Finding the best face on a Voronoi polyhedron—the strong dodecahedral conjecture revisited

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Abstract

In this paper we prove the following theorem. The surface area density of a unit ball in any face cone of a Voronoi cell in an arbitrary packing of unit balls of Euclidean 3-space is at most

$$\frac{-9\pi + 30\arccos\left(\frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{5}\right)\right)}{5\cdot\tan\left(\frac{\pi}{5}\right)} = 0.77836\ldots,$$

and so the surface area of any Voronoi cell in a packing with unit balls in Euclidean 3-space is at least

$$\frac{20\pi\cdot\tan\left(\frac{\pi}{5}\right)}{-9\pi + 30\arccos\left(\frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{5}\right)\right)} = 16.1445\ldots.$$

This result and the ideas of its proof support the Strong Dodecahedral Conjecture according to which the surface area of any Voronoi cell in a packing with unit balls in Euclidean 3-space is at least as large as 16.6508... the surface area of a regular dodecahedron of inradius 1.

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1 Introduction

A family of non-overlapping 3-dimensional balls of radii 1 in Euclidean 3-space, $\mathbb{E}^3$ is called a unit ball packing in $\mathbb{E}^3$. The density of the packing is the proportion of space covered by these unit balls. The sphere packing problem asks for the densest packing of unit balls in $\mathbb{E}^3$. The conjecture that the density of any unit ball packing in $\mathbb{E}^3$ is at most $\frac{\pi}{\sqrt{18}} = 0.74078\ldots$ is often attributed to Kepler. Using an ingenious argument which works in any dimension, Rogers [13] obtained the upper bound $0.77963\ldots$ for the density of unit ball packings in $\mathbb{E}^3$. This bound has been improved by Lindsey [10], and Muder [11], [12] to $0.773055\ldots$. Hsiang [8], [9] proposed an elaborate line of attack (along the ideas of L. Fejes Tóth suggested 40 years earlier), but his claim that he settled Kepler's conjecture seems exaggerated. However, so far no one has found any serious gap in the approach of Hales [3], [4], [5], [6], although no one has been able to fully verify it either. This is not too surprising, given that the detailed argument is described in several papers and relies on long computer aided calculations of more than 5000 subproblems.

For several of the above mentioned papers Voronoi cells of unit ball packings play a central role. Recall that the Voronoi cell of a unit ball in a packing of unit balls in $\mathbb{E}^3$ is the set of points that are not farther away from the center of the given ball than from any other ball's center. As it is well-known, the Voronoi cells of a unit ball packing in $\mathbb{E}^3$ form a tiling of $\mathbb{E}^3$. One of the most attractive problems on Voronoi cells is the Dodecahedral Conjecture first phrased by L. Fejes Tóth in [2]. According to this the volume of any Voronoi cell in a packing of unit balls in $\mathbb{E}^3$ is at least as large as the volume of a regular dodecahedron with inradius 1. Very recently Hales and McLaughlin [7] announced a solution to this problem.

In this paper we study the following stronger version of the Dodecahedral Conjecture often called the Strong Dodecahedral Conjecture. It was first phrased in [1] and it says that the surface area of any Voronoi cell in a packing with unit balls in $\mathbb{E}^3$ is at least as large as $16.6508\ldots$ the surface area of a regular dodecahedron of inradius 1. It is easy to see that if true, then it implies the Dodecahedral Conjecture. The best known result in connection with the Strong Dodecahedral Conjecture is Theorem 4 in [1] according to which the surface area of any Voronoi cell in a packing with unit balls in $\mathbb{E}^3$ is at least $16.1433\ldots$. In this paper we improve this lower bound by using ideas of Muder [11] and some methods from [1]. In order to phrase our main result we introduce a bit of terminology. A face cone of a Voronoi cell in a packing with unit balls in $\mathbb{E}^3$ is the convex hull of the face chosen and the center of the unit ball sitting in the given Voronoi cell. The surface area density of a unit ball in a face cone is simply the spherical area of the region of the unit sphere (centered at the apex of the face cone) that belongs to the face cone divided by the Euclidean area of the face. It should be clear from these definitions that if we have an upper bound for the surface area density in face cones of Voronoi cells, then the reciprocal of this upper bound times 4π

(the surface area of a unit ball) is a lower bound for the surface area of Voronoi cells. In the remaining sections of this paper we give a detailed proof of the following statement.

Theorem 1.1 *The surface area density of a unit ball in any face cone of a Voronoi cell in an arbitrary packing of unit balls of* $\mathbb{E}^3$ *is at most*

$$\frac{-9\pi + 30\arccos\left(\frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{5}\right)\right)}{5\tan\left(\frac{\pi}{5}\right)} = 0.77836\ldots,$$

and so the surface area of any Voronoi cell in a packing with unit balls in E^3 *is at least*

$$\frac{20\pi\tan\left(\frac{\pi}{5}\right)}{-9\pi + 30\arccos\left(\frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{5}\right)\right)} = 16.1445\ldots.$$

Moreover, the above upper bound $0.77836\ldots$ *for the surface area density is best possible in the following sense. The surface area density in the face cone of any* n*-sided face with* $n = 4, 5$ *of a Voronoi cell in an arbitrary packing of unit balls of* $\mathbb{E}^3$ *is at most*

$$\frac{3(2-n)\pi + 6n\cdot\arccos\left(\frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{n}\right)\right)}{n\tan\left(\frac{\pi}{n}\right)}$$

and equality is achieved when the face is a regular n*-gon inscribed in a circle of radius* $\frac{1}{\sqrt{3}\cdot\cos\left(\frac{\pi}{n}\right)}$ *and positioned such that it is tangent to the corresponding unit ball of the packing at its center.*

2 Notations

Let $\mathbb{P}$ be an arbitrary packing of unit balls in 3-dimensional Euclidean space $\mathbb{E}^3$. Let $B(A)$ (resp., $B_r(A)$) denote the closed ball of radius 1 (resp., r) centered at the point A of $\mathbb{E}^3$. We often will need to refer to the boundary of $B(A)$ (resp., $B_r(A)$) by writing $S(A)$ (resp., $S_r(A)$). We will need the notation $\mathbb{C}$ for the set of the centers of the unit balls in $\mathbb{P}$ and C will stand for an arbitrary center in $\mathbb{C}$. The most important geometric object for our investigation of $\mathbb{P}$ is the Voronoi cell $V(C)$ centered at $C \in \mathbb{C}$ which is the collection of all points of $\mathbb{E}^3$ that are not closer to any $C' \in \mathbb{C}$, $C' \neq \mathbb{C}$ than to C. Also, let $\mathbb{V} = \{V(C) \mid C \in \mathbb{C}\}$. Finally, we will need some further notations as well.

$\mathcal{F}$:	an arbitrary face of the Voronoi cell $V(C)$;
$\mathbb{F}$:	the family of all faces of $V(C)$;
$\mathcal{P}$:	the plane of the arbitrary face $\mathcal{F}$ of $V(C)$;
M :	the orthogonal projection of C onto $\mathcal{P}$;
$\mathcal{R}$:	the closed region bounded by a simple polygon of $\mathcal{P}$;
$C(\mathcal{R})$:	the Voronoi cone with apex $C \in \mathbb{C}$ and base $\mathcal{R}$ (i.e. the union of line segments connecting C and $\mathcal{R}$);
$d(\cdot,\cdot)$:	the Euclidean distance between two points (resp., sets) of $\mathbb{E}^3$;
h :	the height $d(C, M)$ of the cone $C(\mathcal{R})$;
$\|\cdot\|$:	the Euclidean norm of vectors in $\mathbb{E}^3$;
area$(\cdot)$:	the area measure of planar regions;
Sarea$(\cdot)$:	the spherical area measure of regions on a sphere;
vol$(\cdot)$:	the volume measure in $\mathbb{E}^3$;
$\Delta(\mathcal{R})$:	the volume density in the Voronoi cone $C(\mathcal{R})$ defined as $\mathrm{vol}(C(\mathcal{R}) \cap B(C))/\mathrm{vol}(C(\mathcal{R}))$;
$\hat{\Delta}(\mathcal{R})$:	the surface area density in the Voronoi cone $C(\mathcal{R})$ defined as $\mathrm{Sarea}(C(\mathcal{R}) \cap S(C))/\mathrm{area}(\mathcal{R})$.

3 The surface area density in face cones of Voronoi cells

Let $\mathcal{F}$ be an arbitrary face of the Voronoi cell $V(C) \in \mathbb{V}$ with center $C \in \mathbb{C}$ of the unit ball packing $\mathbb{P}$ in $\mathbb{E}^3$. The following is a rather well-known fact (see [13]).

Lemma 3.1 *If $e = P_0P_1$ is an arbitrary edge of $\mathcal{F}$ and ℓ denotes the line spanned by e, then*

$$d(C, \ell) \geq \frac{2}{\sqrt{3}} \quad \textit{and} \quad d(C, P_i) \geq \sqrt{\frac{3}{2}}, \quad i = 0, 1. \tag{1}$$

Another simple observation is the following.

Lemma 3.2 *Let $\mathcal{R}$ be an arbitrary closed polygonal region bounded by a simple polygon in the plane $\mathcal{P}$ of the face $\mathcal{F}$ of the Voronoi cell $V(C) \in \mathbb{V}$. If $\mathcal{R} \cap B_r(C) = \emptyset$ for some $r \geq 1$, then*

$$\hat{\Delta}(R) = \frac{\mathit{Sarea}(C(\mathcal{R}) \cap S(C))}{\mathit{area}(\mathcal{R})} < \frac{1}{r^2}.$$

Proof. Note that $B(C) \subset V(C)$. Let $\Sigma = C(\mathcal{R}) \cap S(C)$ and $\Sigma_r = C(\mathcal{R}) \cap S_r(C)$. If $h = d(C, M) = d(C, \mathcal{P}) \leq r$, then $\frac{1}{3} r \cdot \mathrm{Sarea}(\Sigma_r) < \frac{1}{3} h \cdot \mathrm{area}(\mathcal{R}) \leq \frac{1}{3} r \cdot \mathrm{area}(\mathcal{R})$ and therefore $\mathrm{Sarea}(\Sigma_r) < \mathrm{area}(\mathcal{R})$. In this case let $\mathcal{R}'$ denote $\mathcal{R}$. However, if $h > r$, then let $\mathcal{P}'$ be the plane that is tangent to $B_r(C)$ and strictly separates C from the plane $\mathcal{P}$ and let $\mathcal{R}' = C(\mathcal{R}) \cap \mathcal{P}'$. As $r = d(C, \mathcal{P}') < d(C, \mathcal{P}) = h$ therefore $\mathrm{Sarea}(\Sigma_r) < \mathrm{area}(\mathcal{R}') < \mathrm{area}(\mathcal{R})$. Thus, in both cases we obtain that

$$\hat{\Delta}(R) = \frac{\mathrm{Sarea}\,(\Sigma)}{\mathrm{area}(R)} = \frac{\frac{1}{r^2}\mathrm{Sarea}\,(\Sigma_r)}{\mathrm{area}(\mathcal{R})} < \frac{\mathrm{area}(R')}{r^2 \mathrm{area}(R)} \leq \frac{1}{r^2}.$$

finishing the proof of Lemma 3.2. ■

Now, we are in a position to prove the following important observation.

Lemma 3.3 *If $M \notin \mathcal{F}$ or $h \geq \frac{2}{\sqrt{3}}$, then $\hat{\Delta}(\mathcal{F}) \leq \frac{3}{4}$.*

Proof. If $M \notin F$, then (1) implies that $\mathcal{F} \cap B_{2/\sqrt{3}}(C) = \emptyset$. If $h > \frac{2}{\sqrt{3}}$, then obviously $\mathcal{F} \cap B_{2/\sqrt{3}}(C) = \emptyset$. Thus, in both cases Lemma 3.2 implies that $\hat{\Delta}(\mathcal{F}) < \frac{3}{4}$. Finally, the inequality $\hat{\Delta}(\mathcal{F}) \leq \frac{3}{4}$ follows again from Lemma 3.2 in the case left namely, when $h = \frac{2}{\sqrt{3}}$. ■

As we look only for those faces $\mathcal{F}$ of the Voronoi cones whose surface area density is high i.e. for which $\hat{\Delta}(\mathcal{F}) > \frac{3}{4}$, Lemma 3.3 yields that from now on we need to look at only those faces $\mathcal{F}$ of the Voronoi cells in the unit ball packing $\mathbb{P}$ for which $M \in \mathcal{F}$ and $1 \leq h < \frac{2}{\sqrt{3}}$.

4 Wedges

Following Muder [11] we introduce wedges lying in the plane $\mathcal{P}$ of the face $\mathcal{F}$ of the Voronoi cell $V(C) \in \mathbb{V}$ as follows. We say that the triangle $\Delta MPQ \subset \mathcal{P}$ is a wedge if $d(M, P) \geq \sqrt{\frac{3}{2} - h^2}$, $d(M, Q) \geq \sqrt{\frac{3}{2} - h^2}$ and the distance between M and the line of the side PQ is at least $\sqrt{\frac{4}{3} - h^2}$. (Recall that M is the orthogonal projection of C onto $\mathcal{P}$ and that if PQ is a side of the face $\mathcal{F}$, then according to Lemma 3.1 and the assumption that $1 \leq h < \frac{2}{\sqrt{3}}$ ΔMPQ is a wedge.) M is called the basepoint of the wedge ΔMPQ and the angle $\angle PMQ$ is its baseangle. Then let $\eta(h)$ be the baseangle of the wedge ΔMPQ with the property that $d(M, P) = d(M, Q) = \sqrt{\frac{3}{2} - h^2}$ and that the distance between M and the line of the side PQ is equal to $\sqrt{\frac{4}{3} - h^2}$. An easy computation yields that

$$\eta(h) = 2 \arccos \sqrt{\frac{8 - 6h^2}{9 - 6h^2}}, \quad 1 \leq h < \frac{2}{\sqrt{3}}. \tag{2}$$

Now, with the help of $\eta(h)$ we can define standard wedges as follows. Pick an arbitrary polar coordinate system in the plane $\mathcal{P}$ such that the origin with coordinates $(0,0)$ is identical to M. Then we call the wedge ΔMPQ with baseangle Θ, $0<\Theta<\pi$ a standard wedge, labelled somewhat vaguely as $W_h(\Theta)$, if in the case of $0<\Theta\le\eta(h)$ the vertices P and Q have polar coordinates $\left(\sqrt{\frac{3}{2}-h^2},\frac{1}{2}\Theta\right)$ and $\left(\sqrt{\frac{3}{2}-h^2},-\frac{1}{2}\Theta\right)$ and if in the case of $\eta(h)<\Theta<\pi$ the vertices P and Q have polar coordinates $\left(\sqrt{\left(\frac{4}{3}-h^2\right)\left(1+\tan^2\frac{\Theta}{2}\right)},\frac{1}{2}\Theta\right)$ and $\left(\sqrt{\left(\frac{4}{3}-h^2\right)\left(1+\tan^2\frac{\Theta}{2}\right)},-\frac{1}{2}\Theta\right)$. It is easy to check that $W_h(\Theta)$ is indeed a wedge. (Strictly speaking $W_h(\Theta)$ denotes the family of pairwise congruent standard wedges in $\mathcal{P}$ that have the basepoint M and baseangle Θ.) Finally, take a standard wedge $W_h(\Theta)$ and take its Voronoi cone i.e. the convex hull of C and $W_h(\Theta)$ and denote the volume density in this cone by $\Delta(W_h(\Theta))$. Muder [11] (Lemma 3.3) proved the following

Lemma 4.1 *Let $1\le h<\frac{2}{\sqrt{3}}$. Then*

$$\Delta(W_h(\Theta))=\begin{cases}\dfrac{4(\Theta-\pi)+8\arccos\left(\dfrac{h\sqrt{2(1-\cos\Theta)}}{\sqrt{3(1+\cos\Theta)+2h^2(1-\cos\Theta)}}\right)}{h(3-2h^2)\sin\Theta}, & 0<\Theta\le\eta(h);\\[2ex] \dfrac{3\left(\Theta-\pi\right)+6\arccos\left(\frac{\sqrt{3}}{2}h\sin\left(\frac{\Theta}{2}\right)\right)}{h(4-3h^2)\tan\left(\dfrac{\Theta}{2}\right)}, & \eta(h)<\Theta<\pi.\end{cases}$$

Although the explicit formula for $\Delta(W_h(\Theta))$ looks complicated the function $\delta(\Theta,h)=\Delta(W_h(\Theta))$ has some pleasant properties described by Muder [11] (Lemma 3.5) as follows.

Lemma 4.2 *The function $\delta(\Theta,h)=\Delta(W_h(\Theta))$ with $0<\Theta<\pi$ and $1\le h<\frac{2}{\sqrt{3}}$ possesses the following monotonicity properties:*

(i) for a fixed h the function $\delta(\Theta,h)$ is increasing in Θ for $0<\Theta\le\eta(h)$;

(ii) for a fixed h the function $\delta(\Theta,h)$ is decreasing in Θ for $\eta(R)<\Theta<\pi$;

(iii) for a fixed Θ the function $d(\Theta,h)$ is decreasing in h for $h_\Theta\le h<\frac{2}{\sqrt{3}}$, where $h_\Theta=\eta^{-1}(\Theta)$ for $\eta(1)\le\Theta<\pi$ and $h_\Theta=1$ for $0<\Theta<\eta(1)$;

(iv) the function $\delta(\eta(h),h)$ is decreasing in h for $1\le h<\frac{2}{\sqrt{3}}$.

Remark 4.3 *It was conjectured by Muder [11] (Conjecture 3.7) that for any fixed* Θ *with* $0 < \Theta < \pi$ *the function* $\delta(\Theta, h)$ *is decreasing in* h *for* $1 \leq h < \frac{2}{\sqrt{3}}$. *Without going into details we just mention here that this conjecture is false namely, for any fixed* Θ *with* $2.1456 \leq \Theta < \pi$ *the function* $\delta(\Theta, h)$ *is not a decreasing function of* h *with* $1 \leq h < \frac{2}{\sqrt{3}}$.

5 Wedge clusters and their surface area densities

The ordered family $(W_1, W_2, \ldots, W_n)$ of the wedges $W_1, W_2, \ldots, W_n$ lying in the plane $\mathcal{P}$ of the face $\mathcal{F}$ of the Voronoi cell $V(C) \in \mathbb{V}$ is called a wedge cluster of order n if the sum of their baseangles is 2π and, if the intersection of any two wedges is either equal to M or a line segment (of positive length) with M as one endpoint depending on whether the two wedges are non-adjacent or adjacent according to the given cyclic ordering. Finally, a wedge cluster of order n is called a standard wedge cluster of order n if all of its wedges are standard wedges. Thus, a standard wedge cluster of order n lying in the plane $\mathcal{P}$ of the face $\mathcal{F}$ of the Voronoi cell $V(C) \in \mathbb{V}$ can be represented as $(W_h(\Theta_1), W_h(\Theta_2), \ldots, W_h(\Theta_n))$. Now, take the Voronoi cone of this standard wedge cluster of order n i.e. take the union of line segments connecting C and $W_h(\Theta_1) \cup W_h(\Theta_2) \cup \ldots \cup W_h(\Theta_n)$ and denote the surface area density in this cone by $\hat{\delta}_{n,h}(\Theta_1, \Theta_2, \ldots, \Theta_n)$. In general, if $(W_1, W_2, \ldots, W_n)$ is a wedge cluster of order n in the plane $\mathcal{P}$ of the face $\mathcal{F}$ of the Voronoi cell $V(C) \in \mathbb{V}$, then its Voronoi cone is the union of line segments connecting C and $W_1 \cup W_2 \cup \ldots \cup W_n$ and the surface area density in this Voronoi cone is denoted by $\hat{\Delta}(W_1, W_2, \ldots, W_n)$. (In some cases we will prefer the notation $\hat{\Delta}(W_1 \cup W_2 \cup \ldots \cup W_n)$ over $\hat{\Delta}(W_1, W_2, \ldots, W_n)$.) In particular, if W is a wedge in the plane $\mathcal{P}$ of the face $\mathcal{F}$ of the Voronoi cell $V(C) \in \mathbb{V}$, then $\hat{\Delta}(W)$ denotes the surface area density in the Voronoi cone of W. We close this section with two inequalities that show the importance of standard wedges.

Lemma 5.1 *If* Θ *is the baseangle of the wedge* W *with* $\frac{3}{4} < \hat{\Delta}(W)$, *then*

$$\hat{\Delta}(W) \leq \hat{\Delta}(W_h(\Theta)) = \hat{\delta}(\Theta, h) \tag{3}$$

Proof. First, note that if $\mathcal{R}_0 \subset \mathcal{P}, \mathcal{R}_1 \subset \mathcal{P}$ are given such that $\mathcal{R}_0 \cap \mathcal{R}_1 = \emptyset$ and $\hat{\Delta}(\mathcal{R}_1) \leq \hat{\Delta}(\mathcal{R}_0 \cup \mathcal{R}_1)$, then an easy computation yields that $\hat{\Delta}(\mathcal{R}_0 \cup \mathcal{R}_1) \leq \hat{\Delta}(\mathcal{R}_0)$. Second, a statement of Muder [11] (Lemma 3.1) implies in a straightforward way that there are $\mathcal{R}_0 \subset W$, $\mathcal{R}_1 \subset W$ with $\mathcal{R}_0 \cap \mathcal{R}_1 = \emptyset$ and $\mathcal{R}_0 \cup \mathcal{R}_1 = W$ having the following properties:

$$\text{area}(\mathcal{R}_0) = \text{area}(W_h(\Theta)), \tag{4}$$

$$\text{Sarea}(C(\mathcal{R}_0) \cap S(C)) = \text{Sarea}(C(W_h(\Theta)) \cap S(C)), \tag{5}$$

$$d(\mathcal{R}_1, C) \geq \frac{2}{\sqrt{3}}. \tag{6}$$

Now, taking the above partition of W into $\mathcal{R}_0$ and $\mathcal{R}_1$ we get via Lemma 3.2 and (6) that $\hat{\Delta}(\mathcal{R}_1) \leq \frac{3}{4} < \hat{\Delta}(W) = \Delta(\mathcal{R}_0 \cup \mathcal{R}_1)$. Then, applying our first observation we get via (4) and (5) that $\hat{\Delta}(\mathcal{R}_0 \cup \mathcal{R}_1) \leq \hat{\Delta}(\mathcal{R}_0) = \hat{\Delta}(W_h(\Theta)) = \hat{\delta}(\Theta, h)$, finishing the proof of (3). ■

Lemma 5.2 *If $(W_1, W_2, \dots, W_n)$ is a wedge cluster of order n with baseangles $\Theta_1, \Theta_2, \dots, \Theta_n$ such that $\frac{3}{4} < \hat{\Delta}(W_1, W_2, \dots, W_n)$, then*

$$\hat{\Delta}(W_1, W_2, \dots, W_n) \leq \hat{\delta}_{n,h}(\Theta_1, \Theta_2, \dots, \Theta_n). \tag{7}$$

Proof. Partitioning W_i into $\mathcal{R}_{i,0}$ and $\mathcal{R}_{i,1}$ as in (4), (5), and (6) we get that

$$\mathrm{area}(\mathcal{R}_{i,0}) = \mathrm{area}(W_h(\Theta_i)) \tag{8}$$

$$\mathrm{Sarea}(C(\mathcal{R}_{i,0}) \cap S(C)) = \mathrm{Sarea}(C(W_h(\Theta_i)) \cap S(C)), \tag{9}$$

$$d(\mathcal{R}_{i,1}, C) \geq \frac{2}{\sqrt{3}} \tag{10}$$

hold for all $i = 1, 2, \dots, n$. Thus, (10) and Lemma 3.2 imply that $\hat{\Delta}(\mathcal{R}_{i,1}) \leq \frac{3}{4}$, $i = 1, 2, \dots, n$ and so $\hat{\Delta}\left(\bigcup_{i=1}^{n} \mathcal{R}_{i,1}\right) \leq \frac{3}{4}$. However, by assumption $\frac{3}{4} < \hat{\Delta}(W_1, W_2, \dots, W_n) = \hat{\Delta}\left(\left(\bigcup_{i=1}^{n} \mathcal{R}_{i,0}\right) \cup \left(\bigcup_{i=1}^{n} \mathcal{R}_{i,1}\right)\right)$ and therefore by applying (8) and (9) we get that $\hat{\Delta}\left(\left(\bigcup_{i=1}^{n} \mathcal{R}_{i,0}\right) \cup \left(\bigcup_{i=1}^{n} \mathcal{R}_{i,1}\right)\right) \leq \hat{\Delta}\left(\bigcup_{i=1}^{n} \mathcal{R}_{i,0}\right) = \hat{\delta}_{n,h}(\Theta_1, \Theta_2, \dots, \Theta_n)$, finishing the proof of (7). ■

6 The monotonicity properties of $\hat{\delta}(\Theta, h) = \hat{\Delta}(W_h(\Theta))$

It is easy to see that if $\mathcal{R}$ is an arbitrary region in the plane $\mathcal{P}$ of the face $\mathcal{F}$ of the Voronoi cell $V(C) \in \mathbb{V}$, then

$$\hat{\Delta}(\mathcal{R}) = h\Delta(\mathcal{R}). \tag{11}$$

Then (11) and Lemma 4.1 imply in a straightforward way the following important formula for $\hat{\delta}(\Theta, h)$.

Lemma 6.1 *Let $1 \le h < \frac{2}{\sqrt{3}}$. Then*

$$\hat{\Delta}(W_h(\Theta)) = \begin{cases} \dfrac{4(\Theta-\pi) + 8\arccos\left(\dfrac{h\sqrt{2(1-\cos\Theta)}}{\sqrt{3(1+\cos\Theta)+2h^2(1-\cos\Theta)}}\right)}{(3-2h^2)\sin\Theta}, & 0 < \Theta \le \eta(h); \\[2ex] \dfrac{3(\Theta-\pi) + 6\arccos\left(\frac{\sqrt{3}}{2}\cdot h\cdot \sin\left(\frac{\Theta}{2}\right)\right)}{(4-3h^2)\tan\left(\frac{\Theta}{2}\right)}, & \eta(h) < \Theta < \pi. \end{cases}$$

It is remarkable that the function $\hat{\delta}(\Theta, h)$, just like the function $\delta(\Theta, h)$, has some pleasant monotonicity properties described in the following statement.

Lemma 6.2 *The function $\hat{\delta}(\Theta, h) = \hat{\Delta}(W_h(\Theta))$ with $0 < \Theta < \pi$ and $1 \le h < \frac{2}{\sqrt{3}}$ possesses the following monotonicity properties:*

(i) for a fixed h the function $\hat{\delta}(\Theta, h)$ is increasing in Θ for $0 < \Theta \le \eta(h)$;

(ii) for a fixed h the function $\hat{\delta}(\Theta, h)$ is decreasing in Θ for $\eta(h) < \Theta < \pi$;

(iii) for a fixed Θ the function $\hat{\delta}(\Theta, h)$ is decreasing in h for $h_\Theta \le h < \frac{2}{\sqrt{3}}$, where $h_\Theta = \eta^{-1}(\Theta)$ for $\eta(1) \le \Theta < \pi$ and $h_\Theta = 1$ for $0 < \Theta < \eta(1)$;

(iv) the function $\hat{\delta}(\eta(h), h)$ is decreasing in h for $1 \le h < \frac{2}{\sqrt{3}}$.

Proof. (i) and (ii) follow from Lemma 4.2 via (11) in a direct way. However, (iii) and (iv) of Lemma 6.2 together with (11) imply the properties (iii) and (iv) in Lemma 4.2 and so not surprisingly for the proof of (iii) and (iv) in Lemma 6.2 we need an approach very different from that of Muder [11]. The details are as follows.

The proofs of (iii) and (iv) are based on the following statement which is a special case of an extension of Rogers' lemma proved by Bezdek in [1].

Lemma 6.3 *Let the triangles $\Delta U_1U_2U_3 = U$ and $\Delta V_1V_2V_3 = V$ be given by the position vectors u_1, u_2, u_3 and v_1, v_2, v_3 pointing from the origin O to the corresponding vertices of the triangles in $\mathbb{E}^3$. If $1 \le \|u_i\| \le \|v_i\|$ for all $i = 1, 2, 3$ moreover, u_1 (resp., u_2) is orthogonal to the plane of U (resp., to the line of the segment U_2U_3) and $\|v_1\| = d(O, V)$, $\|v_2\| = d(O, V_2V_3)$, then*

$$\hat{\Delta}(U) = \frac{Sarea(conv(O \cup U) \cap S(O)}{area(U)} \ge \hat{\Delta}(V) = \frac{Sarea(conv(O \cup V) \cap S(O))}{area(V)},$$

where $conv(\cdot)$ refers to the convex hull of the corresponding set.

In order to prove (iii) let Θ be fixed and let $1 \leq h_\Theta \leq h_1 \leq h_2 < \frac{2}{\sqrt{3}}$. Then of course, $\Theta \leq \eta(h_1) \leq \eta(h_2) < \pi$. Let $W_{h_1}(\Theta)$ (resp., $W_{h_2}(\Theta)$) be the standard wedge identical to $\Delta M_1P_1Q_1$ (resp., $\Delta M_2P_2Q_2$) with baseangle Θ. Moreover, let N_i be the midpoint of P_iQ_i, $i = 1, 2$. Then it is easy to see that

$$1 \leq d(C, M_1) = h_1 \leq h_2 = d(C, M_2);$$

$$h_1 < d(C, N_1) = \sqrt{h_1^2 \sin^2\left(\frac{\Theta}{2}\right) + \frac{3}{2}\cos^2\left(\frac{\Theta}{2}\right)} \leq$$

$$\sqrt{h_2^2 \sin^2\left(\frac{\Theta}{2}\right) + \frac{3}{2}\cos^2\left(\frac{\Theta}{2}\right)} = d(C, N_2);$$

$$d(C, N_1) < d(C, P_1) = \sqrt{\frac{3}{2}} = d(C, P_2) \quad \text{and}$$

CM_1 (resp., CM_2) is orthogonal to the plane of $\Delta M_1N_1P_1$ (resp., $\Delta M_2N_2P_2$) and CN_1 (resp., CN_2) is orthogonal to N_1P_1 (resp., N_2P_2). Thus, using the fact that $\Delta M_1P_1Q_1$ (resp., $\Delta M_2P_2Q_2$) is symmetric about the line of M_1N_1 (resp., M_2N_2) Lemma 6.3 implies that $\hat{\delta}(\Theta, h_1) = \hat{\Delta}(W_{h_1}(\Theta)) \geq \hat{\Delta}(W_{h_2}(\Theta)) = \hat{\delta}(\Theta, h_2)$, finishing the proof of (iii).

For the proof of (iv) let $1 \leq h_1 \leq h_2 < \frac{2}{\sqrt{3}}$ and let $W_{h_1}(\eta(h_1))$ and $W_{h_2}(\eta(h_2))$ be the corresponding standard wedges identical to $\Delta M_1P_1Q_1$ anre $\Delta M_2P_2Q_2$ with baseangles $\eta(h_1) \leq \eta(h_2)$. Moreover, as above let N_i be the midpoint of P_iQ_i, $i = 1, 2$. Then it is easy to see that

$$1 \leq d(C, M_1) = h_1 \leq h_2 = d(C, M_2);$$

$$h_1 < d(C, N_1) = \frac{2}{\sqrt{3}} = d(C, N_2);$$

$$d(C, N_1) < d(C, P_1) = \sqrt{\frac{3}{2}} = d(C, P_2) \quad \text{and}$$

CM_1 (resp., CM_2) is orthogonal to the plane of $\Delta M_1N_1P_1$ (resp., $\Delta M_2N_2P_2$) and CN_1 (resp., CN_2) is orthogonal to N_1P_1 (resp., N_2P_2). Thus, using the fact that $\Delta M_1P_1Q_1$ (resp., $\Delta M_2P_2Q_2$) is symmetric about the line of M_1N_1 (resp., M_2N_2) Lemma 6.3 implies that $\hat{\delta}(\eta(h_1), h_1) = \hat{\Delta}(W_{h_1}(\eta(h_1))) \geq \hat{\Delta}(W_{h_2}(\eta(h_2)) = \hat{\delta}(\eta(h_2), h_2)$, finishing the proof of (iv). ■

Remark 6.4 *Without going into details we just mention here that (iii) of Lemma 6.2 cannot be extended over a larger interval namely, for any fixed* Θ *with* $1.649 \leq \Theta < \pi$ *the function* $\hat{\delta}(\Theta, h)$ *is not a decreasing function of* h *with* $1 \leq h < \frac{2}{\sqrt{3}}$.

7 The surface area density in the Voronoi cone of an arbitrary n-sided face of a Voronoi cell with $n = 3, 4, 5$ is at most $\hat{\delta}\left(\frac{2\pi}{5}, 1\right) = 0.77836\ldots$

Recall that the function $\hat{\delta}_{n,h}(\Theta_1, \Theta_2, \ldots, \Theta_n)$ is well defined for arbitrary $0 < \Theta_1 < \pi$, $0 < \Theta_2 < \pi, \ldots, 0 < \Theta_n < \pi$ and for any given $1 \leq h < \frac{2}{\sqrt{3}}$ and $n = 3, 4, 5, \ldots$. Clearly, fixing n and h the function $\hat{\delta}_{n,h}$ can be extended continuously to allow $\Theta_i = 0$ since the degenerate wedges have 0 area and do not effect the calculation of $\hat{\delta}_{n,h}$. We can also extend $\hat{\delta}_{n,h}$ to allow $\Theta_i = \pi$ be defining $\hat{\delta}_{n,h}(\Theta_1, \Theta_2, \ldots, \Theta_n) = 0$ in these cases. This is a continuous extension since $\lim_{\Theta \to \pi} \text{area}(W_h(\Theta)) = +\infty$, while all other significant quantities remain finite. These extensions make the domain of $\hat{\delta}_{n,h}$ compact and guarantee that $\hat{\delta}_{n,h}$ achieves its maximum.

Lemma 7.1 *If $\hat{\delta}_{n,h}$ achieves a maximum greater than $\frac{3}{4}$ at $(\Theta_1, \Theta_2, \ldots, \Theta_n)$ and $\Theta_i > \eta(h)$ for some $1 \leq i \leq n$, then $\Theta_j = \frac{2\pi}{n}$ for all $1 \leq j \leq n$.*

Proof. Without loss of generality we may assume that $\Theta_1 > \eta(h)$. Using our earlier notation for the standard wedge $W_h(\Theta_1)$, this means that

$$d(M, P_1) = d(M, Q_1) = \frac{\sqrt{\frac{4}{3} - h^2}}{\cos\left(\frac{\Theta_1}{2}\right)} > \sqrt{\frac{3}{2} - h^2}. \tag{12}$$

If $\Theta_j = \Theta_1$ for all $1 \leq j \leq n$, then of course, we are done. Thus, without loss of generality we may assume that

$$\Theta_1 > \Theta_2. \tag{13}$$

Also, we can assume that the consecutive standard wedges $W_h(\Theta_1)$ and $W_h(\Theta_2)$ are identical to MP_1Q_1 and ΔMP_2Q_2 such that M, Q_1 and P_2 are on a line. If $\Theta_2 \leq \eta(h)$, then $d(M, P_2) = \sqrt{\frac{3}{2} - h^2}$ and therefore (12) implies that $d(M, Q_1) > d(M, P_2)$. If $\Theta_2 > \eta(h)$, then (12) and (13) yield that

$$d(M, P_2) = \frac{\sqrt{\frac{4}{3} - h^2}}{\cos\left(\frac{\Theta_2}{2}\right)} < \frac{\sqrt{\frac{4}{3} - h^2}}{\cos\left(\frac{\Theta_1}{2}\right)} = d(M, Q_1).$$

Thus, always $d(M, P_2) < d(M, Q_1)$ i.e., P_2 is always strictly between the points M and Q_1. As a result we get that the line of P_2Q_2 must intersect $MP_1 \cup P_1Q_1$ in a point say, T. Now, either $T \in P_1Q_1$ or $T \in MP_1$. If $T \in P_1Q_1$, then it is easy to check that $\Delta MP_1T = W_1$

and $\Delta MTQ_2 = W_2$ are wedges moreover, if $\mathcal{R} = \Delta Q_1TP_2$, then $d(C,\mathcal{R}) \geq \frac{2}{\sqrt{3}}$. Finally, if $T \in MP_1$, then $\Delta MTP_2 = W_1$ and $\Delta MP_2Q_2 = W_2$ are wedges moreover, using the notation $\mathcal{R}$ for the quadrilateral $TP_1Q_1P_2$ we get that $d(C,\mathcal{R}) \geq \frac{2}{\sqrt{3}}$. Hence, in both cases we have that

$$\bigcup_{i=1}^{n} W_h(\Theta_i) = W_1 \cup W_2 \cup \left(\bigcup_{i=3}^{n} W_h(\Theta_i) \right) \cup \mathcal{R}. \tag{14}$$

Thus, Lemma 3.2 implies that $\hat{\Delta}(Q) \leq \frac{3}{4} < \hat{\Delta}\left(\bigcup_{i=1}^{n} W_h(\Theta_i)\right)$ and therefore (14) implies that

$$\hat{\delta}_{n,h}(\Theta_1, \Theta_2, \ldots, \Theta_n) = \hat{\Delta}\left(\bigcup_{i=1}^{n} W_h(\Theta_i) \right) < \hat{\Delta}(W_1 \cup W_2 \cup \left(\bigcup_{i=3}^{n} W_h(\Theta_i)) \right),$$

a contradiction (via Lemma 5.2). ∎

As an easy consequence we get the following important observation.

Corollary 7.2 *If $\eta(h) < \frac{2\pi}{n}$ and $\hat{\delta}_{n,h}(\Theta_1, \Theta_2, \ldots, \Theta_n) > \frac{3}{4}$ for some $(\Theta_1, \Theta_2, \ldots, \Theta_n)$, then*

$$\max_{(\Theta_1, \Theta_2, \ldots, \Theta_n)} \hat{\delta}_{n,h}(\Theta_1, \Theta_2, \ldots, \Theta_n) = \hat{\delta}\left(\frac{2\pi}{n}, h \right).$$

The following claim which is an analogue of Lemma 4.4 of Muder [11] will play an important role in the rest of this section.

Lemma 7.3 *Let $0 < \Theta < \pi$ be fixed. Let $\alpha_0 = 1$ and define the following recursion:*

$$\begin{aligned} A_n &= \sqrt{3} \cdot \alpha_n \cdot \sin \frac{\Theta}{2}, \\ B_n &= \left(\Theta - \pi + 2 \arccos\left(\frac{A_n}{2} \right) \right) \cdot \sqrt{4 - A_n^2}, \\ \alpha_{n+1} &= \frac{-B_n + \sqrt{B_n^2 + 16 \cdot \sin^2\left(\frac{\Theta}{2}\right)}}{2 \cdot \sqrt{3} \sin\left(\frac{\Theta}{2}\right)}. \end{aligned}$$

If $1 = \alpha_0 < \alpha_1 < \cdots < \alpha_N$, then for the fixed Θ the function $\hat{\delta}(\Theta, h)$ is decreasing in h for $1 \leq h \leq \alpha_N$.

Proof. Lemma 6.2 (iii) implies that we may assume that $\eta(h) < \Theta$. Then Lemma 6.1 yields that $\hat{\delta}(\Theta, h) = f(h) = \frac{a(h)}{b(h)}$ with $a(h) = 3(\Theta - \pi) + 6 \arccos\left(\frac{\sqrt{3}}{2} \cdot h \cdot \sin\left(\frac{\Theta}{2}\right) \right)$ and

$b(h) = (4 - 3h^2)\tan\left(\frac{\Theta}{2}\right)$. It is easy to see that $a(h)$ as well as $b(h)$ are decreasing in h for $1 \le h < \frac{2}{\sqrt{3}}$. Now, $f(h)$ is decreasing in h for $1 \le h < \frac{2}{\sqrt{3}}$ if

$$f'(h) = \frac{a'(h) - b'(h) \cdot f(h)}{b(h)} \le 0. \tag{15}$$

As $b'(h) < 0 < b(h)$ holds for all $1 \le h < \frac{2}{\sqrt{3}}$ we get that (15) holds whenever $f(h) \le \frac{a'(h)}{b'(h)}$ i.e. whenever

$$\frac{3(\Theta - \pi) + 6\arccos\left(\frac{\sqrt{3}}{2} \cdot h \cdot \sin\left(\frac{\Theta}{2}\right)\right)}{(4 - 3h^2)\tan\left(\frac{\Theta}{2}\right)} \le \frac{\sqrt{3}\cos\left(\frac{\Theta}{2}\right)}{h\sqrt{4 - 3h^2 \cdot \sin^2\left(\frac{\Theta}{2}\right)}}. \tag{16}$$

In order to insure (16) for all $\alpha_n \le h \le \alpha_{n+1}$ it is sufficient to guarantee the validity of the following inequality

$$\frac{3(\Theta - \pi) + 6\arccos\left(\frac{\sqrt{3}}{2}\alpha_n \cdot \sin\left(\frac{\Theta}{2}\right)\right)}{(4 - 3\alpha_{n+1}^2)\tan\left(\frac{\Theta}{2}\right)} \le \frac{\sqrt{3}\cos\left(\frac{\Theta}{2}\right)}{\alpha_{n+1}\sqrt{4 - 3\alpha_n^2\sin^2\left(\frac{\Theta}{2}\right)}}. \tag{17}$$

Now, after introducing the notations $A_n = \sqrt{3} \cdot \alpha_n \cdot \sin\left(\frac{\Theta}{2}\right)$ and

$$B_n = \left(\Theta - \pi + 2\arccos\left(\frac{A_n}{2}\right)\right)\sqrt{4 - A_n^2}$$

we get that (17) is equivalent to

$$3\sqrt{3} \cdot \sin\left(\frac{\Theta}{2}\right)\alpha_{n+1}^2 + 3B_n\alpha_{n+1} - 4\sqrt{3}\sin\left(\frac{\Theta}{2}\right) \le 0. \tag{18}$$

Solving (18) for α_{n+1} we get that the best (i.e. largest possible) choice for α_{n+1} is the following

$$\alpha_{n+1} = \frac{-B_n + \sqrt{B_n^2 + 16 \cdot \sin^2\left(\frac{\Theta}{2}\right)}}{2 \cdot \sqrt{3} \cdot \sin\left(\frac{\Theta}{2}\right)}. \tag{19}$$

Obviously, (19) finishes the proof of Lemma 7.3. ■

Lemma 7.4 *If $(W_1, W_2, \ldots, W_n)$ is a wedge cluster of order n lying in the plane $\mathcal{P}$ of the face $\mathcal{F}$ of the Voronoi cell $V(C)$ with $d(C, \mathcal{P}) = h \ge 1$, then*

$$\hat{\Delta}(W_1, W_2, \ldots, W_n) \le \hat{\delta}\left(\frac{2\pi}{n}, 1\right) \quad \textit{for} \quad n = 4, 5. \tag{20}$$

Proof. As $\hat{\delta}\left(\frac{2\pi}{4},1\right) = 0.75804\ldots > \frac{3}{4}$ and $\hat{\delta}\left(\frac{2\pi}{5},1\right) = 0.77836\ldots > \frac{3}{4}$ we may assume that $\hat{\Delta}(W_1, W_2, \ldots, W_n) > \frac{3}{4}$. Thus, Lemma 5.2 and Lemma 6.2 (i) and (ii) imply that

$$\hat{\Delta}(W_1, W_2, \ldots, W_n) \le \hat{\delta}_{n,h}(\Theta_1, \Theta_2, \ldots, \Theta_n) \le \hat{\delta}(\eta(h), h) \tag{21}$$

where $\Theta_1, \Theta_2, \ldots, \Theta_n$ denote the baseangles of $W_1, W_2, \ldots, W_n$. Now, let $n = 5$. Then let $h_5 = 1.005$. Then (2) implies that $\eta(h_5) = 1.24539\ldots$ moreover, Lemma 6.1 yields that

$$\hat{\delta}(\eta(h_5), h_5) = 0.7778\ldots < 0.77836\ldots = \hat{\delta}\left(\frac{2\pi}{5}, 1\right). \tag{22}$$

Thus, if $h > h_5$, then Lemma 6.2 (iv) implies that $\hat{\delta}(\eta(h), h) < \hat{\delta}(\eta(h_5)h_5)$ and therefore (22) and (21) finish the proof of (20). Hence, the case left is when $1 \le h \le h_5$. As $\eta(h_5) = 1.24539\ldots < \frac{2\pi}{5} = 1.25663\ldots$ therefore (21) and Corollary 7.2 imply that

$$\hat{\Delta}(W_1, W_2, W_3, W_4, W_5) \le \hat{\delta}\left(\frac{2\pi}{5}, h\right). \tag{23}$$

Now, apply Lemma 7.3 for $\Theta = \frac{2\pi}{5}$. Here we get that $\alpha_1 = 1.0062 > h_5$ which means that

$$\hat{\delta}\left(\frac{2\pi}{5}, h\right) \le \hat{\delta}\left(\frac{2\pi}{5}, 1\right) \quad \text{for all} \quad 1 \le h \le h_5. \tag{24}$$

Then (24) and (23) imply (20) in a straightforward way. Now, let $n = 4$. We proceed in the same way with a different choice of the proper constant as in the case $n = 5$. Namely, let $h_4 = 1.063$. Then (2) implies that $\eta(h_4) = 1.4714\ldots$ moreover, Lemma 6.1 yields that

$$\hat{\delta}(\eta(h_4), h_4) = 0.7577\ldots < \hat{\delta}\left(\frac{2\pi}{4}, 1\right) = 0.7580\ldots. \tag{25}$$

Thus, if $h > h_4$, then Lemma 6.2 (iv) implies that $\hat{\delta}(\eta(h), h) < \hat{\delta}(\eta(h_4), h_4)$ and therefore (25) and (21) finish the proof of (20). Hence, the case left is when $1 \le h \le h_4$. As $\eta(h_4) = 1.4714\ldots < \frac{2\pi}{4} = 1.5707\ldots$ therefore (21) and Corollary 7.2 imply that

$$\hat{\Delta}(W_1, W_2, W_3, W_4) \le \hat{\delta}\left(\frac{2\pi}{4}, h\right). \tag{26}$$

Now, apply Lemma 7.3 for $\Theta = \frac{2\pi}{4} = \frac{\pi}{2}$. After some tedious calculations we find that $1 = \alpha_0 < \alpha_1 = 1.0030\ldots < \alpha_{28} = 1.0632\ldots$ and so $h_4 < \alpha_{28}$ and therefore Lemma 7.3 implies that

$$\hat{\delta}\left(\frac{\pi}{2}, h\right) \le \hat{\delta}\left(\frac{\pi}{2}, 1\right) \quad \text{for all} \quad 1 \le h \le h_4. \tag{27}$$

Then, (27) and (26) imply (20) in a straightforward way. This completes the proof of Lemma 7.4. ■

Lemma 7.5 *If $(W_1, W_2, \dots, W_n)$ is a wedge cluster of order n with $n = 3, 4, 5$ lying in the plane $\mathcal{P}$ of the face $\mathcal{F}$ of the Voronoi cell $V(C)$ with $1 \le h = d(C, \mathcal{P})$, then*

$$\hat{\Delta}(W_1, W_2, \dots, W_n) \le \hat{\delta}\left(\frac{2\pi}{5}, 1\right) = 0.77836\dots. \tag{28}$$

Proof. (20) implies (28) for $n = 4, 5$. So we are left with the case $n = 3$. First, recall that Muder ([11], Theorem 5.9) proved that

$$\Delta(W_1 \cup W_2 \cup W_3) \le \delta\left(\frac{2\pi}{3}, 1\right) = 0.6898\dots. \tag{29}$$

Second, according to (11) we have that

$$\hat{\Delta}(W_1, W_2, W_3) = h \cdot \Delta(W_1 \cup W_2 \cup W_3). \tag{30}$$

Third, recall that under the condition $1 \le h < \frac{2}{\sqrt{3}}$ Muder ([11], Proposition 3.5) showed also that

$$\Delta(W_1 \cup W_2 \cup W_3) \le \delta(\eta(h), h). \tag{31}$$

Now, if $1 \le h < 1.11$, then (29) and (30) imply that

$$\hat{\Delta}(W_1, W_2, W_3) \le h \cdot \delta\left(\frac{2\pi}{3}, 1\right) < 1.11 \cdot 0.6899 = 0.765789 < \hat{\delta}\left(\frac{2\pi}{5}, 1\right). \tag{32}$$

Second, if $1.11 \le h < \frac{2}{\sqrt{3}} = 1.1547\dots$, then (30), (31) and Lemma 4.2 (iv) imply that

$$\hat{\Delta}(W_1, W_2, W_3) \le h\delta(\eta(h), h) < \frac{2}{\sqrt{3}}\delta(\eta(1.11), 1.11) < 0.773 < \hat{\delta}\left(\frac{2\pi}{5}, 1\right). \tag{33}$$

Finally, if $\frac{2}{\sqrt{3}} \le h$, then Lemma 3.3 implies that $\hat{\Delta}(W_1, W_2, W_3) \le \frac{3}{4}$. This completes the proof of Lemma 7.5. ■

Remark 7.6 *For the sake of completeness we note that somewhat surprisingly $\hat{\delta}\left(\frac{2\pi}{3}, 1\right)$ is not the maximum value of $\hat{\Delta}(W_1, W_2, W_3)$.*

8 The surface area density in the Voronoi cone of an arbitrary n-sided face of a Voronoi cell with $n \geq 6$ is at most $\hat{\delta}\left(\frac{2\pi}{5}, 1\right) = 0.77836\ldots$

Let $\mathcal{R}$ be a closed region bounded by a simple polygon of the plane $\mathcal{P}$ of the arbitrary face $\mathcal{F}$ of the Voronoi cell $V(C)$. Then let

$$\hat{\varepsilon}(\mathcal{R}) = Sarea(C(\mathcal{R}) \cap S(C)) - \hat{\delta} \cdot area(\mathcal{R}) \tag{34}$$

where $\hat{\delta} = \hat{\delta}\left(\frac{2\pi}{5}, 1\right) = 0.77836\ldots$. In particular, if $W_h(\Theta)$ is a standard wedge in $\mathcal{P}$, then let

$$\hat{\varepsilon}(\Theta, h) = \hat{\varepsilon}(W_h(\Theta)). \tag{35}$$

The following statement summarizes the basic properties of $\hat{\varepsilon}(\mathcal{R})$ and $\hat{\varepsilon}(\Theta, h)$.

Lemma 8.1 *(i) If $\mathcal{R}$ and $\mathcal{R}'$ are closed regions bounded by simple closed polygons in $\mathcal{P}$ with $\mathcal{R} \cap \mathcal{R}' = \emptyset$, then $\hat{\varepsilon}(\mathcal{R} \cup \mathcal{R}') = \hat{\varepsilon}(\mathcal{R}) + \hat{\varepsilon}(\mathcal{R}')$ and $\hat{\varepsilon}(\mathcal{R}) = (\hat{\Delta}(\mathcal{R}) - \hat{\delta})area(\mathcal{R})$.*

(ii) If W is a wedge with baseangle $0 < \Theta < \pi$ in $\mathcal{P}$ with $1 \leq d(C, \mathcal{P}) = h < \frac{2}{\sqrt{3}}$, then $\hat{\varepsilon}(W) \leq \hat{\varepsilon}(\Theta, h)$.

(iii) If $\hat{\varepsilon}(\Theta, h) > 0$, then $1 \leq h < 1.01$ and $1.21 < \Theta < \frac{2\pi}{5} = 1.25\ldots$.

(iv) If $1 \leq h < \frac{2}{\sqrt{3}}$and $0 < \Theta < \pi$, then $\hat{\varepsilon}(\Theta, h) < 0.000245 \cdot \Theta$.

Proof.

(i) This is immediate from the definition of $\hat{\varepsilon}(\mathcal{R})$.

(ii) Let $\mathcal{R}_0 \subset W$, $\mathcal{R}_1 \subset W$ with $\mathcal{R}_0 \cap \mathcal{R}_1 = \emptyset$ and $\mathcal{R}_0 \cup \mathcal{R}_1 = W$ be defined as in the proof of Lemma 5.1. Then $\hat{\Delta}(\mathcal{R}_0) = \hat{\Delta}(W_h(\Theta))$ and $d(C, \mathcal{R}_1) \geq \frac{2}{\sqrt{3}}$. Thus, Lemma 3.2 implies that $\hat{\Delta}(\mathcal{R}_1) \leq \frac{3}{4} < \hat{\delta}$. Therefore (i) implies that $\hat{\varepsilon}(\mathcal{R}_1) \leq 0$ and so $\hat{\varepsilon}(W) \leq \hat{\varepsilon}(\mathcal{R}_0) = \hat{\varepsilon}(W_h(\Theta)) = \hat{\varepsilon}(\Theta, h)$.

(iii) $\hat{\varepsilon}(\Theta, h)$ is defined for $0 < \Theta < \pi$ and $1 \leq h < \frac{2}{\sqrt{3}}$. If $1.01 \leq h < \frac{2}{\sqrt{3}} = 1.1547\ldots$, then Lemma 6.2 (i), (ii), (iv) imply that $\hat{\delta}(\Theta, h) \leq \hat{\delta}(\eta(h), h) \leq \hat{\delta}(\eta(1.01), 1.01) = 0.7760\ldots < \hat{\delta}$. Hence, (i) implies that $\hat{\varepsilon}(\Theta, h) < 0$. Thus, from now on we can assume that $1 \leq h < 1.01$. If in addition $0 < \Theta \leq 1.21 < \eta(1) = 1.23\ldots$, then Lemma 6.2 (i) and (iii) imply that $\hat{\delta}(\Theta, h) \leq \hat{\delta}(\Theta, 1) \leq \hat{\delta}(1.21, 1) = 0.7781\ldots < \hat{\delta}$. If however,

$\frac{2\pi}{5} \leq \Theta < \pi$, then $\eta(h) < \Theta$ for all $1 \leq h < 1.01$ and therefore Lemma 6.2 (ii) and Lemma 7.3 imply that $\hat{\delta}(\Theta, h) \leq \hat{\delta}\left(\frac{2\pi}{5}, h\right) \leq \hat{\delta}\left(\frac{2\pi}{5}, 1\right) = \hat{\delta}$. Thus, in both cases $\hat{\delta}(\Theta, h) \leq \hat{\delta}$ and therefore in both cases (i) implies that $\hat{\varepsilon}(\Theta, h) \leq 0$.

(iv) Based on (iii) without loss of generality we may assume that $1 \leq h < 1.01$ and $1.21 < \Theta < \frac{2\pi}{5} = 1.2566\ldots$. By (i) $\hat{\varepsilon}(\Theta, h) = \left(\hat{\delta}(\Theta, h) - \hat{\delta}\right) area(W_h(\Theta))$, where by the definition of the standard wedge $W_h(\Theta)$ we get that for $0 < \Theta \leq \eta(h)$ $area(W_h(\Theta)) = \frac{1}{4}(3 - 2h^2)\sin\Theta$ and for $\eta(h) < \Theta < \pi$ $area(W_h(\Theta)) = \frac{1}{3}(4 - 3h^2)\tan\left(\frac{\Theta}{2}\right)$. Then, as in Proposition 5.4 of Muder [11] we easily get that

$$\frac{area(W_h(\Theta))}{\Theta} \leq \frac{area(W_1(1.21))}{1.21} = \frac{\sin(1.21)}{4.121}.$$

Moreover, a recent result of Bezdek [1] implies that $\hat{\delta}(\Theta, h) < \sqrt{2}\arccos\left(\frac{23}{27}\right)$. Thus,

$$\frac{\hat{\varepsilon}(\Theta, h)}{\Theta} < \frac{\left(\sqrt{2}\arccos\left(\frac{23}{27}\right) - \hat{\delta}\right)\sin(1.21)}{4 \cdot 1.21} = 0.0002449\ldots < 0.000245,$$

finishing the proof of Lemma 8.1.

■

Now, we turn to wedge clusters.

Lemma 8.2 *Let W be a wedge cluster of arbitrary order in the plane $\mathcal{P}$ of the face $\mathcal{F}$ of the Voronoi cell $V(C)$ with $1 \leq d(C, \mathcal{P}) = h < \frac{2}{\sqrt{3}} = 1.1547\ldots$ and with Θ as one of its baseangles satisfying $0.15 \leq \Theta \leq \frac{\pi}{3} = 1.0471\ldots$. Then $\hat{\varepsilon}(W) < 0$.*

Proof. Via Lemma 8.1 (i), (ii) and (iii) we can assume that $1 \leq h < 1.01$. Then using Lemma 8.1 (i), (ii) and (iv) we easily get that $\hat{\varepsilon}(W) < \hat{\varepsilon}(\Theta, h) + 0.000245 \cdot (2\pi - \Theta) = (\hat{\delta}(\Theta, h) - \hat{\delta})area(W_h(\Theta)) + 0.000245 \cdot (2\pi - \Theta) = \frac{1}{4}(\hat{\delta}(\Theta, h) - \hat{\delta})(3 - 2h^2)\sin\Theta + 0.000245(2\pi - \Theta)$. As $\Theta \leq \frac{\pi}{3} = 1.0471 < \eta(1)$ Lemma 4.2 (iii) implies that $\hat{\delta}(\Theta, h) - \hat{\delta} \leq \hat{\delta}(\Theta, 1) - \hat{\delta}$ and therefore

$$\hat{\varepsilon}(W) < \left(\hat{\delta}(\Theta, 1) - \hat{\delta}\right)\frac{3 - 2(1.01)^2}{4} \cdot \sin\Theta + 0.000245(2\pi - \Theta) = f(\Theta). \qquad (36)$$

Notice that the expression on the right of (36) is a function of Θ only with $0.15 \leq \Theta \leq \frac{\pi}{3} = 1.0471$. Exactly in the same way using the same partition points as in Proposition 5.5 of Muder [11] one can easily show that $f(\Theta) < 0$ for all $0.15 \leq \Theta \leq \frac{\pi}{3}$. This and (36) complete the proof of Lemma 8.2. ■

As an immediate consequence we get the following very fundamental observation.

Corollary 8.3 *If W is an arbitrary wedge cluster in $\mathcal{P}$ with $\hat{\varepsilon}(W) > 0$, then W has a wedge of baseangle Θ with $0 < \Theta < 0.15$.*

Proof. According to Lemma 7.5 the order of W must be at least 6 but, then W must have a wedge of baseangle Θ with $\Theta \leq \frac{2\pi}{6} = \frac{\pi}{3}$. Thus, Lemma 8.2 implies that $0 < \Theta < 0.15$ indeed. ■

Finally, taking over the proof of Proposition 5.7 of Muder [11] word by word and using again Lemma 8.2 at the proper place, one ends up with the following geometrically rather natural observation.

Lemma 8.4 *If the wedge cluster W is a face of the Voronoi cell $V(C)$ having a point P with $d(M,P) = 0.76$, then $\hat{\varepsilon}(W) < 0$.*

Now, we are in a position to finish the proof of Theorem 1.1 by showing that the surface area density in the Voronoi cone of an arbitrary n-sided face $\mathcal{F}$ of the Voronoi cell $V(C)$ with $n \geq 6$ is at most $\hat{\delta} = \hat{\delta}\left(\frac{2\pi}{5}, 1\right) = 0.77836\ldots$. We prove this by contradiction. So, assume that $\hat{\Delta}(\mathcal{F}) > \hat{\delta}$. Of course, $\mathcal{F}$ can be represented as a wedge cluster W of order n with $\hat{\varepsilon}(W) > 0$. Thus, Corollary 8.3 implies that W has a wedge say, ΔMPQ of baseangle $\Theta = \angle PMQ$ with $0 < \Theta < 0.15$. Moreover, applying Lemma 8.1 (ii) and (iii) to W get that

$$1 \leq d(C,M) = h < 1.01. \tag{37}$$

As ΔMPQ is a wedge and $\hat{\varepsilon}(W) > 0$, Lemma 8.4 implies that

$$0.69 < \sqrt{\frac{3}{2} - (1.01)^2} < \sqrt{\frac{3}{2} - h^2} \leq d(M,P) < 0.76, \tag{38}$$

$$0.69 < \sqrt{\frac{3}{2} - (1.01)^2} < \sqrt{\frac{3}{2} - h^2} \leq d(M,Q) < 0.76. \tag{39}$$

As $0 < \Theta < 0.15$ it is easy to check using (38) and (39) that

$$d(P,Q) < 0.13. \tag{40}$$

Using again (38) and (39) and also (37) we get that

$$d(C,P) < \sqrt{(1.01)^2 + (0.76)^2} < 1.27, \tag{41}$$

$$d(C,Q) < \sqrt{(1.01)^2 + (0.76)^2} < 1.27. \tag{42}$$

Now, recall Proposition 5.1 of Muder [11] according to which if $\alpha = \max\{d(C,P), d(C,Q)\}$ and $\beta = d(P,Q)$, then

$$4 \leq 2\alpha^2 + 2\alpha\beta + \beta^2.$$

Using (40), (41) and (42) we get that

$$4 < 2(1.27)^2 + 2 \cdot 1.27 \cdot 0.13 + 0.13^2 < 3.58,$$

a contradiction.

This completes the proof of Theorem 1.1.

References

[1] K. Bezdek, On a stronger form of Rogers' lemma and the minimum surface area of Voronoi cells in unit ball packings, *J. reine angew. Math.* **518** (2000), 131–143.

[2] L. Fejes Tóth, Über die dichteste Kugellagerung, *Math. Z.* **48** (1943), 676–684.

[3] T.C. Hales, Sphere packings 1, *Discrete Comput. Geom.* **17** (1997), 1–51.

[4] T.C. Hales, Sphere packings 2, *Discrete Comput. Geom.* **18** (1997), 135–149.

[5] T.C. Hales, Overview of the Kepler conjecture, *Discrete Comput. Geom.* (to appear), see arXiv: math. MG/9811071.

[6] T.C. Hales, A proof of the Kepler conjecture, *Annals of Math.* (to appear).

[7] T.C. Hales and S. McLaughlin, A proof of the dodecahedral conjecture, http://arXiv:math.MG/9811079.

[8] W.-Y. Hsiang, On the sphere packing problem and the proof of Kepler's conjecture, *Int. J. Math.* 4/5 (1993), 739–831.

[9] W.-Y. Hsiang, Least action principle of crystal formation of dense packing type and Kepler's conjecture, World Sci. Publishing 2001.

[10] J.H. Lindsey, Sphere packing in $\mathbb{R}^3$, *Mathematika*, 33 (1986), 417–421.

[11] D.J. Muder, Putting the best face on a Voronoi polyhedron, *Proc. London Math. Soc.* 3/56 (1988), 329–348.

[12] D.J. Muder, A new bound on the local density of sphere packings, *Discrete Comput. Geom.* **10** (1993), 351–375.

[13] C.A. Rogers, The packing of equal spheres, *J. London Math. Soc.* 3/8 (1958), 609–620.

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