THE UNIVERSITY OF CALGARY
Department of Mathematics

# THE BROADEST CURVE OF LENGTH 1

by

J. Schaer

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## 1. Introduction.

Consider all curves of length 1 in the euclidean plane $E$. Congruence establishes an equivalence relation among them. Let $\tau$ be the set of all equivalence classes. Leo Moser raised the question about a set $A \subset E$ with as small an area as possible such that for all $C \in \tau$ there exists a $c \in C$ with $c \subset A$, i.e. $A$ shall contain at least one curve of every class. Of course $A$ is determined at most up to congruence.

It was conjectured that in analogy to the Kakeya problem maybe the area could be made arbitrarily small. Besicovitch and Rado have proved [1], that a set may be constructed which has measure 0 and contains circles of all radii between 0 and $\frac{1}{2\pi}$, say.

Here we shall solve another simplified question: What is the minimum width of an (infinitely long) strip $S \subset E$ such that for all $C \in \tau$ there exists $c \in C$ with $c \subset S$? If $w(c,\phi)$ denotes the width of the curve $c$ in a direction given by $\phi$ (i.e. the distance between two supporting lines to $c$ that are perpendicular to $\phi$), let us call the *breadth* of $c$, $b_c = \min w(c,\phi)$. Our problem is to find $b_0 = \sup_c b_c$ and if possible, a curve $c$ for which $b_c = b_0$.

In 2 we prove a theorem which we shall need later but which is of interest in itself. We then assume that there is a broadest curve $c$ and deduce certain properties it must have.

In 3 we shall show that $c$ does not cross itself, in 4 that it must be a convex arc. The proofs of these intuitively obvious facts seem to be rather long, but so far I have been unable to find a more direct argument.

Finally, in 5 and 6 we show that $c$ does exist and is unique.

Strictly speaking, a plane curve $c$ is a map $f: I \to E$ from the unit interval of real numbers into the euclidean plane $E$. For convenience we shall however call $f(I)$ a curve $c$ and $f$ a parametrization of $c$. In the following we assume $f$ to be continuous and $f(I)$ rectifiable. Quite often we shall have to consider reparametrizations, e.g. changing the orientation of a loop or linking up parts of the curve in a different order.

The method by which we show that a certain type of curve $c$ does not have maximal breadth is to show that there is a shorter curve $c'$ such that

$$\tilde{c}' \supset \tilde{c} \qquad (*)$$

where $\tilde{c}$ denotes the convex hull of $c$. For (*) implies that $b_{c'} \geq b_c$, and then $c'$ may be dilated to length 1 to produce a curve whose breadth exceeds $b_c$.

## 2. Theorem.

Let $v$ be a closed continuous and rectifiable curve in the Euclidean plane $E$, and $\partial\tilde{v}$ the boundary of its convex hull $\tilde{v}$. $\partial\tilde{v}$ is also a closed continuous and rectifiable curve. Then

$$l(v) \geq l(\partial\tilde{v}),$$

where $l(v)$ denotes the length of $v$, and $l(v) = l(\partial\tilde{v})$ only if $v = \partial\tilde{v}$.

*Proof*: Although it is claimed to be well known [2] let us prove the result first for polygons. $\partial\tilde{v}$ is then also a polygon, and we shall show that the unique shortest closed curve containing the set $V$ of vertices of $\partial\tilde{v}$ is $\partial\tilde{v}$ itself. If $K$ denotes the set of all polygons having $|V|$ vertices and $V$ as a set of vertices, then for any closed curve $u \notin K$, but $V \subset u$, one can find a polygon $p \in K$ which is shorter. {Let $u = f(I)$, where $f\colon I \to E$, and $f(0) = f(1)$. Then there exists $|V|$ values $t_1 < t_2 < \ldots < t_{|V|} \in I$ such that $f(t_i) \in V$ and $f(t_i) \neq f(t_j)$ for $i \neq j$. Let $p$ be the polygon which connects $f(t_1), f(t_2), \ldots, f(t_{|V|}), f(t_1)$ in this order}. Since $K$ is finite, the existence of a shortest curve is evident. If $k \in K$, but $k \neq \partial\tilde{k} = \partial\tilde{v}$, then $k$ passes through $V$ in a different order than $\partial\tilde{v}$, and therefore has chords. It is easy to see that in this case some chords must intersect, and therefore there exists a shorter polygon in $K$ than $k$. $\partial\tilde{v}$ is the only polygon of $K$ which has no shorter one. Hence it is the shortest, and in fact every other closed curve containing $V$ is (strictly) longer.

If now $v$ is an arbitrary closed continuous curve of length 1, say, we can approximate it by polygons $P_n$ whose vertices are, along $v$, $2^{-n}$ apart. From the preceding we know that

$l(\partial\tilde{P}_n) \leq l(P_n)$, and it is clear that $\lim_{n\to\infty} l(P_n) = l(v)$. In order to establish the inequality it is therefore sufficient to prove that $\lim_{n\to\infty} l(\partial\tilde{P}_n) = l(\partial\tilde{v})$. Obviously $\tilde{P}_n \subset \tilde{v}$; but also the Blaschke distance $\delta(\tilde{P}_n,\tilde{v}_n) \leq 2^{-n-1}$, since for any point $M \in v$ there exists a point (actually a vertex) $N \in P_n$ such that the distance $d(M,N) \leq 2^{-n-1}$; and if $X \in \tilde{v}$, then $X$ is a convex linear combination of points $M_1, M_2, \ldots, \in v$, and if $Y$ is the same linear combination of the corresponding points $N_1, N_2, \ldots, \in P_n$, then also $d(X,Y) \leq 2^{-n-1}$. Thus $\tilde{P}_n \to \tilde{v}$ as $n \to \infty$, and also $\partial\tilde{P}_n \to \partial\tilde{v}$, and hence $l(\partial\tilde{P}_n) \to l(\partial\tilde{v})$, q.e.d.

If $\partial\tilde{v} \neq v$, then, for some $n, \partial\tilde{P}_n \neq P_n$ and hence $l(\partial\tilde{P}_n) < l(P_n)$. We shall show that, for $m > n, l(P_m) - l(\partial\tilde{P}_m) \geq l(P_n) - l(\partial\tilde{P}_n)$ and so prove the second part of the theorem. Let $P'_k$ denote a sequence of approximating polygons containing the sequence $P_k$ as a subsequence: $P_k = P'_{2^k}$, and obtained by introducing one new vertex at a time. Let $Q$ be the new vertex of $P'_{k+1}$, introduced between the vertices $P,R$ of $P'_k$. If $Q \in \tilde{P}'_k$, the result is obvious. If however $Q \notin \tilde{P}'_k$, i.e. $Q$ is a vertex of $\partial\tilde{P}'_{k+1}$, denote by $S,T$ the intersections of the segments $PQ$, $QR$ with $\partial\tilde{P}'_k$, and by $A,B$ the adjacent vertices of $\partial\tilde{P}'_{k+1}$. See fig. 1.

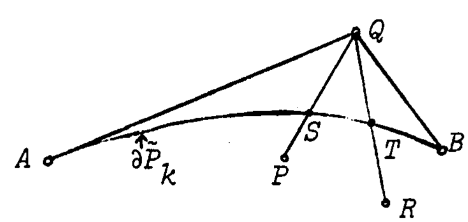


Figure 1.

Now $l(\partial P'_{k+1}) - l(\partial P'_k) \leq d(A,Q) + d(Q,B) - d(A,S) - d(S,T) - d(T,B)$, and

$l(P'_{k+1}) - l(P'_k) = d(P,Q) + d(Q,R) - d(P,R)$. Hence

$$[l(P'_{k+1}) - l(\partial P'_{k+1})] - [l(P'_k) - l(\partial P'_k)] \geq d(P,S) + d(S,Q) + d(Q,T) + $$
$$+ d(T,R) - d(P,R) - d(A,Q) - d(Q,B) + d(A,S) + d(S,T) + d(T,B)$$
$$= (d(A,S) + d(S,Q) - d(A,Q)) + (d(Q,T) + d(T,B) - d(Q,B)) +$$
$$+ (d(P,S) + d(S,T) + d(T,R) - d(P,R)) \geq 0.$$

## 3. A broadest curve need not cross itself.

In this section we establish various properties of a broadest curve $c$ which may be summarized by the title.

(a) *The endpoints $E_0$, $E_1$ of $c$ lie on the boundary $\partial\tilde{c}$ of the convex hull $\tilde{c}$ of $c$.*

Otherwise they would have a positive distance from the compact $\partial\tilde{c}$, and $\tilde{c}$ would be the convex hull of a shorter curve $c' \subset c$.

(b) *The set of points of $c$ in the interior of $\tilde{c}$ is the union of open "line" segments with endpoints on $\partial\tilde{c}$.*

Otherwise an arc of $c$ lying in the interior of $\tilde{c}$ with endpoints $P,Q$ on $\partial\tilde{c}$ could be replaced by the shorter line segment joining $P$ and $Q$ without diminishing $\tilde{c}$. Such a segment together with its endpoints shall be called a *chord* of $c$.

(c) *The chords of $c$ do not cross each other*, i.e. no chord has points on both sides of another chord (i.e. in both open half-planes determined by this other chord).

Otherwise the curve would contain a loop, starting and ending at the point of intersection of the two chords. This point would be in the interior of $\tilde{c}$. Changing the orientation of the loop, the two crossing chords might then be replaced by two opposite sides of the quadrilateral of which they are diagonals, and the curve thus shortened (cf. (b)).

Note that the chords may still have endpoints in common, or they may even fully coincide.

(d) *We may assume that $c$ does not cross any chord at all*, i.e. there exists a parametrization of $c$ such that if the image of the parameter interval $[p,q]$ is a chord, then both the images of $[0,p]$ and $[q,1]$ lie each completely in one of the closed half-planes determined by the chord (and therefore actually in opposite ones).

In order to prove this statement we assume there is a chord $u$ with endpoints $P,Q$. The continuity of $f$ secures that the images of maximal subintervals of $I$ which are disjoint from $u$ are each completely on one side of $u$ (i.e. in one of the open half-planes determined by $u$). We distinguish between three different types of such parts: "Ends" which contain an endpoint of $c$ lying outside $u$; "loops" with coinciding endpoints; and "arcs" connecting $P$ and $Q$.

(i) First we can rule out "loops".

Indeed, if there were a loop $v$ with coinciding endpoints at $Q$, say, then a suitable parametrization of $c$ would let $v$ immediately follow $u$. We shall represent this situation shortly by $PuQvQ$. The convex hull of this piece of $c$ is bounded by the chord $u$ and an arc $w$, both joining $P$ to $Q$. By the theorem of $\underset{\sim}{2}$ we can conclude that $l(u) + l(w) < l(PuQvQ) + l(QuP)$, and hence $l(w) < l(PuQvQ)$.

An immediate consequence of this result is that the number of parts of $c$ outside $u$ is finite, since there are at most two "ends" and the length of any "arc" is greater than that of $u$.

(ii) On each side of $u$ there is at most one "arc".
Indeed, if there were two arcs $v,w$ connecting $P$ and $Q$, then the convex hull of their union would be bounded by $u$ and an arc $\bar{v}$. By the theorem of section 2 $l(u) + l(\bar{v}) < l(v) + l(w)$, and hence the two "arcs" $v,w$ could be replaced by $u$ and $\bar{v}$.

(iii) $c$ does not contain two identical (and identically oriented) chords $PuQ$ and $PvQ$, because there would have to be some "arc" $QwP$, forming a loop together with $PvQ$, say, which could be ruled out by the method of (i).

So far, however, we cannot rule out two chords $PuQ$ and $QvP$. Yet in this case no "arc" can be present (for the same reason), and there must be an "end" on each side of $u$. In this case clearly $c$ does not cross the chord. See fig. 2.

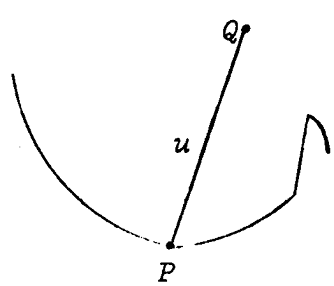


Figure 2.

In the following we can therefore restrict our attention to the case where a chord occurs (as point-set) only once.

(iv) $c$ must have two "ends" w.r.t. any chord $u$.

Indeed, if there were no "ends", then there would have to be an "arc" on each side of $u$. Together they would have the same convex hull as $c$. Linking them up and omitting $u$ altogether, $c$ could be shortened.

If there were exactly one "end", then there would have to be an "arc" on the opposite side of $u$; there might or might not be an "arc" on the same side. In both cases the chord could be omitted after a suitable parametrization, without diminishing the convex hull of $c$.

(v) The two "ends" must lie on opposite sides of $u$.

If in the contrary the two "ends" were both on the same side of $u$, there would have to be an "arc" on the opposite side. Now because of (iv) the endpoints $P,Q$ of $u$ must each be endpoints of an even number of parts of $c$ (including $u$). Hence, if the two "ends" would join the chord $u$ one at $P$ and one at $Q$, then there would have to be also an "arc" on the same side of $u$ as the "ends". In any case we would have two parts $c_1,c_2$ (either two "ends" or an "end" and an "arc") of $c$ emerging from $P$ say, to the same side of $u$, whereas there is an "arc" $v$ on the opposite side; see fig. 3. The parametrization may be chosen such that $c_1PvQuPc_2$. Now there exists a line $l$ through $Q$, intersecting both $c_1$ and $c_2$. Let $R$ and $S$ be the intersection points closest to $P$ along $c_1,c_2$ respectively. Denoting by $d(Q,R)$ the distance between $Q$ and $R$, we may

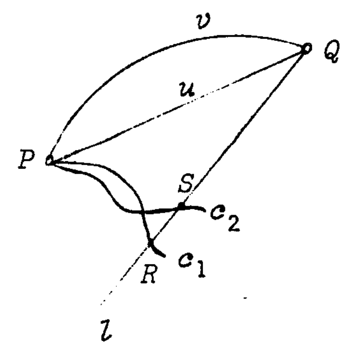


Figure 3.

assume $d(Q,R) \geq d(Q,S)$. The convex hull of $Rc_1PvQuPc_2S$ is bounded by an arc $RwP$ and $PvQlSlR$. By the theorem of section 2 $l(RwPvQlS) + d(S,R) < l(Rc_1PvQuPc_2S) + d(S,R)$. Hence $RwPvQlS$ is shorter than $Rc_1PvQuPc_2S$ and has the same convex hull.

(vi) *c does not have two "arcs" at any chord.*

Else we would again have the situation of fig. 3, which was ruled out in (v).

Therefore the only remaining possibilities, besides the one mentioned in (iii) (see fig. 2: "end" $PuQuP$ "end") are: "end" $PuQ$ "arc" $P$ "end" and "end" $PuQ$ "end", shown in fig. 4. In all these cases there exists a parametrization such that $c$ does not cross $u$. This completes the proof of (d).

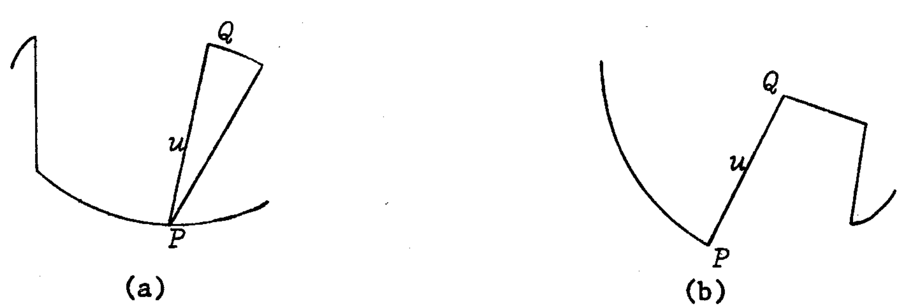


Figure 4.

*Conclusion.*

With this parametrization the set of chords is ordered in a natural way. It is actually even a (possibly infinite) sequence, because by (2dv) every chord separates the two endpoints $E_0$, $E_1$ of $c$ and has therefore one endpoint on $\partial_1$ and the other on $\partial_2$, if $\partial_1, \partial_2$ denote the two arcs into which $\partial\tilde{c}$ is divided by $E_0$ and $E_1$. The convexity of $\tilde{c}$ together with $b_0 > 0$ secures that chords of arbitrarily small length can occur only close to $E_0$ and $E_1$, and since $c$ is of finite length, accumulations of infinitely many chords can occur only there.

Therefore $\tilde{c}$ is the union of a possibly infinite sequence of sets $S_i$ with disjoint interiors, in such a way that every $S_i$ is the convex hull of a convex arc $c_i \subset c$, $c_i = f[a_i, b_i]$, i.e. $A_i c_i B_i$, and the intersections $S_i \cap S_{i+1} = c_i \cap c_{i+1} = f[a_{i+1}, b_i]$ are the chords. See fig. 5.

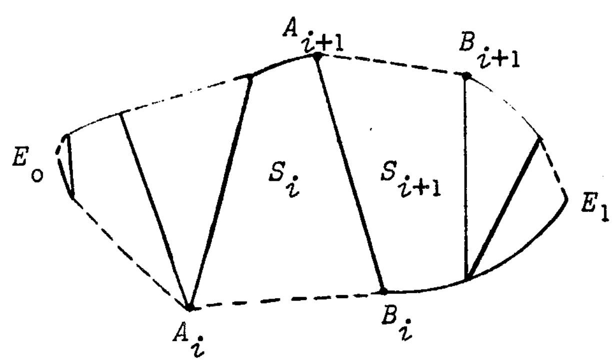


Figure 5.

## 4. A broadest curve must be a convex arc.

In this section we shall show that the sequences $\{S_i\}$ and $\{c_i\}$ are finite, in fact that they consist just of one element, $\tilde{c}$ and $c$ respectively.

*Lemma* 1: If $E_o eP$ is an "end" with respect to a chord $PuQ$, then $d(E_o,Q) \geq d(P,Q)$.

*Proof*: Otherwise the curve $PeE_o lQ$, where $l$ denotes the line segment between $E_o$ and $Q$, would span the same convex hull as $E_o ePuQ$, but would be shorter.

*Lemma* 2: If $c_i$, $c_{i+1}$ are two consecutive arcs of the sequence $\{c_i\}$, $P = A_{i+1}$ and $Q = B_i$ the endpoints of their common chord, and $s_{i+1}$ a line of support to $\tilde{c}$ at the endpoints $P = A_{i+1}$ and $B_{i+1}$ of $c_{i+1}$ (unique, if $A_{i+1} \neq B_{i+1}$ else arbitrary), and if $\alpha_{i+1}$ denotes the angle between the chord $PQ$ and $s_{i+1}$ (in which $E_1$ lies), then $c_i$ (and therefore $S_i$) is confined within an angle $\pi-2\alpha_{i+1}$ at $P$ (see fig. 6, shaded area).

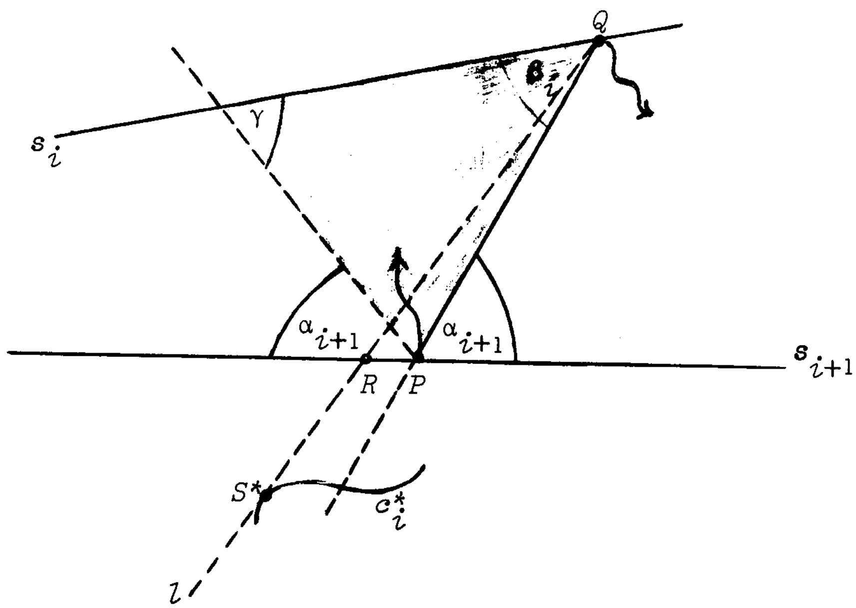


Figure 6.

*Proof*: Let * denote reflection at $s_{i+1}$. If $c_i$ would contain a point outside the mentioned region, then one could find a line of support $l$ to $c_i^*$, passing through $Q$ and intersecting $s_{i+1}$ at a smaller angle than $\alpha_{i+1}$, in the point $R$, say. If $S^*$ denotes a point of $c_i^*$ on $l$ then a new curve $c'$, identical with $c$ except for the piece between $S = S^{**}$ and $Q$, which is replaced by the line segments $SR$ and $RQ$, would be shorter than $c$ and moreover $\tilde{c}' \supset \tilde{c}$.

*Corollary*: $\alpha_{i+1} < \frac{\pi}{2}$ , and hence (if $c$ has chords) $A_{i+1} \neq B_{i+1}$. Therefore, if $c$ has chords, then $f$ is one-to-one and curves of the types of figs. (1) and (3a) with double points are excluded; also, every $c_i$ has a uniquely defined line of support $s_i$ joining its endpoints $A_i, B_i$. We shall denote the angles of the chords of $c_i$ with $s_i$ by $\alpha_i, \beta_i$. (For a first arc $c_1, \alpha_1$ would not be defined; similarly for a last arc).

*Lemma* 3: The sequence $\{c_i\}$ is finite.

*Proof*: Assume in the contrary that we have an infinite sequence of arcs $c_i$. Let $\delta_i = \alpha_{i+1} - \beta_i$; note that $|\delta_i|$ is the angle between $s_i$ and $s_{i+1}$. The arrangement of the supporting lines $s_i$ implies that the sequence of the $\delta_i$ is non-increasing:

$$\delta_i \geq \delta_{i+1} \quad [\text{all } i] \tag{1}$$

Without loss of generality we may assume that the angles $\alpha_{i+1}, \beta_i$ at a particular chord $c_i \cap c_{i+1}$ satisfy $\delta_i \geq 0$. If $\delta_i > 0$, then because of (1) the angles $\alpha_{i+1}, \alpha_i, \alpha_{i-1}, \ldots,$ increase at least in an arithmetic progression. Since none of them can, by the corollary of Lemma 2, exceed $\frac{\pi}{2}$, the sequence $\alpha_{i+1}, \alpha_i, \alpha_{i-1}, \ldots,$ must come to an end. If however $\delta_j = 0$ for $j \leq i$, then the supporting lines $s_{i+1}, s_i$ are parallel, and moreover $s_{i+1} = s_{i-1} = \ldots,$ $s_i = s_{i-2} = \ldots$ . The finite length of $c$ then secures that the sequence $c_{i+1}, c_i, c_{i-1}, \ldots,$ ends. In both cases there is therefore a first arc $c_1$ of the infinite sequence $\{c_i\}$.

If for some $j$ ($> i$) $\delta_j \leq 0$, then the same argument secures that the sequence $\{c_i\}$ would also have a last element. The assumption that the sequence is infinite hence implies that $\delta_j > 0$ for all $j$. But then the length of every chord $c_j \cap c_{j+1}$ must exceed $d(A_2,B_1)\cos\alpha_2$, and so the length of $c$ must be infinite. This contradiction shows that the sequence $\{c_i\}$ is finite, say, $\{c_1, \ldots, c_n\}$.

*Lemma* 4: If $c$ contains a chord $c_i \cap c_{i+1}$, then $\alpha_{i+1} < \frac{\pi}{3}$ (and for the same reason $\beta_i < \frac{\pi}{3}$).

*Proof*: Without loss of generality we may assume $\beta_i \leqslant \alpha_{i+1}$. Then, from the proof of lemma 3, $\alpha_{i+1} \leqslant \alpha_i \leqslant \ldots \leqslant \alpha_2$, and it is sufficient to show $\alpha_2 \leqslant \frac{\pi}{3}$. If we put $i = 1$ in fig. 6, we may conclude from lemma 1 that $\gamma \leqslant \pi - 2\alpha_2$, where $\gamma$ denotes the angle opposite to the first chord $c_1 \cap c_2$ in the shaded triangle to which $c_1$ is confined by lemma 2. But $\gamma = 2\alpha_2 - \beta_1$, and since $0 \leqslant \delta_i \leqslant \ldots \leqslant \delta_1 = \alpha_2 - \beta_1$, we have $2\alpha_2 - \alpha_2 \leqslant \gamma \leqslant \pi - 2\alpha_2$, hence $\alpha_2 \leqslant \frac{\pi}{3}$.

*Lemma* 5: Let $\phi_i$ denote the direction perpendicular to $s_i$ $(i = 1, \ldots, n)$. Then there exists $k$ such that $w(c_k, \phi_k) = w(c, \phi_k)$.

*Proof*: If there is no chord the lemma is obvious. If there are chords we define the angles $\alpha_i$ $(i = 2, \ldots, n)$, $\beta_i$ $(i = 1, \ldots, n-1)$ and $\delta_i = \alpha_{i+1} - \beta_i$ $(i = 1, \ldots, n-1)$ as above. If $\delta_j = 0$ for some $j$, then obviously $k = j$ and $k = j+1$ both satisfy the assertion of the lemma. So we are left to consider the case where $\delta_i \neq 0$ $(i = 1, \ldots, n-1)$.

If $\delta_j > 0$ for some $j$, then $\bigcup_{l=1}^{j} c_l$ lies completely in the triangle bounded by $s_j$, $s_{j+1}$, and $c_j \cap c_{j+1}$. See fig. 7. Similarly, if $\delta_j < 0$, then $\bigcup_{l=j+1}^{n} c_l$ lies completely in that triangle. Therefore if $\delta_j > 0$ then $w\left(\bigcup_{l=1}^{j} c_l,\ \phi_{j+1}\right) \leq w(c_{j+1},\phi_{j+1})$, and if

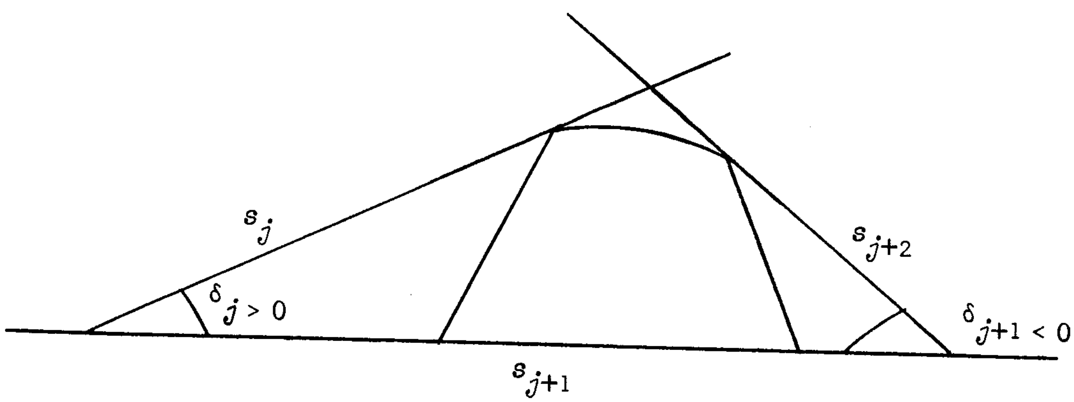


Figure 7.

$\delta_j > 0$ then $w\left(\bigcup_{l=j+1}^{n} c_l,\phi_{j+1}\right) \leq w(c_{j+1},\phi_{j+1})$. Now if $\delta_i > 0$ for all $i$, then $w(c \backslash c_n,\phi_n) \leq w(c_n,\phi_n)$, and so $w(c_n,\phi_n) = w(c,\phi_n) \geq b_0$, i.e. the lemma holds with $k = n$. Similarly, if $\delta_i < 0$ for all $i$, then $w(c_1,\phi_1) = w(c,\phi_1)$. Since the $\delta_i$ $(i = 1,\ldots,n-1)$ are monotomically decreasing, the only case left is $\delta_i > 0$ $(i = 1,\ldots,j-1)$, and $\delta_i < 0$ $(i = j,\ldots,n-1)$. In this case the lemma holds with $k = j$ (see fig. 7).

*Proposition* 1: The sequence $c_1, \ldots, c_n$ consists of just one element, i.e. $n = 1$: A broadest curve must be a convex arc.

*Proof*: We shall show that if a curve $c$ has a chord, then the breadth of $c$, $b_c < b_o$. According to lemma 5 there exists $k$ such that $w(c,\phi_k) = w(c_k,\phi_k)$.

(a) $k = 1$ or $n$. If in the contrary $1 < k < n$, then $c_k$ would contain two chords, $c_{k-1} \cap c_k$ and $c_k \cap c_{k+1}$. The angles $\alpha_k, \beta_k$ form with $s_k$ are, by lemma 4, less than $\frac{\pi}{3}$. The length $l$ of $c_k$ would therefore satisfy

$$1 \geq l \geq 2w(c_k,\phi_k)/\sin\frac{\pi}{3} \geq \frac{4}{\sqrt{3}}\, b_o.$$

But this would imply $b_o \leq \frac{\sqrt{3}}{4}$, whereas we know a curve with breadth $> \frac{\sqrt{3}}{4}$ (see 5, equation (9)). In the following we shall assume $w(c_n,\phi_n) = w(c,\phi_n)$.

(b) Here we want first to show that $w(c_{n-1},\phi_n) \neq w(c_n,\phi_n)$, i.e. that $w(c_{n-1} \cap c_n,\phi_n) < w(c_n,\phi_n)$. Indeed, if we assume $w(c_{n-1} \cap c_n,\phi_n) = w(c_n,\phi_n)$, and if we denote the length of $c_{n-1} \cap c_n$ by $l$, then $l \sin\alpha_n \geq b_o$. But by lemma 1 $d(A_n,E_1) \geq d(A_n,B_{n-1}) = l$, and so $d(B_{n-1},E_1) \geq 2l \sin\frac{\alpha_n}{2}$. Then the length of $c_n$ would be at least $l + 2l \sin\frac{\alpha_n}{2}$, and thus

$$1 \geq l + 2l \sin\frac{\alpha_n}{2} \geq b_o\ (1 + 2\sin\frac{\alpha_n}{2})/\sin\alpha_n \geq \frac{4}{\sqrt{3}}\, b_o;$$

the last inequality is readily proved for $0 < \alpha_n \leq \frac{\pi}{3}$. This would again imply $b_o \leq \frac{\sqrt{3}}{4}$.

(c) Let $R$ denote a point of $c$ which has maximum distance from $s_n$. By (b) $R$ must lie on $c_n$, and by (a) this distance is at least $b_o$. Here we shall show that the angle $\alpha' = \sphericalangle(R,A_n,E_1)$ between the lines $A_nR$ and $s_n$ must exceed $\frac{\pi}{4}$. Indeed, if $\alpha' \leq \frac{\pi}{4}$, then the length $l(c_n)$ of $c_n$ alone would be

$l(c_n) \geq d(A_n,R) + d(R,E_1) \geq b_o\left(\frac{1}{\sin\alpha'} + 1\right) \geq b_o(\sqrt{2}+1)$, and

therefore $b_o \leq \frac{l(c_n)}{1+\sqrt{2}} < \sqrt{2}-1$, a contradiction to (9).

(d) By (c) we may assume $\alpha' > \frac{\pi}{4}$. See fig. 8.

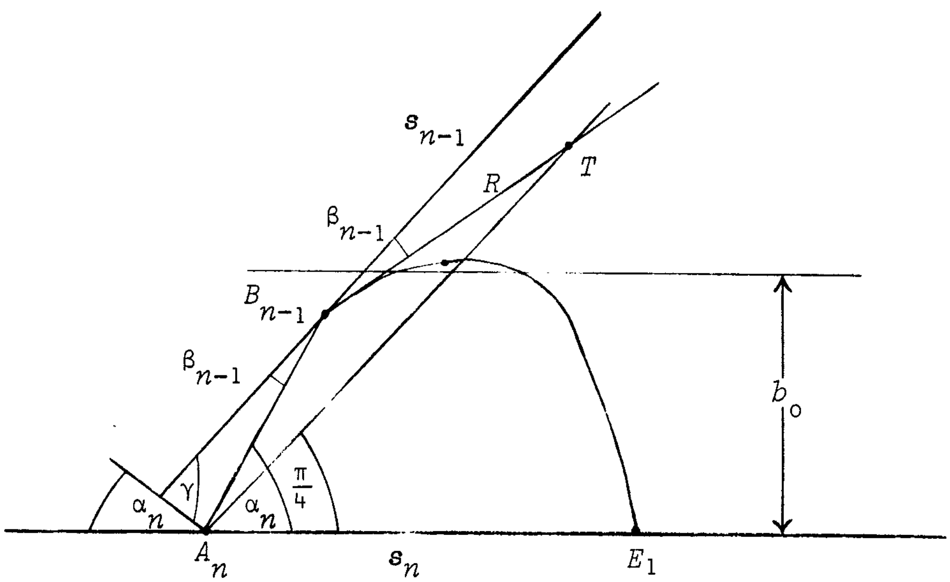


Figure 8.

By (b) the distance of $B_{n-1}$ from $s_n$ is less than $b_o$, and since $R$ must lie in the region described in lemma 2 we must have $2\beta_{n-1} < \alpha_n$. Furthermore we know

$$\frac{\pi}{4} < \alpha' < \alpha_n \leq \frac{\pi}{3} \tag{2}$$

If $n = 2$, then by lemmas 1 and 2, $\gamma \leq \pi - 2\alpha_n$; if $n > 2$ then by lemma 4, $\gamma \leq \frac{\pi}{3} \leq \pi - 2\alpha_n$. So in any case $\gamma = 2\alpha_n - \beta_{n-1} \leq \pi - 2\alpha_n$. Therefore

$$4\alpha_n - \pi \leq \beta_{n-1} < \frac{\alpha_n}{2} . \tag{3}$$

We shall now construct a contradiction by estimating the length $l(c_{n-1} \cup c_n)$.

(e) For this reason we first find a lower bound for $l(c_{n-1} \setminus c_n)$. If $n = 2$, then $E_0 \in c_{n-1}$, and by lemmas 1 and 2

$$l(c_{n-1} \setminus c_n) \geq d(E_0, A_n) \geq 2d(A_n, B_{n-1}) \sin \frac{\beta_{n-1}}{2} .$$

If $n > 2$, then by lemma 4, $l(c_{n-1} \setminus c_n) \geq d(A, A_n)$, where $A$ is the point on $s_{n-1}$ for which $\sphericalangle (A_n, A, B_{n-1}) = \frac{\pi}{3}$ . But then

$\sphericalangle (B_{n-1}, A_n, A) = \frac{2\pi}{3} - \beta_{n-1} > \frac{\pi}{3}$ , [since $\beta_{n-1} < \frac{\alpha_n}{2} \leq \frac{\pi}{6}$] and therefore $l(c_{n-1} \setminus c_n) \geq d(A, A_n) > 2d(A_n, B_{n-1}) \sin \frac{\beta_{n-1}}{2}$ also in this case. By (3) $\sin \frac{\beta_{n-1}}{2} \geq \sin(2\alpha_n - \frac{\pi}{2}) = - \cos 2\alpha_n$. So in any case

$$l(c_{n-1} \setminus c_n) \geq - 2d(A_n, B_{n-1}) \cos 2\alpha_n .$$

(f) Next we show that $d(A_n, B_{n-1}) > d(B_{n-1}, T)$, where $T$ is the point on the ray $A_n\, B_{n-1}$, reflected at $s_{n-1}$, for which $\sphericalangle (T, A_n, E_1) = \frac{\pi}{4}$ (see fig. 8). Indeed, if $\sphericalangle (T, A_n, B_{n-1})$ is denoted by $\varepsilon = \alpha_n - \frac{\pi}{4}$ , then by (3) $\beta_{n-1} \geq 4\varepsilon$ and $\sphericalangle (A_n, T, B_{n-1}) = 2\beta_{n-1} - \varepsilon \geq 7\varepsilon > \varepsilon$. Therefore we have in addition $d(A_n, B_{n-1}) > d(B_{n-1}, R)$.

(g) Now we can estimate

$$1 \geq l(c_{n-1} \cup c_n) = l(c_{n-1} \setminus c_n) + l(c_n)$$
$$\geq -2d(A_n, B_{n-1}) \cos 2\alpha_n + d(A_n, B_{n-1}) + d(B_{n-1}, R) + d(R, E_1).$$

By (f) $2d(A_n, B_{n-1}) \geq d(A_n, B_{n-1}) + d(B_{n-1}, R) > d(A_n, R)$, and so

$$1 > d(A_n, R)[-\cos 2\alpha_n + 1] + d(R, E_1),$$

and by (c)

$$1 > \frac{b_o}{\sin \alpha'}(-\cos 2\alpha_n + 1) + b_o \geq \frac{b_o}{\sin \alpha_n}(1 - \cos 2\alpha_n + \sin \alpha_n)$$
$$= \frac{b_o}{\sin \alpha_n}(2 \sin^2 \alpha_n + \sin \alpha_n) = b_o(1 + 2 \sin \alpha_n) > b_o(1 + \sqrt{2}).$$

But this implies again $b_o < \sqrt{2} - 1$.

Thus the assumption that $n > 1$, i.e. that $c$ has a chord, leads to a contradiction, and the proposition is proved.

## 5. The broadworm.

In this section we shall determine the actual shape of a broadest curve. From 3 we may conclude that there exists a supporting line $s$ passing through both endpoints $E_o$, $E_1$ of $c$.

*Proposition* 2: The two lines of support perpendicular to $s$ pass through $E_o$ and $E_1$.

*Proof*: Suppose $E_o$ were not on any of these two lines. Starting at $E_o$, let $A$ be the first point along $c$ on one of these lines, say $a$. Then the piece of $c$ between $E_o$ and $A$ could be replaced by the shortest line segment from $A$ to $s$ (along $a$), thus shortening the curve and yet not diminishing its convex hull.

*Corollary*: $d(E_0, E_1) \geq b_0$.

*Proposition* 3: The width of $c$ perpendicular to $s$ is exactly $b_0$.

*Proof*: Clearly this width is $\geq b_0$. If it were $> b_0$, then the arc of $c$ above the parallel to $s$ at the distance $b_0$ could be replaced by a straight line segment. See fig. 9. This would

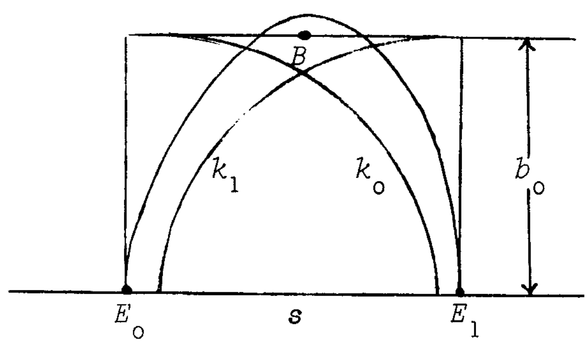


Figure 9.

decrease the length; it also would diminish the convex hull, but the new curve would obviously still have breadth $b_0$.

*Conclusion*: $c$ is confined to a rectangle of width $d(E_0,E_1) \geqslant b_0$ and height $b_0$. Actually a necessary and sufficient condition for $c$ to have breadth $\geqslant b_0$ is that it intersects all tangents of the two quarter-circles $k_0$, $k_1$ with radius $b_0$ and centres $E_0$ and $E_1$ respectively.

*Proposition* 4: If $B$ denotes a point of $c$ on the common tangent of $k_0$ and $k_1$, then the piece of $c$ between $E_0$ and $B$ must stay outside $k_1$, and in the same way every point of the piece of $c$ between $B$ and $E_1$ must have a distance $\geqslant b_0$ from $E_0$.

*Proof*: If the piece of $c$ between $E_0$ and $B$ had a point $D$ inside $k_1$, then the line of support to $\tilde{c}$ at $D$ would be parallel to some tangent of $k_1$. This tangent could not be intersected by $c$.

We shall now solve the problem in two more steps. First, we assume $d(E_0,E_1) \geqslant b_0$ given and find the shortest curve, having the properties of the propositions 3 and 4, under this restriction. Its length will still depend on $d(E_0,E_1)$. Afterwards we shall vary $d(E_0,E_1)$, or rather an equivalent parameter, in order to minimize the length of the shortest curves of step 1.

The first step is easy: If $d(E_0,E_1) \geqslant \frac{2\sqrt{3}}{3}\, b_0$, then $c$ consists of the two equal sides of an isosceles triangle with base $d(E_0,E_1)$ and height $b_0$. If $d(E_0,E_1) \leqslant \frac{2\sqrt{3}}{3}\, b_0$ then $c$ will consist of two arcs on $k_0$ and $k_1$ and pieces of four tangents to these circles, as shown in fig. 10.

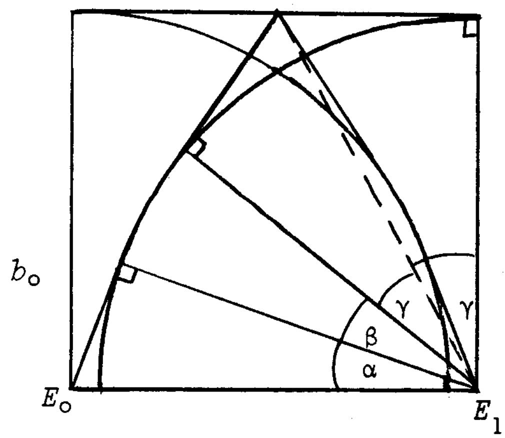


Figure 10.

It is obvious, that for the second step only the possibility $b_o \leq d(E_o,E_1) \leq \frac{2\sqrt{3}}{3} b_o$ need be considered. With the angles shown fig. 10, we have for the length $l$ of $c$:

$$l = 2b_o(\tan\alpha + \beta + \tan\gamma) \tag{4}$$

where

$$2\tan\gamma = \frac{1}{\cos\alpha} \qquad \text{and} \qquad \beta = \frac{\pi}{2} - \alpha - 2\gamma \tag{5}$$

The equation $\frac{dl}{d\alpha} = 0$ reduces to $x^3 - x^2 - 5x + 3 = 0$, where $x = 2\sin\alpha$. This cubic has three real solutions, but only one of them lies between 0 and 2. This solution reads explicitly

$$x = 2\sin\alpha = \frac{1}{3}(1 + 8\sin\varepsilon) \tag{6}$$

with

$$\varepsilon = \frac{1}{3}\arcsin\frac{17}{64}\ . \tag{7}$$

Requiring that for this shortest curve $l = 1$, we find

$$\underline{b_0 = \frac{1}{2}(\tan\alpha + \beta + \tan\gamma)}\ , \qquad (8)$$

where $\alpha$ is determined by (6) and (7), and $\gamma$ and $\beta$ by (5).

Numerically we find

$$\begin{aligned} \varepsilon &\simeq 0.89617 \\ x &\simeq 0.57199 \\ \tan\alpha &\simeq 0.29846 \\ \tan\gamma &\simeq 0.52180 \\ \beta &\simeq 0.31888 \\ \underline{b_0} &\simeq \underline{0.43893} \qquad (9) \end{aligned}$$

## 6. The existence.

The curve constructed in 5 is actually the broadest curve of length 1. Indeed, in 5 we have shown that it is the broadest convex arc, whereas in 3 and 4 we have shown any other curve has a smaller breadth.

## 7. Acknowledgement.

The results of this paper have been found mainly at the 1966 Summer Research Institute of the Canadian Mathematical Congress in Halifax, N.S. But without R.K. Guy's constant reminding me of the Diet of Worms, they had a fair chance to get me before I got them.

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*University of Calgary*
*Alberta, Canada*