

# RIGIDITY OF BALL-POLYHEDRA IN EUCLIDEAN 3-SPACE

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ABSTRACT. In this paper we introduce ball-polyhedra as finite intersections of congruent balls in Euclidean 3-space. We define their duals and study their face-lattices. Our main result is an analogue of Cauchy's rigidity theorem.

## 0. INTRODUCTION

Take the intersection of finitely many but at least three closed balls of given radius, say  $R > 0$ , in Euclidean 3-space and assume that removing any of the balls yields that the intersection of the remaining balls becomes a larger set. Also assume that this intersection has non-empty interior. Such an intersection we call a *ball-polyhedron* of radius  $R$  in Euclidean 3-space.

The boundary of a ball-polyhedron is the union of vertices, edges and faces defined as follows. A boundary point is called a *vertex* if it belongs to at least three of the closed balls defining the ball-polyhedron. A *face* of the ball-polyhedron is the intersection of one of the closed balls with the boundary of the ball-polyhedron. Finally, if the intersection of two faces is non-empty, then it is the union of (possibly degenerate) circular arcs. The non-degenerate arcs are called *edges* of the ball-polyhedron. Obviously, by definition every ball-polyhedron possesses vertices, edges and faces. Note, that each convex polyhedron of Euclidean 3-space can be approximated by ball-polyhedra of radius  $R$  if we let  $R$  tend to infinity. This means that several properties of ball-polyhedra are strongly connected to some special features of convex polyhedra.

Last but not least, ball-polyhedra are natural objects of study for several important problems of discrete geometry such as the Kneser-Poulsen conjecture [1], the Borsuk problem [2] and the problem of approximation of sets of constant width by Reuleaux polyhedra [5].

For the purpose of this paper we assume that  $R = 1$  i.e. we are going to study ball-polyhedra of unit radius in Euclidean 3-space. One of the best known results on the geometry of convex polyhedra is the *Cauchy's rigidity theorem*: If two convex polyhedra  $P$  and  $P'$  are combinatorially equivalent with corresponding facets being

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congruent, then also the angles between corresponding pairs of adjacent facets are equal and thus  $P$  is congruent to  $P'$ .

For more details on Cauchy's rigidity theorem and on its extensions we refer the reader to Connolly's excellent survey paper [3] (see also [4], [6] and [7]).

The main goal of this paper is to prove an extension of Cauchy's rigidity theorem for ball-polyhedra. In order to phrase it we need to introduce the following rather natural terminology. The *face-lattice* of a ball-polyhedron is just the algebraic lattice formed by its vertices, edges and faces. Moreover, to each edge we can assign an *inner dihedral angle* the following way. Take any point  $p$  in the relative interior of the edge and take the two unit-balls that contain the two faces of the ball-polyhedron meeting along the edge. Now, the inner dihedral angle along this edge is the angle of the two half-spaces supporting the two balls at  $p$ . It is obviously independent of the choice of  $p$ . Finally, at each vertex of a face of a ball-polyhedron there is a *face angle* formed by the two edges meeting at the given vertex (which is the angle between the tangent half-lines of the two edges at the vertex).

The following statement summarizes some basic properties of face-lattices of ball-polyhedra.

**Theorem 0.1.**

(1) Euler-Poincaré Theorem: *For any ball-polyhedron with  $v$  vertices,  $e$  edges and  $f$  faces we have that*

$$v - e + f = 2.$$

(2) Duality: *Take a ball-polyhedron which has no face bounded by two edges only. Then the intersection of the closed unit balls centered at the vertices of the ball-polyhedron is a ball-polyhedron, the face-lattice of which is dual to that of the original ball-polyhedron.*

We say that the ball-polyhedron  $P$  is (globally) *rigid with respect to its face angles* (resp. *inner dihedral angles*) if the following holds: If  $Q$  is another ball-polyhedron whose face-lattice is isomorphic to that of  $P$  and whose face angles (resp. inner dihedral angles) are equal to the corresponding face angles (resp. inner dihedral angles) of  $P$ , then  $Q$  is congruent to  $P$ . A ball-polyhedron is called *triangulated* if all the faces are bounded by three edges, and it is called *simple* if at every vertex exactly three edges meet.

**Theorem 0.2.** *Let  $P$  be a triangulated ball-polyhedron. Then  $P$  is rigid with respect to its face angles.*

Although we do not know whether any triangulated ball-polyhedron is rigid with respect to its inner dihedral angles, we note that there is a quite large family of non-rigid ball-polyhedra.

**Proposition 0.1.** *Any ball-polyhedron that has two faces meeting at more than one edge is not rigid neither with respect to its face angles nor with respect to its inner dihedral angles, that is, in short, it is flexible.*

This suggests the following

**Rigidity Conjecture for Simple Ball-Polyhedra.** Let  $P$  be a simple ball-polyhedron. Then  $P$  is rigid with respect to its inner dihedral angles if and only if  $P$  has no two faces meeting at more than one edge.

According to Claim 5.1 the Rigidity Conjecture for Simple Ball-Polyhedra implies that if  $P$  is a simple ball-polyhedron, then  $P$  is rigid with respect to its face angles if and only if  $P$  has no two faces meeting at more than one edge.

Note that the centers of any ball-polyhedron must be in a strictly convex position that is each center is a vertex of the convex hull of the center points (otherwise one of the unit balls can be removed without changing the ball-polyhedron).

**Proposition 0.2.** *The Rigidity Conjecture for Simple Ball-Polyhedra is equivalent to the following:*

- (\*) *Any triangulated polyhedron with the property that for the three vertices of any of its faces there is a unit ball containing those three vertices on its boundary and all the other vertices in its interior is rigid, that is if  $K$  and  $L$  are triangulated polyhedra with the above property and with isomorphic face-lattices such that the edge lengths of  $K$  are equal to the corresponding edge lengths of  $L$ , then  $K$  is congruent to  $L$ .*

(\*) appears to be a rather difficult problem that is partly connected to problems discussed in [8] and [9].

Finally, we prove the Rigidity Conjecture for Simple Ball-polyhedra in the following special case.

**Theorem 0.3.** *Let  $K$  be a triangulated 3-dimensional convex polyhedron. Then there exists an  $\varepsilon > 0$  depending on  $K$  such that the intersection of the closed unit balls centered at the vertices of the convex polyhedron  $\delta K$  is a simple ball-polyhedron that is rigid with respect to its inner dihedral angles for all  $0 < \delta < \varepsilon$ .*

Last but not least we remark that it would be highly interesting to extend our results to higher dimensions and in particular, to spherical and hyperbolic spaces.

## 1. PRELIMINARIES

All of our work takes place in the Euclidean 3-space,  $\mathbb{R}^3$ . A *ball* is the compact convex set formed by points at a distance not greater than a given positive distance from a given point. A *sphere* is the boundary of a ball. If the radius is 1, we denote them by  $\mathbf{B}(x) := \{y \in \mathbb{R}^3 \mid d(x, y) \leq 1\}$  and  $\mathbb{S}(x) := \{y \in \mathbb{R}^3 \mid d(x, y) = 1\}$ , respectively.

**Definition.** Let  $X \subseteq \mathbb{R}^3$ . Denote by  $\mathbf{B}X := \bigcap_{x \in X} \mathbf{B}(x)$ .

We make the following rather trivial

**Observation 1.1.**

(1)  $X \subseteq Y \implies \mathbf{B}X \supseteq \mathbf{B}Y$ ,

(2)  $X \subseteq \mathbf{B}X$ .

We need the following notions from spherical geometry.

**Definition.** The shortest geodesic arc connecting two non-antipodal points of a sphere is called a *spherical segment*.

A subset of a sphere is *spherically convex*, if it is empty, or it is contained in an open hemisphere and the spherical segment connecting any two points of it is also contained in it.

The first two claims of the following lemma are due to Sallee [5]. This simple collection of statements turns out to be extremely useful in the investigation of ball-polyhedra.

**Lemma 1.1.**

(SL1) If  $p, q$  are the endpoints of a circular arc say,  $c$  of radius at least 1, and  $c$  is not longer than a semicircle, then  $\mathbf{B}(\{p, q\}) \subseteq \mathbf{B}(c)$ .

(SL2)  $\mathbf{B}X \cap \mathbb{S}(p)$  is either  $\mathbb{S}(p)$  itself or spherically convex on  $\mathbb{S}(p)$  for any  $p \in \mathbb{R}^3$  and for any  $X \subset \mathbb{R}^3$ .

(SL3) If a sphere  $\mathbb{S}$  of radius 1 contains a non-degenerate arc of the intersection  $\mathbb{S}_1 \cap \mathbb{S}_2$  of two distinct spheres of radius 1, then either  $\mathbb{S} = \mathbb{S}_1$  or  $\mathbb{S} = \mathbb{S}_2$ .

*Proof.* (SL1) Let  $r \in c$ . We have to show that  $\mathbf{B}(\{p, q\}) \subseteq \mathbf{B}(r)$ , or equivalently that  $\mathbf{BB}(\{p, q\}) \ni r$ . Since  $\mathbf{BB}(\{p, q\})$  is the intersection of all unit balls containing  $p$  and  $q$  and any such ball obviously contains  $c$  the claim holds.

(SL2) is a consequence of (SL1).

(SL3) is obvious. □

We are now in the position to refresh the definitions of ball-polyhedra and of their face-lattice.

Let  $X \subset \mathbb{R}^3$  be a finite set consisting of at least three points and assume that  $\mathbf{B}X$  has non-empty interior, moreover assume that  $X$  is a *non-redundant set of centers* i.e. removing any of the balls yields that the intersection of the remaining balls is a larger set. Take a sphere  $\mathbb{S}(x)$ , where  $x \in X$ . According to (SL2)  $\mathbb{S}(x) \cap \text{bd}(\mathbf{B}X)$  is spherically convex on  $\mathbb{S}(x)$ . Also, it is clear that  $\mathbb{S}(x) \cap \text{bd}(\mathbf{B}X)$  contains at least three spherically non-collinear points and so, it is a convex domain on  $\mathbb{S}(x)$  with a non-empty interior. We call it the *face* corresponding to the center  $x$ . Two faces can meet in one point or along a number of (possibly degenerate) circular arcs (unlike in the case of convex polyhedra, they indeed can meet along two disjoint arcs). The non-degenerate arcs are the *edges* of  $\mathbf{B}X$ . A *vertex* is a point on the boundary that belongs to at least three faces.

In the following  $\mathcal{F}(\mathbf{B}X)$  denotes the faces and  $\mathcal{V}(\mathbf{B}X)$  the vertices of the ball-polyhedron  $\mathbf{B}X$ .

**Definition.** Let  $\mathbf{B}X$  be a ball-polyhedron and let  $u, v$  be two points on the edge  $e$  of  $\mathbf{B}X$ , and let  $F_1, F_2$  be the two faces meeting in  $e$ . Then the two faces belong to the unit spheres say,  $\mathbb{S}(x), \mathbb{S}(y)$  with  $x, y \in X$ . Take the two spherical segments connecting  $u$  and  $v$  on  $\mathbb{S}(x)$  and  $\mathbb{S}(y)$ . Now we have a biangle on the boundary of  $\mathbf{B}(\{x, y\})$ . This biangle bounds a region on the boundary of  $\mathbf{B}(\{x, y\})$  in the middle of which we have the edge  $e$ . This region is the *wedge* of  $u$  and  $v$ .

**Claim 1.1.** *In the above setting the wedge of  $u$  and  $v$  is part of the boundary of  $\mathbf{B}X$ . Moreover, there are no vertices of  $\mathbf{B}X$  in the wedge other than possibly  $u$  and  $v$ .*

*Proof.* First, note that the two sides of the biangle are shorter than a semicircle since  $u$  and  $v$  are not antipodal on either spheres. Since  $u, v \in \mathbf{B}X$ , the biangle is also contained by  $\mathbf{B}X$ . This is due to (SL1). So is the edge,  $e$ . But the wedge is the union of two spherically convex regions, one on each sphere bounded by the edge and either spherical segment. The bounding curve of both parts is in  $\mathbf{B}X$ , and the

two parts are spherical convex hulls of their respective boundaries. Applying (SL2) we get that both parts are in  $\mathbf{B}X$ .

Now, let  $p$  be a point in the wedge with  $p \neq u, v$ . If  $p$  is on the edge  $e$ , then it is obviously not a vertex. If  $p$  is not on  $e$ , then  $p$  is a point either on  $\mathbb{S}(x)$  or on  $\mathbb{S}(y)$ , say it is on  $\mathbb{S}(x)$ . Then any other sphere, say  $\mathbb{S}(z)$  (where  $z \in X$ ), that would contain  $p$  should intersect  $\mathbb{S}(x)$  in a circle of radius less than 1. This circle would then go through  $p$  and it would have a non-degenerate arc in the wedge. But that means that some points of the wedge outside the circle are not in  $\mathbf{B}(z)$ . This contradicts the first part of the claim.  $\square$

Although we do not need the following statements in the sequel we mention them since they reflect basic properties of ball-polyhedra.

**Proposition 1.1.** *Let  $X \subset \mathbb{R}^3$  be a set of at least 3 points. Then  $X$  is a non-redundant set of centers for the ball-polyhedron  $\mathbf{B}X$  if and only if any point  $v$  of  $X$  can be separated from the other points by a unit ball i.e. there is a unit ball that contains  $X \setminus \{v\}$  and does not contain  $v$ .*

*Proof.* Suppose that  $v \in X$  cannot be omitted from  $X$  when generating  $\mathbf{B}X$ . Then there is a point  $p \in \mathbf{B}(X \setminus \{v\})$  which is not in  $\mathbf{B}(v)$ . Thus  $\mathbf{B}(p)$  separates  $v$  from  $X \setminus \{v\}$ . The argument read from right to left yields the other direction of the proposition.  $\square$

**Corollary 1.0.1.** *Let  $K$  be a 3-dimensional convex polyhedron. Then there exists an  $\varepsilon > 0$  depending on  $K$  such that the vertices of  $\delta K$  (denoted by  $\mathcal{V}(\delta K)$ ) form a non-redundant set of centers for the ball-polyhedron  $\mathbf{B}(\mathcal{V}(\delta K))$ .*

*Proof.* Denote by  $X := \mathcal{V}(K)$ . Since  $X$  is the set of vertices of a convex polyhedron we have that  $v$  and  $X \setminus \{v\}$  can be strictly separated by a plane for any  $v \in X$ . Hence, if  $\delta$  is small enough, then there is a unit ball that separates  $\delta v$  and  $\delta X \setminus \{\delta v\}$ .  $\square$

## 2. EULER-POINCARÉ FORMULA, PROOF OF THEOREM 0.1 (1)

*Proof.* Let  $\mathbf{B}X$  be a ball-polyhedron in  $\mathbb{R}^3$  with  $v$  vertices,  $e$  edges and  $f$  faces. Since  $\mathbf{B}X$  is a compact convex set with nonempty interior, its boundary is homeomorphic to a sphere, so all we have to prove is that the face-structure is a CW-decomposition of  $bd(\mathbf{B}X)$ . In the definition of ball-polyhedra we made sure that it has at least one vertex by requiring at least three intersecting balls. The set of vertices is the *0-skeleton* of the CW-decomposition. Edges are circular arcs ending in vertices, thus they form the *1-skeleton* of the decomposition. In fact, the 1-skeleton is connected as one can check easily by induction on the number of intersecting balls. Faces are *2-cells*, since they are compact spherically convex domains with non-empty interior lying on unit spheres. Since any ball-polyhedron must have vertices,  $bd(\mathbf{B}X)$  cannot be one single face, and it is easy to see that all faces are bounded by a closed curve formed by a sequence of edges. So, faces build up the *2-skeleton* of  $bd(\mathbf{B}X)$ .  $\square$

## 3. DUALITY, PROOF OF THEOREM 0.1 (2)

In this section we define a mapping between the face-lattices of  $\mathbf{B}X$  and  $\mathbf{B}\mathcal{V}(\mathbf{B}X)$  and show that it is bijective and it reverses containment. This mapping is the *duality mapping* between the two ball-polyhedron.

In order this duality to work we are forced to make one additional assumption. Namely, we have to assume that there are *no biangles in  $\mathbf{B}X$*  i.e. all faces contain at least three vertices. No duality can work without this assumption, since the dual of this condition gives that in each vertex of the dual ball-polyhedron at least three faces meet which follows from the definition of a vertex. This assumption is also necessary for the rigidity of a simple ball-polyhedron as pointed out in section 4.

The duality mapping consists of the following two mappings (edges will be discussed later on).

The *vertex-face mapping* is:

$$\mathcal{V}(\mathbf{B}X) \ni \mathbf{v} \mapsto V \in \mathcal{F}(\mathbf{B}\mathcal{V}(\mathbf{B}X)),$$

where  $V$  is the face of  $\mathbf{B}\mathcal{V}(\mathbf{B}X)$  with  $\mathbf{v}$  as its center.

The *face-vertex mapping* is:

$$\mathcal{F}(\mathbf{B}X) \ni F \mapsto \mathbf{f} \in \mathcal{V}(\mathbf{B}\mathcal{V}(\mathbf{B}X)),$$

where  $\mathbf{f}$  is the center of face  $F$ .

We need to prove that the above defined mappings are well defined, their images are in the given sets and they are one-to-one and onto. Then we also have to show that edges are mapped properly, i.e. two vertices in  $\mathbf{B}\mathcal{V}(\mathbf{B}X)$  are connected by an edge if and only if the corresponding faces of  $\mathbf{B}X$  meet in an edge.

**3.1. The vertex-face mapping.** It is well defined: Since  $\mathbf{v}$  is a vertex of  $\mathbf{B}X$ , there are three different centers  $x, y, z \in X$  such that  $\mathbf{v}$  is at distance 1 from all the three, i.e.  $x, y, z \in \mathbb{S}(\mathbf{v})$ . According to (SL1) they can be chosen such that they are not on a great circle of  $\mathbb{S}(\mathbf{v})$ . Applying Observation 1.1, we get that  $X \subseteq \mathbf{B}\mathcal{V}(\mathbf{B}X)$ , so  $x, y, z \in \mathbf{B}\mathcal{V}(\mathbf{B}X)$ .  $\mathbf{v}$  makes them points on the boundary, so, we have three spherically non-collinear points on  $\mathbb{S}(\mathbf{v})$ , all belonging to the boundary of  $\mathbf{B}\mathcal{V}(\mathbf{B}X)$ . Using (SL2) we get that  $\mathbb{S}(\mathbf{v})$  contains a face of  $\mathbf{B}\mathcal{V}(\mathbf{B}X)$ .

The vertex-face mapping is one-to-one since the center of a face is unique. It is also onto since any face of a ball-polyhedron corresponds to a center of the ball-polyhedron.

**3.2. The face-vertex mapping.** It is well defined: If  $F$  is a face of  $\mathbf{B}X$ , then its center,  $\mathbf{f}$  is in  $X$ . Since  $X \subseteq \mathbf{B}\mathcal{V}(\mathbf{B}X)$ ,  $\mathbf{f} \in \mathbf{B}\mathcal{V}(\mathbf{B}X)$ . We use our assumption that all faces of  $\mathbf{B}X$  have at least three vertices. Let  $u, v, w$  be vertices of  $F$ . They are on  $\mathbb{S}(\mathbf{f})$  and are spherically non-collinear, so  $\mathbf{f}$  is going to be a vertex of  $\mathbf{B}\mathcal{V}(\mathbf{B}X)$ . Note that we proved that  $X \subseteq \mathcal{V}(\mathbf{B}\mathcal{V}(\mathbf{B}X))$ .

The mapping is one-to-one since two faces of a ball-polyhedron cannot have the same center. This is a consequence of the definition of a face.

Showing that the mapping is onto is not obvious. First, it is obviously onto on  $X$  since  $X$  is minimal. We will show that it is onto  $\mathcal{V}(\mathbf{B}\mathcal{V}(\mathbf{B}X))$ , what – as a byproduct – gives us the equality  $X = \mathcal{V}(\mathbf{B}\mathcal{V}(\mathbf{B}X))$ . Let  $\mathbf{f} \in \mathcal{V}(\mathbf{B}\mathcal{V}(\mathbf{B}X))$ . We

want to find a face of  $\mathbf{B}X$  the center of which is  $\mathbf{f}$ . Since  $\mathbf{f}$  is a vertex there are three points  $u, v, w \in \mathcal{V}(\mathbf{B}X)$  such that  $u, v, w \in \mathbb{S}(\mathbf{f})$ . We complete the proof by showing that  $u, v, w$  are vertices of the *same* face of  $\mathbf{B}X$ . Consider  $\mathbb{S}(\mathbf{f})$  in the neighbourhood of  $v$ . Let  $e_1, e_2, \dots, e_k$  be the edges of  $\mathbf{B}X$  adjacent to  $v$ .  $\mathbb{S}(\mathbf{f})$  certainly contains  $v$ . What does  $\mathbb{S}(\mathbf{f})$  look like around  $v$ ?

- If  $\mathbb{S}(\mathbf{f})$  also contains an edge adjacent to  $v$ , then according to (SL3) it is one of the spheres containing the two faces of  $\mathbf{B}X$  that meet at the edge. So  $\mathbf{f} \in X$ .
- Suppose that the beginnings (i.e. some non-degenerate arcs) of  $e_1, e_2, \dots, e_k$  run in the interior of  $B(\mathbf{f})$ . Taking these arcs  $a_1 \subseteq e_1, a_2 \subseteq e_2, \dots, a_k \subseteq e_k$  we have  $k$  "legs" on the boundary of  $\mathbf{B}X$  with a common starting point,  $v$ . They have a cyclic order  $1, 2, \dots, k, 1, \dots$ . Any two consecutive legs are on a face of  $\mathbf{B}X$  containing  $v$ . Take the spherical convex hulls of all consecutive pairs of the legs on their corresponding face. The union of the  $k$  regions together with  $v$  is a neighborhood of  $v$  on the boundary of  $\mathbf{B}X$ . According to (SL1) this union is in the interior of  $B(\mathbf{f})$ . (Precisely, (SL1) needs to be stated for intersections of open balls.) So,  $\mathbb{S}(\mathbf{f})$  meets  $\mathbf{B}X$  only in  $v$  in this neighborhood. However, according to (SL2),  $\mathbb{S}(\mathbf{f}) \cap \mathbf{B}X$  is spherically convex on  $\mathbb{S}(\mathbf{f})$ , and it should also contain  $u$  and  $w$ , which yields a contradiction. So, there is only one case left:
- Suppose that the beginning (i.e. some non-degenerate arc) of some edge, say  $e_1$ , runs outside of  $B(\mathbf{f})$ .  $e_1$  connects  $v$  with some other vertex, say  $z$ . Since  $\mathbf{f} \in \mathbf{B}\mathcal{V}(\mathbf{B}X)$  and  $B(\mathbf{f}) \supseteq \mathcal{V}(\mathbf{B}X)$ , so  $B(\mathbf{f})$  contains  $v$  and  $z$ . Let  $z'$  be the point of intersection of  $\mathbb{S}(\mathbf{f})$  and  $e_1$  other than  $v$ . It may, of course, be  $z$  itself. Now take the wedge of  $v$  and  $z'$ . Since  $v, z' \in \mathbb{S}(\mathbf{f})$ , the biangle bounding this wedge is in  $B(\mathbf{f})$ . Moreover, only the two vertices  $v$  and  $z'$  of the biangle are on  $\mathbb{S}(\mathbf{f})$ , since  $\mathbb{S}(\mathbf{f})$  is different from the two spheres intersecting along the edge. On the other hand, the edge-section  $vz'$  in the middle of the wedge runs outside  $B(\mathbf{f})$  except for the two endpoints,  $v$  and  $z'$ . Then obviously the intersection of the wedge with  $\mathbb{S}(\mathbf{f})$  is a closed curve, so it coincides with the intersection of the complete boundary of  $\mathbf{B}X$  with  $\mathbb{S}(\mathbf{f})$ . But there is no vertex in a wedge (claim 1.1), so  $\mathbb{S}(\mathbf{f})$  can not contain another vertex of  $\mathbf{B}X$ . This is a contradiction.

**3.3. The edge-edge mapping.** That the above defined two mappings reverse containment is easy to see from their definitions. To complete the duality, we will prove that edges are mapped on edges, i.e. if two vertices in  $\mathbf{B}X$  are connected by an edge separating two faces, then the corresponding two faces of  $\mathbf{B}\mathcal{V}(\mathbf{B}X)$  meet in an edge between the corresponding vertices<sup>1</sup>. If it holds, then this mapping of edges is one-to-one. Using the Euler-Poincaré formula, we obtain that it's indeed bijective.

Let  $e$  be an edge of  $\mathbf{B}X$  between vertices  $u, v$ , and faces  $F, G$ . We want to show that the corresponding faces  $U, V$  of  $\mathbf{B}\mathcal{V}(\mathbf{B}X)$  are connected by an edge with vertices  $f, g$ . Take  $\mathbb{S}(u) \cap \mathbb{S}(v)$ . This is a circle on which  $f$  and  $g$  lie. We want to show that one of the two arcs of this circle connecting  $f$  with  $g$  is an edge of  $\mathbf{B}\mathcal{V}(\mathbf{B}X)$ .

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<sup>1</sup>One could think that a weak duality (a duality that only deals with faces and vertices) between two planar graphs induces the appropriate mapping on the edges as well. Another one can produce a counter-example, though.

Let  $p$  be a point running around on the circle. Consider  $\mathbf{B}(p) \cap \mathbf{B}X$ .  $u, v \in \mathbb{S}(p)$ . If  $p = f$  or  $p = g$ , then  $\mathbb{S}(p) \cap bd(\mathbf{B}X)$  is a face. When  $p$  is on exactly one of the two arcs, then  $\mathbb{S}(p) \cap bd(\mathbf{B}X)$  runs in the wedge of  $u$  and  $v$ , hence  $\mathbf{B}(p)$  contains all vertices of  $\mathbf{B}X$ . That shows that this arc is contained by all balls around the vertices of  $\mathbf{B}X$ , so it is in  $\mathbf{BV}(\mathbf{B}X)$ . Thus, this arc is the desired edge.

Note, that since  $\mathbf{B}X$  has no biangles, it has at least three vertices, so  $\mathbf{BV}(\mathbf{B}X)$  will be a ball-polyhedron. Hence, we have the following

*Remark 3.1.* The duality is a bijection of the class of ball-polyhedra without biangles onto itself.

#### 4. FLEXIBLE BALL-POLYHEDRA, PROOF OF PROPOSITION 0.1

**4.1. The construction.** In this subsection we construct a simple ball-polyhedron that is *not rigid*. Indeed, we give a continuous deformation of it during which the ball-polyhedra at each stage are isomorphic to one another and neither the inner dihedral angles nor the face angles change, however the ball-polyhedra at different stages are mutually non-congruent. In fact, we are going to construct a flexible simple ball-polyhedron having no biangle faces at all. (Actually, a simplified version of the following construction yields also flexible simple ball-polyhedra with some biangle faces though.)

The ball-polyhedron will be the intersection of six unit balls. We begin with the intersection of two unit balls close to each other, say at a distance  $\frac{1}{10}$ , and intersect this "big lens" with a third unit ball to cut off a small piece of its edge. To make easy to follow the construction, we let the center of the third unit ball be in the plane of the edge of the lens (step 1 on Figure 4.1). All the following three centers are going to be in the same plane. (Actually, this is not crucial for the construction to work.)

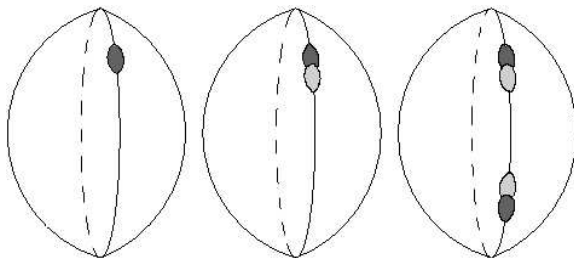


Figure 4.1: The three steps of making a simple flexible ball-polyhedron.

Now, we have a ball-polyhedron with two "big" faces, i.e. the remaining parts of the faces of the lens (both are bounded by two edges) and a "small" third face on the third ball that is bounded by the two new edges. This simple ball-polyhedron has two vertices.

We take a fourth unit ball with its center also in the plane of the old edge of the lens that also cuts off a small piece of the old edge of the lens. We position this fourth ball so that it also cuts off one of the vertices. This way we get a new ball-polyhedron with the following faces. There are the two old "big" faces (formed by the remaining parts of the two faces of the lens), and instead of the third face we



have two faces (small triangles) meeting along a new small edge (step 2. on Figure 4.1). This ball-polyhedron has four vertices, each is in the intersection of exactly three faces, so this is a simple ball-polyhedron.

Now, we look at the remaining part of the edge of the original lens. Since we positioned the third and fourth balls in a way to cut off only small pieces from the edge of the original lens, this edge is still long. Finally, we take a fifth and a sixth ball with centers in the plane of the edge of the lens to cut off another two small pieces from the old edge to get another two small triangles. We position the fifth and sixth ball in a way that these small triangles are "far" from the previous two small triangles, i.e. we do not cut off any of the four vertices that had been constructed before (step 3. on Figure 4.1).

So, now we have a ball-polyhedron with two big faces meeting at *two* edges (the remainings of the original lens) and two pairs of triangles. In each pair the two triangles meet along an edge and the third vertex of each triangle is on one of the two edges of the two big faces. This is the flexible ball-polyhedra. Looking at its face-lattice we see that it is indeed a simple ball-polyhedron.

How to "flex" it? Fix the first four centers and take the line connecting the first two. Now, if we rotate continuously and simultaneously the last two centers about this line, then the second pair of triangles will move "along the old edge" of the old lens. We do not let it hit the first pair of triangles. One can see that during this motion the ball-polyhedra are isomorphic and the inner dihedral angles along edges (resp. face angles) remain constant. The ball-polyhedra at any two stages of the motion are obviously incongruent.

**4.2. Generalization of the construction.** Now we turn to the proof of Proposition 0.1. Let the edges separating the two faces ( $F_1$  and  $F_2$ ) meeting along more than one edge be  $e_1, e_2, \dots, e_k$ . If we take the edge-graph of the ball-polyhedron and delete  $e_1, e_2, \dots, e_k$  we get a graph  $G$  with  $k$  connected components. Now, we can fix the centers of the two faces  $F_1, F_2$  and also fix the centers of the faces corresponding to the first  $k - 1$  connected components of  $G$ . As in the construction, we take the line connecting the centers of  $F_1$  and  $F_2$  and rotate the centers corresponding to the last connected component of  $G$  about this line. We make sure not to hit any of the vertices in the other components. It is obviously a continuous deformation of the ball-polyhedron via isomorphic ball-polyhedra that preserves inner dihedral angles and face angles. If the rotation is small enough no two stages are congruent.

## 5. PROOF OF THEOREM 0.2

We start with the following claim the proof of which follows the idea of Cauchy's proof of his rigidity theorem.

**Claim 5.1.** *The face-lattice and the face angles determine the inner dihedral angles of any ball-polyhedron.*

*Proof.* Suppose that  $P$  and  $P'$  are two ball-polyhedra with isomorphic face-lattices and with equal corresponding face angles. We assign a sign (+, - or 0) to each edge of  $P$  according to whether the inner dihedral angle along that edge of  $P$  is greater,

equal to or less than the inner dihedral angle along the corresponding edge of  $P'$ . We want to show that all the edges of  $P$  have the sign 0. Suppose that this is not true. Then the graph of the edges of  $P$  has at least one edge either with the  $+$  sign or with the  $-$  sign. Now, delete all the edges which have the sign 0. Let the graph obtained this way be denoted by  $G$ . Obviously,  $G$  is a non-empty simple planar graph.

Now, recall Cauchy's combinatorial lemma [10]:

**Lemma 5.1** (Cauchy's combinatorial lemma.). *Let  $G$  be a non-empty simple planar graph. If the edges of  $G$  are two-colored, then there is a vertex of  $G$  with at most two color changes in the cyclic order of the edges around the vertex.*

Thus,  $G$  must have a vertex say,  $v$  with some edges labelled either with  $+$  or with  $-$  such that  $v$  has at most two sign changes in the cyclic order of the edges around  $v$ .

We consider the *vertex-figure* of  $v$  defined as follows. For each edge incident to  $v$  we take the tangent half-line at  $v$ . These tangent half-lines are in a convex position as easily seen when one goes around  $v$  with planes supporting  $P$  at  $v$ . Now, take the intersection of these tangent half-lines with the unit sphere centered at  $v$ . The intersection points yield the vertices of a spherical convex polygon,  $\Gamma$  on the unit sphere. The face angles of  $P$  at  $v$  are the lengths of sides of  $\Gamma$  in the spherical metric. An inner dihedral angle along an edge incident to  $v$  is the angle of the two planes supporting either faces of  $P$  at  $v$ . As these planes are spanned by two consecutive tangent half-lines we get that the inner dihedral angles are the angles of  $\Gamma$ .

Let  $v'$  be the vertex of  $P'$  corresponding to  $v$ . We define the vertex-figure of  $v'$  in  $P'$  in the same way as above and denote it by  $\Gamma'$ .  $\Gamma$  and  $\Gamma'$  are convex spherical polygons with their sides being in a bijection and the corresponding sides being equal. The sign assignment on edges incident to  $v$  gives a sign assignment on the vertices of  $\Gamma$  indicating whether the angle of  $\Gamma$  at the given vertex is greater, equal to or less than the corresponding angle of  $\Gamma'$ . The sign assignment on the edges of  $G$  at the vertex  $v$  implies that some vertices of  $\Gamma$  are labelled with either  $+$  or  $-$  such that the number of sign changes in the cyclic order of the vertices of  $\Gamma$  is at most 2. However this contradicts Cauchy's arm lemma [10] finishing the proof of Claim 5.1.

□

So, we need to prove that the face-lattice, the face angles and the inner dihedral angles determine a triangulated ball-polyhedron. A face of our ball-polyhedron is the intersection of three circles (spherical caps) of radius  $< \frac{\pi}{2}$  on a unit sphere. The radius of one such circle is determined by the corresponding dihedral angle of  $P$ . The angles of the circles are the face angles of  $P$ . We finish our proof by showing that these 6 parameters (the radii of the three circles and their angles) determine the intersection of the three circles up to an isometry of the unit sphere.

Let the three circles be  $C_1, C_2, C_3 \subset \mathbb{S}^2$  with radii  $r_1, r_2$  and  $r_3$ , respectively, where  $\mathbb{S}^2$  denotes the unit sphere centered at the origin of  $\mathbb{R}^3$ . Let the face be  $T := C_1 \cap C_2 \cap C_3$ . Denote by  $T'$  the polar of  $T$  that is let  $T' := \{x \in \mathbb{S}^2 : x \cdot y \leq 0 \text{ for all } y \in T\}$ , where  $\cdot$  refers to the standard inner product of  $\mathbb{R}^3$ . It

is easy to see that  $T'$  is the spherical convex hull of the polars of the circles that is  $T' = Sconv(C'_1 \cup C'_2 \cup C'_3)$ . Note also that  $C'_1, C'_2$  and  $C'_3$  are circles of radii  $\frac{\pi}{2} - r_1, \frac{\pi}{2} - r_2$  and  $\frac{\pi}{2} - r_3$ .

How does the boundary of  $T'$  look like? The polars of the supporting great circles at a vertex of  $T$  form a spherical line segment tangent to and connecting the two circles polar to the two circles that meet at the given vertex. Moreover, the length of one such spherical line-segment is  $\pi - \alpha_i$ , where  $\alpha_i$  is the corresponding inner angle of  $T$  at the given vertex.

So,  $T'$  is a union of three circular arcs connected by three spherical line segments. The radii of the circles are given and so are the lengths of the connecting tangent spherical line segments. We show that these parameters determine  $T'$ , and so  $T$ , up to isometry. It is sufficient to show that the spherical triangle formed by the centers of  $C'_1, C'_2$  and  $C'_3$  is determined up to isometry. Denote these centers by  $c'_1, c'_2$  and  $c'_3$ .

Consider the spherical quadrilateral formed by the following four sides. One is the spherical line segment on the boundary of  $T'$  tangent to  $C'_1$  and  $C'_2$  (call this side the bottom base), the other two are the radii of  $C'_1$  and  $C'_2$  adjacent to the bottom base (call these two sides legs) and the fourth side is the spherical line segment connecting  $c'_1$  with  $c'_2$  (call this side the top base). Now, we have a spherical quadrilateral with the following informations: the length of the bottom base and the lengths of the two legs are given and the legs are perpendicular to the bottom base. All this determines the spherical quadrilateral up to isometry thus, the length of the top base is also determined. Hence, the spherical distance of  $c'_1$  and  $c'_2$  is determined. The same way we get that the other two sides of the spherical triangle  $\Delta_{c'_1 c'_2 c'_3}$  are also given thus, this triangle is determined up to isometry, finishing the proof of Theorem 0.2.

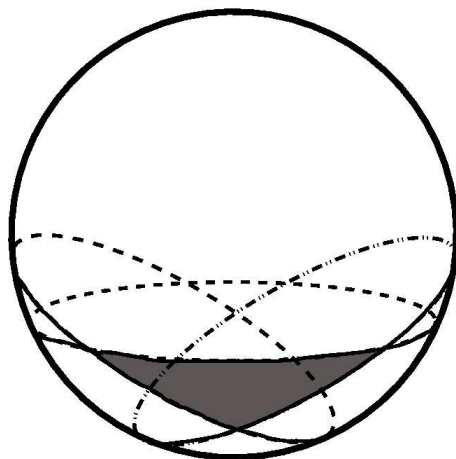


FIGURE 1.  $T$ , the gray region: a face of  $P$  bounded by three spherical circles.

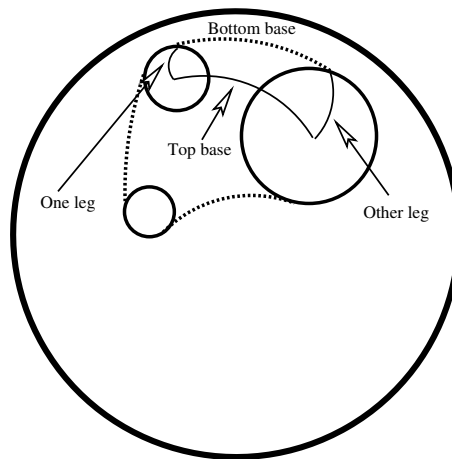


FIGURE 2.  $T'$ : The polar of  $T$  with one of the three spherical quadrilaterals.

## 6. PROOF OF PROPOSITION 0.2

First of all, we give a definition of triangulated polyhedra that are not necessarily convex.

**Definition.** Let  $K$  be a simplicial complex whose underlying space is homeomorphic to a sphere. A *triangulated polyhedron* in Euclidean 3-space is the image of  $K$  under a simplexwise linear map  $T : K \rightarrow \mathbb{R}^3$ . Where it causes no confusion we identify  $K$  with its image  $T(K)$ .

Suppose, that the Rigidity Conjecture for Simple Ball-Polyhedra holds. Let  $K$  be a triangulated polyhedron with the property that for the three vertices of any of its faces there is a unit ball containing those three vertices on its boundary and all the other vertices inside. We want to show that such a polyhedron is rigid according to the definition of rigidity set out in (\*).

Let  $V$  denote the set of vertices of  $K$ . Take the intersection of the unit balls corresponding to the faces of  $K$ . We get a ball-polyhedron  $P$  with the set of vertices containing  $V$ . In fact,  $V$  is identical to  $\mathcal{V}(P)$ . We claim more:  $P$  has the same face-lattice as  $K$ . To verify this, first observe that by the properties of  $K$  two vertices of  $K$  belong to the same face of  $P$  if and only if they are on the same face of  $K$ . As  $P$  has the same number of faces as  $K$  and as  $K$  is triangulated  $P$  cannot have more vertices than  $K$ . Thus,  $V$  is the set of vertices of  $P$ ,  $P$  is triangulated and the triangulation of  $P$  is the same as that of  $K$ . In other words,  $K$  is the underlying polyhedron of the ball-polyhedron  $P$ .

Now, we take the dual of  $P$ . It is a simple ball-polyhedron. If the Rigidity Conjecture for Simple Ball-Polyhedra holds, then the simple ball-polyhedron in question is rigid with respect to its inner dihedral angles. But those angles and the distances of those pairs of vertices of  $P$  that are connected by an edge in  $P$  determine each other. Since  $P$  and  $K$  have the same face-lattice, the inner dihedral angles of the dual of  $P$  and the edge lengths of  $K$  determine each other. Thus  $K$  is rigid.

Now suppose that (\*) holds. We want to prove the Rigidity Conjecture for Simple Ball-Polyhedra. Let  $P$  be a simple ball-polyhedron with no two faces meeting at more than one edge. If  $P$  is the intersection of 3 unit balls, then of course,  $P$  is rigid. So, we may assume that  $P$  is generated by at least four balls. If  $P$  has a face bounded by two edges only, then it has two edges that meet at more than one edge. This follows easily from the fact that exactly three edges meet in each vertex of a simple ball-polyhedron and that the edge graph of a ball-polyhedron is connected.

Thus,  $P$  does not have a biangle face, so the dual ball-polyhedron of  $P$  exists (denote it by  $P'$ ) and it is a triangulated ball-polyhedron the underlying polyhedron of which satisfies the condition in (\*). (The condition that  $P$  has no two faces meeting at more than one edge guarantees that the underlying simplicial complex of  $P'$  is homeomorphic to a sphere i.e. it is a triangulated polyhedron.) Since the inner dihedral angles of  $P$  and the edge lengths of the underlying polyhedron of the dual of  $P$  determine each other, the rigidity of the first implies the rigidity of the second with respect to its inner dihedral angles.

## 7. PROOF OF THEOREM 0.3

Let  $K$  be a triangulated convex polyhedron and  $\delta > 0$  be chosen. For each face of  $\delta K$  we take the unit ball containing the three vertices of the face on the boundary and the other vertices of  $\delta K$  inside (for small enough  $\delta$  it is possible). If  $\delta$  is small enough, then the intersection of these balls is a ball-polyhedron the face-lattice of which is the same as the face-lattice of  $\delta K$ . Thus, for example it has no faces bounded by two edges only. Hence, the dual of this ball-polyhedron is  $\mathbf{B}(\mathcal{V}(\delta K))$ .

We have to show that  $\mathbf{B}(\mathcal{V}(\delta K))$  is rigid with respect to its inner dihedral angles. Its inner dihedral angles determine the (Euclidean) distances of those pairs of points from  $\mathcal{V}(\delta K)$  that are connected by an edge in the dual ball-polyhedron of  $\mathbf{B}(\mathcal{V}(\delta K))$ . But the dual ball-polyhedron has the same face-lattice as  $\delta K$  which is a convex polytope. Hence, the inner dihedral angles of  $\mathbf{B}(\mathcal{V}(\delta K))$  determine the edge lengths of the triangulated convex polyhedron  $\delta K$ . Cauchy's rigidity theorem yields that  $\mathcal{V}(\delta K)$  is determined up to isometry by its edge lengths. Then,  $\mathcal{V}(\delta K)$  determines  $\mathbf{B}(\mathcal{V}(\delta K))$ , finishing our proof.

## REFERENCES

- [1] K. Bezdek and R. Connelly. Pushing disks apart – the Kneser-Poulsen conjecture in the plane, *J. Reine Angew. Math.* **553** (2002), pp.221-236.
- [2] A. Heppes and P. Révész, A splitting problem of Borsuk, *Mat. Lapok* **7** (1956), pp.108-111.
- [3] R. Connelly, Rigidity, in *Handbook of Convex Geometry*, eds.: P.M. Gruber and J.M. Wills, North Holland 1993, pp.223-271.
- [4] H. Gluck, Almost all simply connected surfaces are rigid, *Lecture Notes in Math.* **438**, Springer-Verlag, 1975, pp.225-239.
- [5] G.T. Sallee, Reuleaux Polytopes, *Mathematika* **17** (1970), pp.315-323.
- [6] W. Whiteley, Infinitesimally rigid polyhedra. I. Statics of frameworks, *Trans. Amer. Math. Soc.* **285/2** (1984), pp.431-464.
- [7] W. Whiteley, Infinitesimally rigid polyhedra. II. Modified spherical frameworks, *Trans. Amer. Math. Soc.* **306/1** (1988), pp.115-139.
- [8] L. Rodriguez, H. Rosenberg, Rigidity of certain polyhedra in  $\mathbb{R}^3$ , *Comment. Math. Helv.* **75**, (2000), pp.478-503
- [9] J.-M. Schlenker, A rigidity criterion for non-convex polyhedra in  $\mathbb{R}^3$ , *Manuscript* (2003), pp.1-13
- [10] M. Aigner and G.M. Ziegler *Proofs from the book*, Springer Verlag Berlin 1998.

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