CHAPTER ONE

Elementary Differential Equations

1.1 Differential Equations:

Introduction: Just as integration may be called the reverse of differentiation, a differential equation may be called generalisation of the idea of integration. If C is an arbitrary constant, we may write

$\frac{d}{dx}(x^2+C)=2x$......................(1)

or equivalently

$\int 2x\,dx=x^2+C$.................(2).

Equation (1) says that the derivative of (the function of x) x^2+C is equal to 2x, while the second equation says that the integral of (the function of x) 2x with respect to (w.r.t.) x is equal to the function (of x) x^2+C. If we ask the question "What is that function of x whose derivative with respect to (w.r.t) x is 2x, the answer would be "the function x^2+C, where C is an arbitrary constant". Or equivalently, if we call this unknown function by the symbol y (which is a function of x), "What is that function y (of x) whose derivative with respect to (w.r.t) x is 2x, the answer would be y(x) = x^2+C. The idea of a differential equation is born. This baby will do wonders when it grows up.

We have answered the question "Find a function y (of x) whose derivative w.r.t. x is 2x". The answer is y(x) = x^2+C, where C is an arbitrary constant (this is because the derivative of C is zero). So, once again, assuming that the symbol y stands for some (unknown) function of x, we have found that

if $\frac{dy}{dx}=2x$...................(3)

then $y(x)=x^2+C$............(4).

The function y(x) in eq. (4) is called the *solution* of the differential equation in eq.(3). If we substitute this "solution" into equation (3) ($y=x^2+C$) , then the left and right hand sides are equal (verify) , so that the equation is satisfied.
Let us go very, very slow, regroup, and talk geometrically now.

The subject of differential equations is an extension of the idea of integration. When we integrate a function of x w.r.t x, we are looking for a function whose derivative is known. But the derivative of a function y (of x), which is the slope of the curve y = y(x) in the x-y plane at the point P(x,y), may vary with the point, i.e., with both the coordinates x and y of the point P. How do you find y = y(x) in such a case? In symbols, for a function y = y(x) which we are trying to find, we may know that $\frac{dy}{dx}$ = g(x) in which case we only need to integrate g(x) to find y, or we may know that $\frac{dy}{dx}$ = f(x,y). How do we find y = y(x) now? We illustrate with an example.

Example 1: Find all functions y = y(x) which satisfy $\frac{dy}{dx}$ = x for all values of x?

Solution: $y = \frac{x^2}{2} + c$ where c is a constant which can only be determined if additional information is known about y (if as an example, it is known that y = 5 at x = 0, then C must be equal to 5 (verify)).

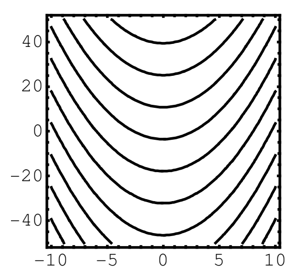
Example 2: Find all functions y = y(x) which satisfy $\frac{dy}{dx} = x + y$ for all values of x and y.

Solution: We do not know how to solve this problem yet! (But try $y = Ae^{x} - x - 1$ for any constant A. The arbitrary constant A can only be determined if we have some additional information about y. If as an example, we know that y = 3 at x = 0, then A must be equal to 4 (verify)).

Solution Curves: It is clear that the two questions are similar. Both the equations are called differential equations and in both cases, the problem is to find a function of x, which function we are calling y, from the information given about the derivative of the function. We shall dwell on example 1 for a while and then go on to other examples. We shall take up example 2 again in a later chapter (it is that difficult!).

The functions $y = \frac{x^2}{2} + c$ for different *constant* values of c, satisfy the equation $\frac{dy}{dx}$ = x (verify). When we 'solve' the equation $\frac{dy}{dx}$ = x , we are looking for a function y = y(x) which is such that in some specified portion of the x-y plane (or the whole plane if not so specified), the slope of curve y = y(x) is equal to the x coordinate of the point P. One such curve passes through every point P in the plane (this requires a proof!). Some of these curves (called the solution curves) are shown in the diagram below.

```
ContourPlot[y - x^2 / 2, {x, -10, 10},
  {y, -50, 50}, ContourShading -> None, PlotPoints -> 100]
```



- ContourGraphics -

The function ContourPlot[F(x,y)=c, {x,xmin,xmax}, {y,ymin,ymax}] in *Mathematica* draws the curves F(x,y) = c for different values of c in the part of the x-y plane covered by the indicated limits of x and y. Unless otherwise indicated, ten values of c are chosen. In this case, we have drawn the curves $y - x^2/2 = c$ (i.e. $y = x^2/2 + c$) for different values of c. Each different value of c specifies a curve.

Initial Conditions: We may also specify a solution curve by prescribing that it pass through a particular chosen point in the x-y plane. If we insist, for example, that we want that curve which passes through the point (2, 4), then at x = 2, y must be 4, and therefore c must be chosen from $4 - 2^2/2 = c$, or c must be 2, so that the solution curve of our differential equation, which solution curve passes through the point (2, 4) is the curve $y = x^2/2 + 2$. The condition that the curve must pass through the point (2,4) may also be written as y(2) = 4, i.e. y = 4 at x = 2. A condition of this type is called an initial condition for such a differential equation. The terminology 'initial condition' comes from the fact that formerly the variable t, denoting time, used to be the favourite name for the independent variable, and the condition was usually put at t = 0.

A Simple case, Separation of variables: Once again, we wish to note that the differential equation itself, the equation $\frac{dy}{dx}$ = f(x,y), simply specifies the value of $\frac{dy}{dx}$ at the point (x,y). 'Solving' this differential equation implies finding all those functions y = y(x), which satisfy this equation, or geometrically speaking, finding all those curves in the (x,y) plane whose slope at the point (x,y) is equal to f(x,y). It is easy if the function f(x,y) on the right hand side involves only the variable x and not the variable y. In that case it is 'only' a question of integration. It is far more difficult if the function f involves both x and y. As a matter of fact, to this day, a general method of 'solving' the equation $\frac{dy}{dx}$ = f(x,y) analytically, is not known. However, the equation is so important for modelling natural and other phenomena that whatever cases we can solve, adds considerably to our capacity for such modelling.

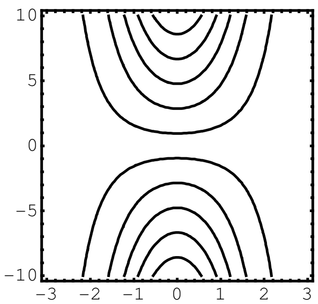
The next simplest case (in which we can 'solve' our equation) is one in which the function f(x,y) can be 'broken up' as $f_1(x)f_2(y)$, so that the function f(x,y) is a product of two functions, one depending only upon x, and the other depending only upon y. Such equations are called 'separable' differential equations. In this case our equation becomes $\frac{dy}{dx}=f_1(x)f_2(y)$. If $f_2(y)$ vanishes for some value of y, y = 2 say, so that $f_2(2)=0$, then the 'curve' y=2 is a solution of our equation, because on y = 2, $\frac{dy}{dx}$ = 0, and this is what our equation demands (verify). For values of y, for which $f_2(y)\neq 0$, we may divide our equation by $f_2(y)$, and it becomes $\frac{1}{f_2(y)}\frac{dy}{dx} = f_1(x)$. Both sides are functions of x, the right hand side is known, and the other one is unknown, yet! We may integrate both sides w.r.t. x. Thus integrating, we obtain $\int\frac{1}{f_2(y)}\frac{dy}{dx}dx=\int f_1(x)dx$. Now by the method of substitution, this equation becomes $\int\frac{1}{f_2(y)}dy = \int f_1(x)dx$. Since both $f_1(x)$ and $f_2(y)$ are known (because our differential equation is presumably known), we can integrate the two sides of this last equation. If we can do so, then we have all the solutions of our equation. After integration, if we can solve the resulting equation for y in terms of x, then we know the solutions as explicitly defined functions, otherwise implicitly defined. We give an example.

Example 3: Find all solutions of the differential equation $\frac{dy}{dx}$ = xy. Also find that particular solution curve which passes through the point (0,2).

Solution: The curve y = 0 is a solution curve of this equation, because along y = 0, $\frac{dy}{dx}$ =0, and the equation is "satisfied" (i.e. both sides are equal). To find other solutions, we assume that $y\neq 0$, and divide both sides of our differential equation by y to obtain $\frac{1}{y}\frac{dy}{dx} = x$. Integrating both sides of this equation, we have $\int\frac{1}{y}\frac{dy}{dx}dx=\int x dx$. By the method of substitution, this equation may be written as $\int\frac{1}{y}dy = \int x dx$. Performing the integration, this gives $\ln|y|+c_1=\frac{x^2}{2}+c_2$ or $\ln|y|=\frac{x^2}{2}+c_2-c_1$. Now since c_1 and c_2 are both unknown, c_2-c_1 is simply an unknown constant, which we may call by a (simpler) symbol c. Our solution now becomes $\ln|y|=\frac{x^2}{2}+c$ or $|y|=e^{x^2/2}.e^c$, or $y=\mp e^c\, e^{x^2/2}$. Since $\mp e^c$ is simply an unknown constant, we denote it by yet another (simple) symbol, A say, and now our solution becomes $y=Ae^{x^2/2}$ where A is an unknown constant, positive or negative. Luckily, the solution we found previously, namely y=0, is included in these solutions for the particular value of A=0 (this does not always happen). We shall now verify that this indeed is a solution of our differential equation for any *constant* A. The equation $y=Ae^{x^2/2}$ implies $\frac{dy}{dx} = Ae^{x^2/2}x$ by the chain rule of differentiation and this last quantity is equal to xy. The verification is done. To find the particular curve that passes through the point (0,2), we apply the condition that y = 2 at x = 0 or y(0) = 2. The constant A therefore must be chosen from $2=Ae^0$ or A=2, so the required solution (curve) is $y = 2e^{x^2/2}$.

Some of the solution curves $y=Ae^{x^2/2}$ for different values of A are shown in the diagram below. Notice the positive slope of the solution curves in the first and third quadrants where the product xy is positive and the negative slope of these curves in the second and fourth quadrants where this product is negative.

```
ContourPlot[y E^(-x^2/2), {x, -3, 3}, {y, -10, 10},
 ContourShading -> False, PlotPoints -> 100]
```



- ContourGraphics -

Another Simple Case, Slope at (x,y) is a function of (y/x): Another class of differential equations $\frac{dy}{dx} = f(x, y)$ which are relatively easy to solve are those in which the function f(x,y) does not depend upon the variables x and y in an arbitrary way, but only on a very special combination of these two variables, namely x/y (or which is the same thing, y/x). So that if f(x,y) is equal to G(y/x) for some function G, then we may easily solve our differential equation. For in this case, our equation is $\frac{dy}{dx}$=G(y/x). The trick is that, instead of looking for the function y in this case, we look for the function v(x), where y = xv(x). Knowledge of v(x) will give us y(x). Now the equation y = xv(x) gives $\frac{dy}{dx} = v(x) + x\frac{dv}{dx}$, so that our differential equation becomes $v(x) + x\frac{dv}{dx} = G(v)$ or $x\left(\frac{dv}{dx}\right) = G(v) - v$. This equation is of the 'separable' type discussed immediately above and may be solved by the same technique. If G(v) = v for some constant v, then v = that constant, is a solution of this reduced equation. Otherwise, we may divide by the quantity G(v) - v and integrate both sides w.r.t. x to obtain $\int \frac{1}{G(v) - v}\, dv = \int \frac{1}{x}\, dx$, and integration gives us v and then y. We give an example.

Example 4: Find all solutions of the differential equation $\frac{dy}{dx} = \frac{y^2 + x^2}{x^2}$.

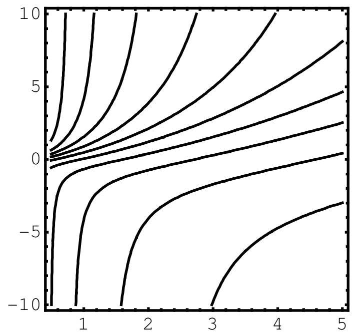
Solution: We notice that the right hand side may be written as $(y/x)^2 + 1$, and therefore depends only upon the combination y/x of the two variables x and y. We now look for v(x)=y(x)/x, which gives $v + x\frac{dv}{dx} = v^2 + 1$ or $x\frac{dv}{dx} = v^2 - v + 1$. Since the right hand side does not vanish for any real value of v, we may divide by $v^2 - v + 1$, bring x on the right hand side, and integrate both sides w.r.t. x. This gives $\int \frac{1}{v^2 - v + 1}\frac{dv}{dx}\, dx = \int \frac{1}{x}\, dx$. This last equation may be written as $\int \frac{1}{v^2 - v + 1}\, dv = \int \frac{1}{x}\, dx$. The left hand side may be integrated by 'completing the square' which gives $v^2 - v + 1 = \left(v - \frac{1}{2}\right)^2 + \frac{3}{4}$ and then substituting $v - \frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$. After integration, we obtain, $v(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\tan\left(A + \frac{\sqrt{3}}{2}\ln|x|\right)$ where A is an arbitrary constant. y = xv(x) gives the solution to our differential equation. Let us verify this solution on *Mathematica*.

```
DSolve[y'[x] == (y[x] / x)^2 + 1, y[x], x]
```

$$\left\{\left\{y[x] \to \frac{1}{2}\left(x + \sqrt{3}\, x\, \mathrm{Tan}\left[\frac{1}{2}\left(\sqrt{3}\, C[1] + \sqrt{3}\, \mathrm{Log}[x]\right)\right]\right)\right\}\right\}$$

The function DSolve[y'[x] = f(x,y),y[x],x] gives the analytical solution of a differential equation (if it can!). The solution given above coincides with the one we have obtained except that the arbitrary constant of integration is written slightly differently. It is written here as $\frac{\sqrt{3}}{2}$ `c[1]` where `c[1]` is arbitrary. We wrote the same constant as A This is immaterial. Here are some of the solution curves.

```
ContourPlot[ArcTan[(2 y - x) / (x √3)] - (√3/2) Log[x], {x, .5, 5},
  {y, -10, 10}, ContourShading -> False, PlotPoints -> 100]
```



`- ContourGraphics -`

The option ContourShading -> False in the function ContourPlot[ ] removes the shading between different contours. Increasing the number of PlotPoints smooths out the curves. The default number of PlotPoints in *Mathematica* is 15.

Exercise 1: Draw the above graphs without prescribing any options about the ContourShading and the ContourPoints.

Exercise 2: Verify by substitution that the solution of the differential equation that we have obtained is correct.

Excercise 3: Give one argument why you think the solution curves in the above graph satisfy the differential equation.

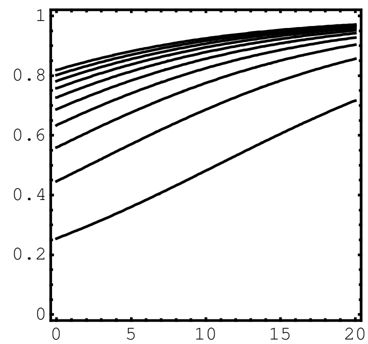
We shall now give more examples of differential equations which are either separable or of the type $\frac{dy}{dx}$ = F(y/x)

Example 5: Find all solutions of the 'logistic' differential equation $\frac{dx}{dt} = \beta x\,(1 - x)$, where β is a known constant.

Solution: Once again, this equation is separable. We notice that (the straight lines) x = 0 and x = 1 are both solutions of this equation (verify). For x ≠ 0 or 1, we bring the term x(1-x) to the left hand side and integrate both sides w.r.t. t. We obtain $\int \frac{1}{x\,(1-x)}\,dx = \int \beta dt$. Integrating both sides (left side by the method of partial fractions) ,

we get $\ln \left| \frac{x}{1-x} \right| = \beta t + c$, where c is an arbitrary constant. Solving for x, we get $x = \frac{Ae^{\beta t}}{1+Ae^{\beta t}}$, where A is an arbitrary constant. Notice that in this case, A = 0 gives the particular solution x = 0 (which solution we noted above) and if we let A go to infinity (or minus infinity), we obtain the particular solution x = 1. Here are some solution curves for β = .1

```
ContourPlot[E^-.1 t (x / (1 - x)), {t, 0, 20},
 {x, 0, 1}, ContourShading -> False, PlotPoints -> 100]
```



- ContourGraphics -

The logistic model is used quite often by biologists to model the growth of populations. In this model the term βx denotes the 'Malthusian' growth of a population, i.e. growth without any constraint of resources. If the environment cannot support more than a certain amount of population (which amount we have denoted by 1, but see example 6 below), then as x approaches one, the growth slows down. This is apparent in the above graph. Notice that in all the curves, x approaches 1 asymptotically as t approaches infinity.

Example 6: Let us assume the model $\frac{dy}{dt} = \alpha\, y^n\,(N - y)$ for the spread of infectious diseases in an 'advanced' society. Here N denotes the total population, and n is a positive number less than one. This is because an infectious disease spreads by an infected member coming into contact with a healthy member. If y is the number of infected members at any time t, so that N-y is the number of healthy members, and an encounter is proportional to y(N-y). However, because of heightened awareness of the disease in an 'advanced' society, all infected members do not come into contact with healthy members. Only a small fraction of them, perhaps $\sqrt{y}$, do so. Hence the model. Take n = 1/2, and compare the spread of the disease in this model with that in the logistic model.

Solution: We write y = Nx to eliminate N as a free parameter from the problem. Comparisons are easier with fewer free parameters. Our equation now becomes $\frac{dx}{dt} = \beta\sqrt{x}\,(1 - x)$ where $\beta = \alpha\sqrt{N}$. This equation is again 'separable'. We bring the term $\sqrt{x}\,(1 - x)$ to the left hand side and integrate both sides w.r.t. t. This gives $\int \frac{1}{\sqrt{x}\,(1-x)}\,dx = \int\beta dt$. To integrate the left hand side , we substitute x = u^2, and then apply the method of partial fractions to the resulting integral. The result is $\ln\left|\frac{1+\sqrt{x}}{1-\sqrt{x}}\right| = \beta t + c$ where c is an arbitrary constant. Notice that these arbitrary constants are absolutely crutial when we use integration to solve a differential equation which we always do. Solving for x, we obtain x $= \left(\frac{Ae^{\beta t}+1}{Ae^{\beta t}-1}\right)^2$,
where now A is an arbitrary constant.

For comparison, we notice that our original model was $\frac{dy}{dt} = \alpha y^n\,(N - y)$, and in this model, if we put y=Nx, then this equation becomes $\frac{dx}{dt} = \beta x^n\,(1 - x)$, where β=αN^n. This implies that, keeping the value of α same, the value of β for n=1/2 is considerably less than its value for n = 1. For N = 1 million say, the value of β for n = 1 is a thousand times more than its value for n = .5. Because of the presence of the term $e^{\beta t}$in the answer, this would imply a considerably faster spread in the case of n = 1 as compared with n = .5. Here is a comparison of two populations in which 16 individuals in a population of 1 million are infected at t = 0. α in both cases is taken to be 10^{-6}. The case in example 5 is taken as case 1 and the one in example 6 as case2. While the number of infected cases in example 5 is rising sharply, the curve in example 6 is rising very slowly in comparison.

Clear[A1, B1, A2, B2]

A1 = .0000160003;

A2 = -1.00803;

B1 = 1;

B2 = .001;

$$\text{Plot}\left[\left\{\frac{\text{A1} * \text{E}^{\text{B1}*\text{t}}}{1 + \text{A1} * \text{E}^{\text{B1}*\text{t}}} * 10^6,\ \left(\left(\frac{\text{A2} * \text{E}^{\text{B2}*\text{t}} + 1}{\text{A2} * \text{E}^{\text{B2}*\text{t}} - 1}\right) \text{^} 2\right) * 10^6\right\},\ \{\text{t}, 0, 5\},\ \text{AxesLabel} \to \{\text{t}, \text{y}\}\right]$$

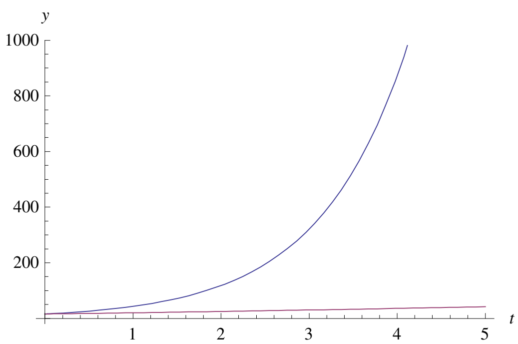


Fig : The faster rising curve in this figure corresponds to n = 1 in the model $\frac{dy}{dt} = \alpha y^n\ (N - y)$. The other curve corresponds to n = .5

Example 7: If a bank loans you money at a certain rate of interest which is compounded 'continuously', then your loan x (= x(t)) goes up according to the rule $\frac{dx}{dt} = \alpha x$. If the year is taken as the unit of time and the interest rate is k per cent per year, then α = k/100. Of course, if you borrow 100 dollars today, then you will owe more than 100 + k dollars at the end of one year because of the compounding effect. How much will you owe at the end of one year? When will you owe 200 dollars?

Solution: The equation $\frac{dx}{dt} = \alpha x$ is a 'separable' equation. x = 0 is a solution of this equation (which says that if you borrow nothing, then you owe nothing!). To determine other solutions, we take x to the left hand side and integrate both sides w..r.t t. This gives $\int \frac{1}{x}\, dx = \int \alpha dt$. Integration gives ln|x| = αt + c where c is an arbitrary constant. This gives x = A$e^{\alpha t}$, where now A is an arbitrary constant, (A=$\mp e^c$). If at t = 0, x = 100, then A=100, so x=100$e^{\alpha t}$. At the end of one year, you owe 100e^{α}. If the interest rate is 10% say, then $\alpha = \frac{10}{100}$, and at the end of one year, you owe 110.517 dollars. (see below). The 'effective' rate (called APR (Annual Percentage Rate) in Banking circles) is more than 10.5%.

```
100 E^.1
110.517
```

The value of t (=T say) when you will owe 200 dollars is given by 200 = 100$e^{T/10}$, which gives T = 10 ln2 which is less than 7 years (see below and talk to a banker about the "rule of seventy").

10 Log[2] // N

6.93147

Example 8: Radioactive substances decay according to the rule $\frac{dx}{dt}$ = -αx, where x is the amount of the radioactive substance in a sample and α is a positive constant depending upon the particular substance. This is because if two grams of a radioactive substance will decay to one gram in a certain period, then two tons of it will decay to one ton in the same period. The rate at which the total amount decays is proportional to the amount present. Notice that x is decreasing with time, so that x'(t) is negative. The half life of a radioactive substance is that period of time in which the amount of substance in a sample is reduced by half and is generally the basis for radiocarbon dating of old samples in archaeology. Determine the half life of a radioactive substance if α is known.

Solution: The equation $\frac{dx}{dt}$ = -αx is a 'separable' equation. x = 0 is a solution of this equation. To determine other solutions, we take x to the left hand side and integrate both sides w..r.t t. This gives $\int \frac{1}{x}\,dx = -\int \alpha dt$. Integration gives ln|x|=-αt+c where c is an arbitrary constant. This gives x = A$e^{-\alpha t}$, where now A is an arbitrary constant. At t = 0, x = A. The time T when x=A/2 is the half life. This time T is given by $\frac{A}{2}$=A$e^{-\alpha T}$ or T=$\frac{\ln 2}{\alpha}$.

Example 9: Find all solutions of the differential equation $\frac{dy}{dt}=\frac{(100-y(t))\,(\alpha)^t}{1000}$, where α is a known constant.

Solution: This is a 'separable' differential equation and y(t)=100 is a particular solution of it. To find other solutions, we take the term (100-y) to the left hand side and integrate both sides w.r.t. t. This gives $\int \frac{1}{100-y}\,dy=\int \frac{(\alpha)^t}{1000}\,dt$. To integrate $\int (\alpha)^t\,dt$, we notice that $\alpha^t=e^{t\ln\alpha}$, and that $\int e^{kt}\,dt = e^{kt}/k$, for any constant k. Integration now gives ln|100-y| = $\frac{-(\alpha)^t}{1000\ln(\alpha)}$+c where c is an arbitrary constant (verify). This gives y=100-AExp($\frac{-(\alpha)^t}{1000\ln(\alpha)}$), where we have written the function e^xas Exp(x), and where now A is an arbitrary constant. A = 0 gives the particular solution y = 100.

Example 10:

A Problem in Ecology: The pollution in the atmosphere is getting too high. It is at 15 parts per million now. The safe limit for human beings is 20 parts per million. It is all because of these engines that burn fossil fuel. A typical engine takes in air from the atmosphere and its exhaust contains 100 parts per million of the pollutants. It is estimated that these engines recycle .1% of the atmosphere this way every year. The governments of the world are trying to stop any further increase in the burning of fossil fuels by stopping any further spread of these engines, but they are not succeeding too well. At present moment, these engine exhausts are going up at the rate of 10% per year, so that in t years from now, the engines will be recycling .1$(1.1)^t$ per cent of the atmosphere every year. If nothing changes, and this increase continues, in how many years will the atmosphere become unsafe for humans? If there is no further increase in the burning of fossil fuels, in how many years then will the atmosphere become unsafe for humans?

Solution: We must first make a Mathematical Model of the problem. With that in mind, we assume that y = y(t) is the total number of units of pollutant in the atmosphere at time t, and that the atmosphere has one million units of polluted air in it at any moment. Now y = y(t) is the pollutant in the atmosphere in parts per million. At present moment, which moment we take to be t = 0, y = 15, so y(0) = 15. We want to know how y = y(t) is changing with time. We take one year to be the unit of time.$\frac{dy}{dt}$ is the rate, *at this moment,* at which y is changing ($\frac{dy}{dt}$is positive if y is increasing with time) per unit of time. Maybe this rate will be different one minute from now, but if this rate stayed the same as now, then after one unit of time (i.e. after one year), the amount of y will be y+$\frac{dy}{dt}$, because if the rate stayed the same, then the average rate of change will be equal to the instantaneous rate. We want to know $\frac{dy}{dt}$ at any moment t.

At moment t from now, some (polluted) air from the atmosphere will be going into the engines. The rate will be, $.1(1.1)^t$ per cent of the atmosphere, i.e. $(1.1)^t$ thousand units of the gas from the atmosphere in one unit of time. This gas will contain $y(t)\,(1.1)^t/1000$ units of pollutant in it. This is the rate at which y(t), the pollutant, is going out of the atmosphere into the engines per unit of time . Also pollutant is coming in from the engines at the rate of $100\,(1.1)^t/1000$ units of pollutant per unit of time. Therefore

$\frac{dy}{dt}$ = rate in - rate out = $\frac{(100-y(t))\,(1.1)^t}{1000}$ ....(1)

This is true at all moments t in $t \geq 0$.
So that in this problem, $\alpha = 1.1$ in example 9 above. y = 15 at t = 0 and we want to know the value of t (=T, say) when y = 20. Here are the calculations:

Clear[A]

$$\text{NSolve}\left[15 == 100 - A\,\text{Exp}\left[\frac{-1}{1000\,\text{Log}[1.1]}\right],\ A\right]\ //\ N$$

{{A -> 85.8965}}

A = 85.8965;

$$\text{NSolve}\left[20 == 100 - A\,\text{Exp}\left[\frac{-(1.1)^T}{1000\,\text{Log}[1.1]}\right],\ T\right]\ //\ N$$

{{T -> 20.0787}}

The atmosphere will become unsafe in a touch more than 20 years.

Example 11: While the water reservoir in a city is kept clean throughout the year by the intake of fresh water from the river flowing through the city, some deposit keeps on accumulating at the bottom of the reservoir throughout the year. The yearly cleaning is done at springtime. At this time, the city brings in huge machines into the reservoir which stir its water thoroughly for a number of days. If on day one, the deposit in the bottom is 1% of the total contents of the reservoir, for how many days should the machines work to bring the deposit down to .01%. Assume that the capacity of the reservoir is 50 million gallons, that the reservoir is always full, and that fresh water flows into it (and flows out of it) at the rate of 10 million gallons per day.

Solution: Let x = x(t) be the total amount of deposit in the water at any time t. The rate at which x is changing is equal to rate in minus the rate out. Assuming that the river water is clean, rate in is zero. The amount of deposit in the reservoir is x/50 (per million gallons of contents) at any time t, and 10 million gallons of thoroughly mixed water is flowing out per day. It follows that the rate out is 10x/50, so that we have $\frac{dx}{dt} = -\frac{x}{5}$. The solution of this (separable) equation is $x=Ae^{-t/5}$(see example 7 in this section), where A is an arbitrary constant. At t = 0, x = .5 (million gallons), so A = .5 and we have $x = (.5)e^{-t/5}$. We want the value of t (=T, say) when x = .5/100. This value of T is given by $\frac{.5}{100} = (.5)\,e^{-T/5}$or T = 5ln(100) = 23 (days) approximately.(see below).

5 Log[100] // N

23.0259

Example 12: While the water reservoir in a city is kept clean throughout the year by the intake of fresh water from the river flowing through the city, some deposit keeps on accumulating at the bottom of the reservoir throughout the year. The yearly cleaning is done in the month of August. At this time, the city brings in huge machines into the reservoir which stir its water thoroughly for a number of days. At this time they fill up the reservoir as well. If on day one, the deposit in the bottom is 1% of the total contents of the reservoir, for how many days should the machines work to bring the deposit down to .01%. Assume that the capacity of the reservoir is 100 million gallons, and that the reservoir is half full when the machines are brought in. Fresh water flows into the reservoir at the rate of 10 million gallons per day, of which 5 million gallons are let out every day till the reservoir is full and then 10 million gallons are let out every day.

Solution: First we make a mathematical model of the problem: Let x = x(t) be the amount of deposit in the reservoir at any time t. Then the rate at which x is changing, $\frac{dx}{dt}$, is equal to rate in minus the rate out. Assuming that the river water is clean, rate in is zero. The reservoir gets filled after 10 days and the rate out is different before and after this time. Before the reservoir is full, the amount of fluid in it at any time t is equal to 50+5t, so the amount of deposit is x(t)/(50+5t) per million gallons of fluid. Five million gallons of fluid are let out every day, so that the rate out is $\frac{5x}{50+5t}$. After the reservoir is full, the amount of fluid in it at any time is 100 million gallons, so that the deposit is x(t)/100 per million gallons of fluid. Ten million gallons of fluid are let out every day, so the rate out is 10x/100 or x/10. Our differential equation now becomes (verify)

$\frac{dx}{dt} = -\frac{5x}{50+5t} = -\frac{x}{10+t}$, in $0 < t < 10$, and
$\frac{dx}{dt} = -\frac{x}{10}$, in $t > 10$.

We want the value of t for which x = .01% of 100 or .01.Both these equations are of the 'separable' type and have solutions

$x(t) = \frac{A}{10+t}$ in $0 \le t < 10$, and

$x = Be^{-t/10}$ in $t > 10$,

where A and B are two arbitrary constants (verify). We know that x(0) = .5 and that this condition only applies to the first solution (the second one is not valid as $x \to 0$). This gives us A = 5. How do we determine B? We determine B by assuming that the amount of deposit in the reservoir does not change *suddenly* as it gets filled up. The amount of deposit in the reservoir changes *continuously.* This assumption is reasonable. We now get that the limit of x as t approaches 10 from the left and from the right must be equal. This gives $\frac{5}{10+10} = Be^{-1}$ or B = e/4. Putting everything together, we have

$x(t) = \frac{5}{10+t}$ in $0 \le t \le 10$, and $x(t) = \frac{e^{(1-t/10)}}{4}$ in $t \ge 10$.

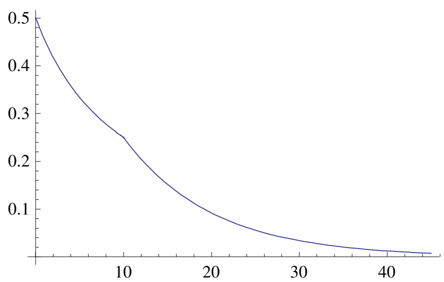
We want the value of t (= T, say) at which x = .01. This occurs in t > 10 and is given by T=10(1-ln(.04)) = 42.1888 (see below). So this time the machines must work for 42 days and more.

```
10*(1 - Log[.04]) // N
```

42.1888

The function x (t) is plotted in the following Figure. Notice the discontinuity in slope (but not in the function) at t=10.

```
F[t_] := Which[0 < t < 10, 5/(10 + t), t > 10, Exp[1 - t/10]/4, True, .5];
Plot[F[t], {t, 0, 45}, PlotRange -> All]
```



Problems on Chapter one:

1. You borrow 100,000 dollars from a bank at the interest rate of 6% per year compounded 'continuously'. You pay back at the rate of 10,000 dollars per year, and if the remaining amount in the last year is less than 10,000, you pay it back as a lump sum. In how many years will the loan be paid? What will be your last payment?

(b) Also show that if you paid only 6000 dollars (the interest?) at the end of every year, you will still owe the bank more than a million dollars at the end of 100 years!

(c) Moral of the story: Pay your mortgage early!

[Hint: If y(0) is the amount of the initial loan, and y(i) is the amount of loan after the loan payment at the end of i-th year, then show that y(i+1) = y(i)$e^{.06}-$10,000. Make a table in *Mathematica* showing y(i) , i = 1,2,3,.....16] .

y[0] = 100000;
y[i_] := y[i - 1] Exp[.06] - 10000; t = Table[y[i], {i, 1, 16}]

{96183.7, 92131.3, 87828.4, 83259.4, 78407.9, 73256.4, 67786.3, 61977.9, 55810.4, 49261.6, 42307.7, 34923.9, 27083.5, 18758.2, 9918.17, 531.471}

y[0] = 100000;
y[i_] := y[i - 1] Exp[.06] - 6000; t = Table[y[i], {i, 90, 100}]

$\{754607., 795270., 838447., 884293., 932975., 984667., 1.03956\times10^6, 1.09784\times10^6, 1.15972\times10^6, 1.22544\times10^6, 1.29521\times10^6\}$

2. The number of bacteria multiplies 'continuously' in a culture in such a manner that it doubles every hour, so that the effective rate of increase is 100% in one hour. What is the nominal (continuous) rate of increase? There are 10,000 bacteria present initially and 4000 are removed from the culture every half hour. How many bacteria are present in the culture at the end of 6 hours? [Hint: The nominal rate of increase is α% per hour such that $\frac{dy}{dt} = \frac{\alpha}{100}\,y$].

3.Estimates about the age of fossils are made on the basis of the amount of a substance called C-14 present in the fossil. This substance starts decaying radioactively at death (of what is the fossil now) and its half life is 5750 years. If 30% of C-14 is still present in a fossil, how old is the fossil? [Hint: If y(t) is the amount of C-14 present at time t, then with radioactive decay, y'(t)=$-\alpha$y(t) for some constant α. Half life determines α.]

4. **The Linear Equation:**

We saw in this chapter that if $\frac{dy}{dt} = 4\,y - 5$, then the solutions of this equation may be found by looking upon it as a separable differential equation. Show that the solutions obtained in this way are all of the type **y(t) =**

$\frac{5}{4}+Ae^{-4t}$, where A is a constant.

5. The solutions of the differential equation in the previous problem, namely, $\frac{dy}{dt} = 4y - 5$, may also be found by the following method which method will generalise to more difficult equations. We go through the following steps:

Step1. Show that $y_1(t)=5/4$ is a solution of our equation. (So that step 1 consists of finding one, any one, solution of the equation. This one solution is sometimes called a particular integral).

Step2. Show that if $y_2(t)$ is *any* other solution of this equation, then the function $z(t) = y_2(t) - y_1(t)$ satisfies the equation $\frac{dz}{dt} = 4z$.

Step3. Show that all solutions of the equation $\frac{dz}{dt} = 4z$ are of the type $z = Ae^{4t}$, where A is an arbitrary constant. You can do this either by looking upon it as a separable differential equation as was done in the text, or by *assuming* that its solutions are of the *type* $z = Ae^{\lambda t}$ for some constant λ and try to determine λ. Substitution of this assumed solution into the differential equation gives $A\lambda e^{\lambda t} = 4Ae^{\lambda t}$ which gives $\lambda = 4$. So that the assumed solution turns out to be Ae^{4t}. This is the same solution. Notice that for the equation $\frac{dz}{dt} = 4z$, any constant multiple of a solution is also a solution, so that you cannot find A unless you have some more information. Also for the equation $\frac{dz}{dt} = 4z$ again, some of any two solutions is also a solution.

Step4. It follows from steps 2 and 3 that $y_2(t) = z(t) + y_1(t) = \frac{5}{4}+Ae^{4t}$. But $y_2(t)$ was *any* solution of the equation $\frac{dy}{dt} = 4y - 5$. It follows that all solutions of this equation are of the type $y(t) = \frac{5}{4}+Ae^{4t}$, where A is an unknown constant. The constant A can be determined if we have additional information on y(t).

Step 5: You have solved an extremely important differential equation. So go through the above four steps again with the equations (a) $\frac{dy}{dt} = 5y - 5$, (b) $\frac{dy}{dt} = 6y - 10$, (c) $\frac{dy}{dt} = 10y - 15$, and (d) $\frac{dy}{dt} = 15y - 1$.

6. In this problem, we shall use the method of the previous problem to find solutions of more difficult differential equations. We take $\frac{dy}{dx} + y = x$.
Step1. Show that $y_1(x) = x - 1$ is a solution of this differential equation. This is our particular integral this time. A particular integral is *any* one solution of a differential equation.
Step2. If $y_2(x)$ is *any* other solution of our differential equation, show that the function $z(x) = y_2(x) - y_1(x)$ satisfies the equation $\frac{dz}{dx} + z = 0$.
Step3. Show that all solutions of the differential equation $\frac{dz}{dx} + z = 0$ are of the type $z(x) = Ae^{-x}$ where A is an arbitrary constant.
Step4. It follows that $y_2(x) = y_1(x) + z(x) = x-1+Ae^{-x}$ where A is an arbitrary constant. But $y_2(x)$ was *any* solution of our equation, so that we have determined all solutions. To determine A, we need more information on y(x). If y(0)=1 for example, then A must be equal to 2.

7. This time we consider the equation $\frac{dy}{dt} + 2y = 3+5e^{7t}$. Look for a particular integral in the form $y_1(t) = A+Be^{7t}$, where A and B are constants. Determine A and B by substituting into the differential equation. We get A=3/2, and B=5/9 (Insist that the equation be satisfied for all values of t, not for any one value of t). Now follow

problem 5 above. All solutions of this equation turn out to be of the type $y(t)=\frac{3}{2}+\frac{5}{9}e^{7t}+Ce^{-2t}$, where C is an arbitrary constant. We need more information on y(t) to determine C.

8. The population of a certain country is 200 million people today. If no immigration is allowed, then this population will decrease continuously at a nominal rate of 2% per year, so that $\frac{dy}{dt}=-.02y$ in that situation. If people immigrate into that country continuously at a constant rate of 5 million per year, show that y(t), the population at time t, follows the law $\frac{dy}{dt}=-.02y+5$, where the unit of y is 1 million, and the unit of time is 1 year. What will be the population of this country 10 years from today? At what level will it stabilise?

9. They want to clean up the lakes in our country. There are two lakes to clean up. One, called lake A, is at level A and the other, called lake B, is at a lower level B. Water from the upstream river flows into lake A and then from lake A into lake B. From lake B, water goes into the ocean. The river water also contains pollutants, but only in the amount of 1 part per million, while the lakes, because of affluent from the industries in the coastal cities, contain pollutants at the level of 1 part in a thousand. For the next few days (or months), the industries will not be allowed to direct their affluent into the lakes. Lake A is much bigger and contains 10 million units of fluid, while lake B contains 5 million. Both lakes are full at this time. The upstream river flows into lake A at the rate of 1 million units of fluid each day, and the same amount of fluid flows into lake B and then into the ocean. We shall assume that the pollutant in both the lakes is well mixed in the water at all times. Show that if $y_1(t)$ and $y_2(t)$ are the total amount of pollutants in lakes A and B respectively, then $y_1'(t)=1-.1y_1(t)$ .... (1), and $y_2'(t)=.1y_1(t)-.2y_2(t)$ ..... (2). Also $y_1(0)=10{,}000$, and $y_2(0)=5{,}000$. How would you go about finding $y_1(t)$ and $y_2(t)$ in $t>0$? [Hint: $y_1'(t)=$ rate in minus rate out. Same for $y_2'(t)$].

10. We shall solve the problem in the previous question here along the lines of prob #5 above.

Step 1. Show that the solution of equation (1) in the previous problem subject to the given initial condition is given by $y_1(t)=10+9990e^{-.1t}$.

Step2. Substitute the value of $y_1(t)$ obtained in step 1 into equation (2) and get an equation in $y_2(t)$ which reads

$$y_2'(t)=.1\left(10+9990e^{-.1t}\right)-.2y_2(t).$$

Step3. Show that $x(t)=5+9990e^{-.1t}$ is a particular integral of the equation in step 2.

Step 4. Show that all other solutions of the equation in step 2 are of the type $y_2(t)=x(t)+Ae^{-.2t}$. This solution will satisfy the initial condition on $y_2(t)$ if $A=-4995$. Therefore $y_2(t)=5+9990e^{-.1t}-4995e^{-.2t}$.

The variable t is being measured in days in this problem. In how many days (correct to 1/10 of a day) will the pollution in both the lakes be less than 10 parts per million? What is the pollution in the upper lake at that time in parts per million?

11.The population of a country is 50 million people today. If no emigration is allowed, this population will increase continuously at the (nominal) rate of 2% per year. However, emigration is taking place to other countries continuously at the constant rate of .5 million people per year. What will be the population of this country in 10 years from now?

12. The population of a country is 50 million people today. It is decreasing continuously at a (nominal) rate of 2%

per year. People are alarmed! The government starts to give incentives for people from other countries to immigrate. In the beginning the immigration is slow but soon the word spreads and immigration picks up. It is estimated that the immigration in any year after that is $\frac{t+1}{20}$ million people where t is being measured in years. Show that the population y(t) of the country satisfies the rule $\frac{dy}{dt} + \frac{y}{50} = \frac{t+1}{20}$. Also y(0)=50. Find the value of y(20).

[Hint: Take the particular integral to be of the type $\alpha t+\beta$ where α and β are constants. Determine these constants by direct substitution into the differential equation. You have now found a particular integral. Now follow problem 5 above.]

13.A population consists of two species. We shall call them predators and prey. In the absence of predators, the prey multiply, while in the absence of prey, the predators die away, because they have no food. Also predators multiply if there are more prey present and more of the prey die if there are more predators present. Show that all this behaviour is captured in the model $y_1'(t) = \alpha y_1(t) - \beta y_2(t)$ and $y_2'(t) = \gamma y_1(t) - \delta y_2(t)$, where α, β, γ, and δ are some *positive* constants, and y_1(t) and y_2(t) denote the numbers of prey and predators respectively. Solutions of these equations can be found by assuming that $y_1(t)=Ae^{\lambda t}$, and $y_2(t)=Be^{\lambda t}$ and looking for the constants A, B, and λ. This is the same procedure that we adopted in the case of one equation, see prob 5 above. Also notice that if you have two solutions of these equations, then the sum of these two solutions is also a solution.We shall visit this model again in a later chapter and suggest some modifications. For now notice that if there are no predators ($y_2(t)$) present, then $y_1'(t)$, the rate of increase of prey with time is positive, i.e. the prey are multiplying, and that this rate decreases in the presencs of predators. Similarly, if there are no prey ($y_1(t)$) present, the rate of increase of $y_2(t)$ is negative but increases in the presence of the prey. What happens if the constants α, β, γ, and δ are such that $\beta\gamma=\alpha\delta$?

14. We shall find solutions to the equations in the previous problem for some specific values of the constants α, β, γ, and δ. We take $\alpha = 2$, $\beta = 1$, $\gamma = 5$, and $\delta = 4$. Substituting the assumed solution ($y_1(t)=Ae^{\lambda t}$, $y_2(t)=Be^{\lambda t}$) into the differential equations and cancelling out the common factor $e^{\lambda t}$, show that the constants A, B, and λ must satisfy A(2-λ)-B=0, and 5A-B(4+λ)=0. Show that both these equations are satisfied if (1)λ=1, A=B and B is arbitratry, or (2) λ=–3, B=5A , and A is arbitrary. Verify that $y_1(t)=Be^{t}$, $y_2(t)=Be^{t}$ and $y_1(t)=Ae^{-3t}$, $y_2(t)=5Ae^{-3t}$ are both solutions to our differential equations. Verify also that $y_1(t)= Be^{t}+Ae^{-3t}$, and $y_2(t)= Be^{t}+5Ae^{-3t}$ is a solution to our differential equations for all (constant) values of A and B. You can find A and B if initial values of $y_1(t)$ and $y_2(t)$ are given. What are A and B if $y_1(0) = 5$ and $y_2(0) = 17$?

15. An inverted conical water tank has a half vertex angle of 45 degrees, so that the radius of any (circular) cross section is the same as the distance of the cross section from the vertex. The amount of water in the tank, when the height of water in the tank is h is, in such a case, $(1/3)\pi h^3$, while its surface area at the top is πh^2 (which is the derivative of the volume w.r.t h. Why is it so?). Water is being pumped into it at a constant rate of 100 cubic meters per minute. Water evaporates from the top of this reservoir. The rate of evaporation is proportional to the surface area of the water at any moment. Show that if v(t) is the volume of water in the tank at any moment t, then v(t) satisfies the rule, $v'(t) = 100-\alpha v^{2/3}$ for some constant α which depends upon the rate of evaporation, and v(t) is in cubic meters. If v(0) = 0, find the value of v(t) in the tank in t > 0. At what moment t does water stop increasing in the tank? [Hint: Integrate the separable differential equation. Put $v = u^3$for integration].

16. According to Newton's Law of cooling, if a hot surface is exposed to cool air, the rate at which the surface cools is proportional to the difference between $T_1$and T_2, the temperatures of the surface and the air respectively. Show that $T_1 = T_1(t)$ is governed by the law $\frac{dT_1}{dt} = -\alpha(T_1 - T_2)$, where α is a positive constant. Suppose that at t = 0, T_1 = 200(degrees). Take T_2 = 0 (degrees), a constant, and $\alpha = 1$. Also let t be in minutes. How long does it take for the surface to cool down to 100 degrees?

17. When current flows through an electric circuit, the charge Q = Q(t) in the circuit satisfies the differential equation $R\frac{dQ}{dt} + \frac{Q}{C} - E = 0$. Here R is the electric 'resistance' of the circuit, C is something called the 'capacitance'

of the circuit, and E is the 'voltage' of the battery which generates the current. Find all solutions of this differential equation in the particular case when R, C, and E are constants. Usually they are not (all constants).

18. In the preceding problem, take R = .5, C = .05, and E = E(t) = sin(60t). Find a particular integral (one solution) of the equation by looking for it in the form $Q_1(t)$=A cos(60t)+B sin(60t), where A and B are constants. Determine A and B by substitution into the differential equation $R\frac{dQ}{dt} + \frac{Q}{C} - E = 0$. This gives A=$\frac{-2}{1601}$ and B=$\frac{80}{1601}$. If $Q_2(t)$ is *any* other solution of this equation, show that z(t)=$Q_2(t) - Q_1(t)$ satisfies the differential equation $R\frac{dz}{dt} + \frac{z}{C} = 0$. Find all solutions z(t) of this equation and hence all solutions of the differential equation $R\frac{dQ}{dt} + \frac{Q}{C} - E = 0$.

19.While discussing the spread of an infectious disease, we argued for the equation$\frac{dy}{dt} = \alpha y^p(N - y)$. Here y = y(t) is the number of infected people at any time t in a total population of N individuals. Initially, 16 individuals in a total population of 1 million are infected. Take $\alpha = 10^{-6}$. Also N = 10^6, and y(0) = 16. For comparison, find y(10) in four different cases when p=1/3, 1/2, 2/3 and 1. [Hint: Put y = Nx, and look for x. Treat the resulting differential equation in x as a separable differential equation and for integration, put x = u^3, u^2, u^3, and u respectively in the four different cases.]

20. Consider all solutions of the equation $\frac{dy}{dx}$ = 2x+3. These solutions are parabolas. Draw some solutions on your screen. Now find all solutions of the equation $\frac{dy}{dx} = \frac{-1}{2x+3}$. Draw some solutions on your screen. Are the two sets orthogonal, i.e. do they intersect at right angles everywhere? Why?

21. Consider all solutions of the equation $\frac{dy}{dx} = x + y$. Draw some on your screen. Now find all solutions of the equation$\frac{dy}{dx} = \frac{-1}{x+y}$. Draw some on your screen. Are the two sets orthogonal. Why? [Hint: To solve the first differential equation, see prob 6 above. In this case the particular integral turns out to be $y_1(x) = -x - 1$. To solve the second differential equation, look upon x as a function of y].

22. We have seen that solutions of the differential equation $\frac{dy}{dx}$ = F(x,y) describes a set of curves in the (x,y) plane whose slope at any point (x,y) is the number F(x,y) at that point. Show that solutions of the equation $\frac{dy}{dx}$ = -1/F(x,y) describes another set of curves whose slope at the point (x,y) is the number -1/F(x,y) at that point. Are the two sets of curves orthogonal to each other everywhere they intersect? Why?

23. Consider the set of curves x^2+y^2 = C in the (x,y) plane. For different values of C, they describes concentric circles centered at the origin. Show that these circles satisfy the differential equation $\frac{dy}{dx}$ = -x/y. Now solve the differential equation $\frac{dy}{dx}$= y/x as a separable differential equation. Show that the solutions of this latter equation are straight lines through the origin which lines are perpendicular to the circles everywhere they intersect.

24. Repeat problem 23 for the set of ellipses $2x^2+3y^2 =$ C where C is an arbitrary constant. This set of curves describes a set of ellipses in the plane. Find a set of curves which are orthogonal to these ellipses everywhere they intersect. Draw some curves (of each set) on your screen. Such sets are important in physical situations where they are (sometimes) called potential curves and lines of force.

25. The body temperature u of a dead body comes down according to the Newton,s Law, $\frac{du}{dt} = -\alpha(u-u_0)$ where u_0 is

the temperature of the surrounding environment and α is a positive constant. Notice that if $u_0 <$ u, i.e if the body is warmer than the environment, then the body temperature u is coming down with time as it should. At t = 0, the person is alive and his/her body temperature is 99 degrees. If this temperature is noted as 55 degrees, 45 degrees and 40 degrees at 2, 3 and 4 P.M., at what time did the person die?