoi cell in a packing with unit balls in E^3 *is at least*

$$\frac{20\pi\tan\left(\frac{\pi}{5}\right)}{-9\pi + 30 \arccos\left(\frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{5}\right)\right)} = 16.1445\ldots.$$

Moreover, the above upper bound $0.77836\ldots$ *for the surface area density is best possible in the following sense. The surface area density in the face cone of any* n*-sided face with* $n = 4, 5$ *of a Voronoi cell in an arbitrary packing of unit balls of* $\mathbb{E}^3$ *is at most*

$$\frac{3(2-n)\pi + 6n \cdot \arccos\left(\frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{n}\right)\right)}{n\tan\left(\frac{\pi}{n}\right)}$$

and equality is achieved when the face is a regular n*-gon inscribed in a circle of radius* $\frac{1}{\sqrt{3}\cdot\cos\left(\frac{\pi}{n}\right)}$ *and positioned such that it is tangent to the corresponding unit ball of the packing at its center.*

The Kelvin problem asks for the surface minimizing partition of $\mathbb{E}^3$ into cells of equal volume. According to Lhuilier's memoir [39] of 1781, the problem has been described as one of the most difficult in geometry. The solution proposed by Kelvin is a natural generalization of the hexagonal honeycomb in $\mathbb{E}^2$. Take the Voronoi cells of the dual lattice giving the densest sphere packing. This gives truncated octahedra, the Voronoi cells of the body centered cubic lattice. A small deformation of the faces produces a minimal surface, which is Kelvin's proposed solution. Just recently Phelan and Weaire [50] produced a remarkable counter-example to the Kelvin conjecture. Their work indicates also that Kelvin's original question is even harder than it was expected. In fact, the following simplier question seems to be still open. One can regard this as the isoperimetric inequality for parallelohedra and one can call the conjecture below the truncated octahedral conjecture. (Recall that a parallelohedron is a 3-dimensional convex polyhedron that tiles $\mathbb{E}^3$ by translation.)

Conjecture 7.5 *The surface area of any parallelohedron of volume* 1 *in* $\mathbb{E}^3$ *is at least as large as the surface area of the truncated octahedral Voronoi cell of the body-centered cubic lattice of volume* 1 *in* $\mathbb{E}^3$.

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