# Rigidity of ball-polyhedra via truncated Voronoi and Delaunay complexes

Károly Bezdek
*Department of Mathematics and Statistics*
*University of Calgary*
*Calgary, Canada*
*Email: bezdek@math.ucalgary.ca*

Márton Naszódi
*Institute of Mathematics*
*Eötvös University*
*Budapest, Hungary*
*Email: nmarci@math.elte.hu*

***Abstract*—A ball-polyhedron is the intersection with non-empty interior of finitely many (closed) unit balls in Euclidean 3-space. A new result of this paper is a Cauchy-type rigidity theorem for ball-polyhedra. Its proof presented here is based on the underlying truncated Voronoi and Delaunay complexes of ball-polyhedra.**

***Keywords*-ball-polyhedron; dual ball-polyhedron; truncated Delaunay complex; (infinitesimally) rigid polyhedron; rigid ball-polyhedron**

## I. Introduction

First, we recall the notation of ball-polyhedra, the central object of study for this paper. Let $\mathbb{E}^3$ denote the 3-dimensional Euclidean space. As in [4] and [5] a *ball-polyhedron* is the intersection with non-empty interior of finitely many closed congruent balls in $\mathbb{E}^3$. In fact, one may assume that the closed congruent 3-dimensional balls in question are of unit radius; that is, they are unit balls of $\mathbb{E}^3$. Also, it is natural to assume that removing any of the unit balls defining the intersection in question yields the intersection of the remaining unit balls becoming a larger set. (Equivalently, using the terminology introduced in [5], whenever we take a ball-polyhedron we always assume that it is generated by a *reduced family* of unit balls.) Furthermore, following [4] and [5] one can represent the boundary of a ball-polyhedron in $\mathbb{E}^3$ as the union of *vertices, edges*, and *faces* defined in a rather natural way as follows. A boundary point is called a *vertex* if it belongs to at least three of the closed unit balls defining the ball-polyhedron. A *face* of the ball-polyhedron is the intersection of one of the generating closed unit balls with the boundary of the ball-polyhedron. Finally, if the intersection of two faces is non-empty, then it is the union of (possibly degenerate) circular arcs. The non-degenerate arcs are called *edges* of the ball-polyhedron. Obviously, if a ball-polyhedron in $\mathbb{E}^3$ is generated by at least three unit balls, then it possesses vertices, edges, and faces. Clearly, the vertices, edges and faces of a ball-polyhedron (including the empty set and the ball-polyhedron itself) are partially ordered by inclusion forming the *vertex-edge-face structure* of the given ball-polyhedron. It was an important observation of [4] as well as of [5] that the vertex-edge-face structure of a ball-polyhedron is not necessarily a lattice (i.e., a partially ordered set (also called a poset) in which any two elements have a unique supremum (the elements' least upper bound; called their join) and an infimum (greatest lower bound; called their meet)). Thus, it is natural to define the following fundamental family of ball-polyhedra, introduced in [5] under the name *standard ball-polyhedra* and investigated in [4] as well without having a particular name for it. Here a ball-polyhedron in $\mathbb{E}^3$ is called a *standard ball-polyhedron* if its vertex-edge-face structure is a lattice (with respect to containment). In this case, we simply call the vertex-edge-face structure in question the *face lattice* of the standard ball-polyhedron. This definition implies among others that any standard ball-polyhedron of $\mathbb{E}^3$ is generated by at least four unit balls.

Second, let us give the main motivation for writing this paper. One of the best known results in the geometry of convex polyhedra is Cauchy's rigidity theorem: If two convex polyhedra $P$ and $Q$ in $\mathbb{E}^3$ are combinatorially equivalent with the corresponding faces being congruent, then the angles between the corresponding pairs of adjacent faces are also equal and thus, $P$ is congruent to $Q$. Putting it somewhat differently the combinatorics of an arbitrary convex polyhedron and its face angles completely determine its inner dihedral angles. For more details on Cauchy's rigidity theorem and on its extensions we refer the interested reader to [1]. In [4] we have been looking for analogues of Cauchy's rigidity theorem for ball-polyhedra. In order to quote properly the relevant results from [4] we need to recall the following terminology. To each edge of a ball-polyhedron in $\mathbb{E}^3$ we can assign an *inner dihedral angle*. Namely, take any point $p$ in the relative interior of the edge and take the two unit balls that contain the two faces of the ball-polyhedron meeting along that edge. Now, the inner dihedral angle along this edge is the angular measure of the intersection of the two half-spaces supporting the two unit balls at $p$. The angle in question is obviously independent of the choice of $p$. Moreover, at each vertex of a face of a ball-polyhedron there is a *face angle* formed by the two edges meeting at the given vertex (which is, in fact, the angle between the two tangent halflines of the two edges meeting at the given vertex). Finally, we say that the standard ball-polyhedron $P$ in $\mathbb{E}^3$ is *globally rigid with respect to its face angles* (resp., *its inner dihedral angles*) if

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the following holds. If $Q$ is another standard ball-polyhedron in $\mathbb{E}^3$ whose face lattice is isomorphic to that of $P$ and whose face angles (resp., inner dihedral angles) are equal to the corresponding face angles (resp. inner dihedral angles) of $P$, then $Q$ is congruent to $P$. Furthermore, a ball-polyhedron of $\mathbb{E}^3$ is called *triangulated* if all its faces are bounded by three edges. It is not hard to see that any triangulated ball-polyhedron is, in fact, a standard one. Now, we are ready to state the main (rigidity) result of [4]: The face lattice and the face angles determine the inner dihedral angles of any standard ball-polyhedron in $\mathbb{E}^3$. In particular, if $P$ is a triangulated ball-polyhedron in $\mathbb{E}^3$, then $P$ is globally rigid with respect to its face angles. The following fundamental analogue question is still an open problem (see [3], p. 63).

*Problem 1.1:* Prove or disprove that the face lattice and the inner dihedral angles determine the face angles of any standard ball-polyhedron in $\mathbb{E}^3$.

One can regard this problem as an extension of the (still unresolved) conjecture of Stoker [8] according to which for convex polyhedra the face lattice and the inner dihedral angles determine the face angles. The following special case of Problem 1.1 has already been put forward as a conjecture in [4]. For this we need to recall that a ball-polyhedron is called a *simple ball-polyhedron*, if at every vertex exactly three edges meet. Now, based on our terminology introduced above the conjecture in question ([4], p. 257) can be phrased as follows.

*Conjecture 1.2:* Let $P$ be a simple and standard ball-polyhedron of $\mathbb{E}^3$. Then $P$ is globally rigid with respect to its inner dihedral angles.

Actually, the restriction that only standard ball-polyhedra are considered in Conjecture 1.2 is a rather natural one. Namely, if $\mathbf{Q}$ is a non-standard ball-polyhedron, then $\mathbf{Q}$ possesses a pair of faces whose intersection consists of at least two connected components. However, such a ball-polyhedron is always flexible (and so, it is not globally rigid) as shown in Section 4 of [4].

In this paper we give a proof of the local version of Conjecture 1.2.

## II. Main Result

We say that the standard ball-polyhedron $P$ of $\mathbb{E}^3$ is *rigid with respect to its inner dihedral angles*, if there is an $\varepsilon > 0$ with the following property. If $Q$ is another standard ball-polyhedron of $\mathbb{E}^3$ whose face lattice is isomorphic to that of $P$ and whose inner dihedral angles are equal to the corresponding inner dihedral angles of $P$ such that the corresponding faces of $P$ and $Q$ lie at distance at most $\varepsilon$ from each other, then $P$ and $Q$ are congruent.

Now, we are ready to state the main result of this paper.

*Theorem 2.1:* Let $P$ be a simple and standard ball-polyhedron of $\mathbb{E}^3$. Then $P$ is rigid with respect to its inner dihedral angles.

Also, it is natural to say that the standard ball-polyhedron $P$ of $\mathbb{E}^3$ is *rigid with respect to its face angles*, if there is an $\varepsilon > 0$ with the following property. If $Q$ is another standard ball-polyhedron of $\mathbb{E}^3$ whose face lattice is isomorphic to that of $P$ and whose face angles are equal to the corresponding face angles of $P$ such that the corresponding faces of $P$ and $Q$ lie at distance at most $\varepsilon$ from each other, then $P$ and $Q$ are congruent. As according to [4] the face lattice and the face angles determine the inner dihedral angles of any standard ball-polyhedron in $\mathbb{E}^3$ therefore Theorem 2.1 implies the following claim in a straightforward way.

*Corollary 2.2:* Let $P$ be a simple and standard ball-polyhedron of $\mathbb{E}^3$. Then $P$ is rigid with respect to its face angles.

In the rest of this paper we give a proof of Theorem 2.1.

## III. Infinitesimally Rigid Polyhedra

Recall that a *convex polyhedron* of $\mathbb{E}^3$ is a bounded intersection of finitely many closed halfspaces in $\mathbb{E}^3$. A *polyhedral complex* in $\mathbb{E}^3$ is a finite family of convex polyhedra such that any vertex, edge, and face of a member of the family is again a member of the family, and the intersection of any two members is empty or a vertex or an edge or a face of both members. In this paper a *polyhedron* of $\mathbb{E}^3$ means simply the union of all members of a polyhedral complex in $\mathbb{E}^3$ possessing the additional property that its boundary is a surface in $\mathbb{E}^3$ (i.e., a 2-dimensional topological manifold embedded in $\mathbb{E}^3$).

We denote the convex hull of a set $C$ by $\operatorname{conv} C$. A polyhedron $Q$ of $\mathbb{E}^3$ is

- *weakly convex* if its vertices are in convex position (i.e., if its vertices are the vertices of a convex polyhedron);
- *decomposable* if it can be triangulated without adding new vertices;
- *co-decomposable* if its complement in $\operatorname{conv} Q$ can be triangulated without adding new vertices;
- *weakly co-decomposable* if it is contained in a convex polyhedron $\tilde{Q}$, such that all vertices of $Q$ are vertices of $\tilde{Q}$, and the complement of $Q$ in $\tilde{Q}$ can be triangulated without adding new vertices.

Clearly, the boundary of every polyhedron in $\mathbb{E}^3$ can be triangulated without adding new vertices. Now, let $P$ be a polyhedron in $\mathbb{E}^3$ and let $T$ be a triangulation of its boundary without adding new vertices. We call the 1-skeloton $G(T)$ of $T$ the *edge graph* of $T$. By an *infinitesimal flex* of the edge graph $G(T)$ in $\mathbb{E}^3$ we mean an assignment of vectors to the vertices of $G(T)$ (i.e., to the vertices of $P$) such that the displacements of the vertices in the assigned directions induce a zero first-order change of the edge lengths: $(p_i - p_j) \cdot (q_i - q_j) = 0$ for every edge $p_i p_j$ of $G(T)$, where $q_i$ is the vector assigned to the vertex $p_i$. An infinitesimal flex is called trivial if it is the restriction of an infinitesimal rigid motion of $\mathbb{E}^3$. Finally, we say that the polyhedron $P$ is *infinitesimally rigid* if every infinitesimal flex of the edge graph $G(T)$ of $T$ is trivial. (It is not hard to see that the infinitesimal rigidity of a polyhedron is a well-defined notion i.e., independent of the triangulation $T$.) We need the following remarkable rigidity theorem of Izmestiev and Schlenker [6] for the proof of Theorem 2.1.

*Theorem 3.1:* Every weakly convex, decomposable and weakly co-decomposable polyhedron of $\mathbb{E}^3$ is infinitesimally rigid.

## IV. Dual Ball-Polyhedron

The closed ball of radius $\rho$ centered at $p$ in $\mathbb{E}^3$ is denoted by $\mathbf{B}(p,\rho)$. Also, it is convenient to use the notation $\mathbf{B}(p) := \mathbf{B}(p,1)$. For a set $C \subseteq \mathbb{E}^3$ we denote the intersection of closed unit balls with centers in $C$ by $\mathbf{B}(C) := \cap\{\mathbf{B}(c)\colon c \in C\}$. Recall that every ball-polyhedron $P = \mathbf{B}(C)$ can be generated such that $\mathbf{B}(C\backslash\{c\}) \neq \mathbf{B}(C)$ holds for any $c \in C$. Therefore whenever we take a ball-polyhedron $P = \mathbf{B}(C)$ we always assume the above mentioned reduced property of $C$. The following duality theorem has been proved in [4] and it is also needed for our proof of Theorem 2.1.

*Theorem 4.1:* Let $P$ be a standard ball-polyhedron of $\mathbb{E}^3$. Then the intersection $P^*$ of the closed unit balls centered at the vertices of $P$ is another standard ball-polyhedron whose face lattice is dual to that of $P$ (i.e., there exists an order reversing bijection between the face lattices of $P$ and $P^*$).

In fact, the proof presented in [4] leads to the following quite general duality theorem (which in this general form however, is not needed for our proof of Theorem 2.1): Let $V$ denote the set of vertices of a ball-polyhedron $P$ in $\mathbb{E}^3$ which has no face bounded by two edges. Then there is a duality (a containment-reversing bijection) between the vertex-edge-face structures of $P$ and the "dual" ball-polyhedron $P^* = \mathbf{B}(V)$ of $P$.

## V. Truncated Voronoi and Delaunay Complexes

Finally, let us give a detailed construction of the so-called *truncated Voronoi and Delaunay complexes* of an arbitrary ball-polyhedron, which are going to be the underlying geometric complexes of the given ball-polyhedron playing a central role in our proof of Theorem 2.1. We leave the proofs of the claims mentioned in this section to the reader partly because they are straightforward and partly because they are also well known (see for example, [2] or [7]).

The *farthest-point Voronoi tiling* corresponding to a finite set $C := \{c_1,\dots,c_n\}$ in $\mathbb{E}^3$ is the family $\mathcal{V} := \{V_1,\dots,V_n\}$ of closed *convex polyhedral sets* $V_i := \{x \in \mathbb{E}^3\colon |x-c_i| \geq |x-c_j| \quad \text{for all } j \neq i, 1 \leq j \leq n\}$, $1 \leq i \leq n$. (Here a closed convex polyhedral set means a not necessarily bounded intersection of finitely many closed halfspaces in $\mathbb{E}^3$.) We call the elements of $\mathcal{V}$ *farthest-point Voronoi cells*. In the sequel we omit the words "farthest-point" as we do not use the other (more popular) Voronoi tiling: the one capturing closest points.

It is known that $\mathcal{V}$ is a tiling of $\mathbb{E}^3$. We call the vertices, (possibly unbounded) edges and (possibly unbounded) faces of the Voronoi cells of $\mathcal{V}$ simply the *vertices*, *edges* and *faces* of $\mathcal{V}$.

The *truncated Voronoi tiling* corresponding to $C$ is the family $\mathcal{V}^t$ of closed convex sets $\{V_1 \cap \mathbf{B}(c_1),\dots,V_n \cap \mathbf{B}(c_n)\}$. Clearly, from the definition it follows that $\mathcal{V}^t = \{V_1 \cap P,\dots,V_n \cap P\}$ where $P = \mathbf{B}(C)$. We call elements of $\mathcal{V}^t$ *truncated Voronoi cells.*

Next, we define the (farthest-point) *Delaunay complex* $\mathcal{D}$ assigned to the finite set $C = \{c_1,\dots,c_n\} \subset \mathbb{E}^3$. It is a polyhedral complex on the vertex set $C$. For an index set $I \subseteq \{1,\dots,n\}$, the convex polyhedron $\operatorname{conv} c_i\colon i \in I$ is a member of $\mathcal{D}$ if, and only if, there is a point $p$ in $\cap\{V_i\colon i \in I\}$ which is not contained in any other Voronoi cell. In other words, $\operatorname{conv} c_i\colon i \in I \in \mathcal{D}$ if, and only if, there is a point $p \in \mathbb{E}^3$ and a radius $\rho \geq 0$ such that $\{c_i\colon i \in I\} \subset \operatorname{bd}\mathbf{B}(p,\rho)$ and $\{c_i\colon i \notin I\} \subset \operatorname{int}\mathbf{B}(p,\rho)$. It is known that $\mathcal{D}$ is a *polyhedral complex*, in fact, it is a tiling of $\operatorname{conv} C$ by convex polyhedra.

*Lemma 5.1:* Let $C = \{c_1,\dots,c_n\} \subset \mathbb{E}^3$ be a finite set, and $\mathcal{V} = \{V_1,\dots,V_n\}$ be the corresponding Voronoi tiling of $\mathbb{E}^3$. Then

(V) For any vertex $p$ of $\mathcal{V}$, there is an index set $I \subseteq \{1,\dots,n\}$ with $\dim\{c_i\colon i \in I\} = 3$ such that $\operatorname{conv} c_i\colon i \in I \in \mathcal{D}$ and $p = \cap\{V_i\colon i \in I\}$.
And *vice versa*: if $I \subseteq \{1,\dots,n\}$ with $\dim\{c_i\colon i \in I\} = 3$ is such that $\operatorname{conv} c_i\colon i \in I \in \mathcal{D}$ then $\cap\{V_i\colon i \in I\}$ is a singleton namely, a vertex of $\mathcal{V}$.

(E) For any edge $\ell$ of $\mathcal{V}$, there is an index set $I \subseteq \{1,\dots,n\}$ with $\dim\{c_i\colon i \in I\} = 2$ such that $\operatorname{conv} c_i\colon i \in I \in \mathcal{D}$ and $\ell = \cap\{V_i\colon i \in I\}$.
And *vica versa*: if $I \subseteq \{1,\dots,n\}$ with $\dim\{c_i\colon i \in I\} = 2$ is such that $\operatorname{conv} c_i\colon i \in I \in \mathcal{D}$ then $\cap\{V_i\colon i \in I\}$ is an edge of $\mathcal{V}$.

(F) For any face $f$ of $\mathcal{V}$, there is an index set $I \subseteq \{1,\dots,n\}$ with $|I| = 2$ such that $\operatorname{conv} c_i\colon i \in I \in \mathcal{D}$ and $f = \cap\{V_i\colon i \in I\}$.
And *vica versa*: if $I \subseteq \{1,\dots,n\}$ with $|I| = 2$ is such that $\operatorname{conv} c_i\colon i \in I \in \mathcal{D}$ then $\cap\{V_i\colon i \in I\}$ is a face of $\mathcal{V}$.

We define the *truncated Delaunay complex* $\mathcal{D}^t$ corresponding to $C$ similarly to $\mathcal{D}$: For an index set $I \subseteq \{1,\dots,n\}$, the convex polyhedron $\operatorname{conv} c_i\colon i \in I$ is a member of $\mathcal{D}^t$ if, and only if, there is a point $p$ in $\cap\{V_i \cap \mathbf{B}(c_i)\colon i \in I\}$ which is not contained in any other truncated Voronoi cell. Note that the truncated Voronoi cells are contained in the ball-polyhedron $\mathbf{B}(C)$. Thus, $\operatorname{conv} c_i\colon i \in I \in \mathcal{D}^t$ if, and only if, there is a point $p \in \mathbf{B}(C)$ and a radius $\rho \geq 0$ such that $\{c_i\colon i \in I\} \subset \operatorname{bd}\mathbf{B}(p,\rho)$ and $\{c_i\colon i \notin I\} \subset \operatorname{int}\mathbf{B}(p,\rho)$.

## VI. Proof of Theorem 2.1

*Lemma 6.1:* Let $P = \mathbf{B}(C)$ be a simple ball-polyhedron in $\mathbb{E}^3$. Then no vertex of the Voronoi tiling $\mathcal{V}$ corresponding to $C$ is on $\operatorname{bd} P$, and no edge of $\mathcal{V}$ is tangent to $P$.

*Proof:* At least four Voronoi cells meet in any vertex of $\mathcal{V}$. Moreover, the intersection of each Voronoi cell with $\operatorname{bd} P$ is a face of $P$. Hence, if a vertex of $\mathcal{V}$ were on $\operatorname{bd} P$ then at least four faces of $P$ would meet at a point, contradicting the assumption that $P$ is simple.

Let $\ell$ be en edge of $\mathcal{V}$, and assume that it contains a point $p \in \operatorname{bd} P$. By the previous paragraph, $p \in \operatorname{relint} \ell$. From Lemma 5.1 (E) it follows that $p$ is in the intersection of some Voronoi cells $\{V_i : i \in I\}$ with $\dim\{c_i : i \in I\} = 2$. Clearly, $\ell$ is orthogonal to the plane $\operatorname{aff}\{c_i : i \in I\}$. Finally, in a neighborhood of $p$, $P$ is the same as $\mathbf{B}(\{c_i : i \in I\})$ and hence, $\ell$ must intersect $\operatorname{int} P$.

*Lemma 6.2:* Let $P = \mathbf{B}(C)$ be a simple ball-polyhedron in $\mathbb{E}^3$. Then $\mathcal{D}^t$ is a sub-polyhedral complex of $\mathcal{D}$, that is $\mathcal{D}^t \subseteq \mathcal{D}$, and faces, edges, and vertices of members of $\mathcal{D}^t$ are again members of $\mathcal{D}^t$.

*Proof:* Clearly, $\mathcal{D}^t \subseteq \mathcal{D}$, and their vertex sets are identical (both are $C$). Let $\operatorname{conv} c_i : i \in I \in \mathcal{D}^t$ be a 3-dimensional member of $\mathcal{D}^t$. Then, the corresponding vertex (Lemma 5.1 (V)) $v$ of $\mathcal{V}$ is in $\operatorname{int} P$ by Lemma 6.1. For a given face of $\operatorname{conv} c_i : i \in I$, there is a corresponding edge (Lemma 5.1 (E)) $\ell$ of $\mathcal{V}$. Clearly, $v$ is an endpoint of $\ell$. Now, $\operatorname{relint} \ell \cap P \neq \emptyset$, and thus the face $\operatorname{conv} c_i : i \in I$ of $\mathcal{V}$ corresponding to $\ell$ is in $\mathcal{D}^t$.

Next, let $\operatorname{conv} c_i : i \in I \in \mathcal{D}^t$ be a 2-dimensional member of $\mathcal{D}^t$. Then, for the corresponding edge $\ell$ of $\mathcal{V}$ we have $\operatorname{relint} \ell \cap P \neq \emptyset$. By Lemma 6.1, $\ell$ is not tangent to $P$, thus $\operatorname{relint} \ell \cap \operatorname{int} P \neq \emptyset$. An edge $[c_i, c_j]$ of $\operatorname{conv} c_i : i \in I$ corresponds to a face (Lemma 5.1 (F)) $f$ of $\mathcal{V}$. Clearly, $\ell$ is an edge of $f$. Now, $\operatorname{relint} f \cap P \neq \emptyset$, and thus $[c_i, c_j]$ is in $\mathcal{D}^t$.

The following lemma helps to understand the 2-dimensional members of $\mathcal{D}^t$.

*Lemma 6.3:* Let $P = \mathbf{B}(C)$ be a simple and standard ball-polyhedron in $\mathbb{E}^3$. Moreover, let $Q$ be the union of the 3-dimensional polyhedra in $\mathcal{D}^t$. Then the 2-dimensional members of $\operatorname{bd} Q$ are triangles, and a triangle $\operatorname{conv} c_1, c_2, c_3$ is in $\operatorname{bd} Q$ if, and only if, the corresponding faces $F_1, F_2, F_3$ of $P$ meet (at a vertex of $P$).

*Proof:* By Lemma 6.2, the 2-dimensional members of $\operatorname{bd} Q$ are 2-dimensional members of $\mathcal{D}^t$. Let $\operatorname{conv} c_i : i \in I \in \mathcal{D}^t$ with $\dim\{c_i : i \in I\} = 2$. Then, clearly, $\operatorname{conv} c_i : i \in I \in \mathcal{D}$ and, by Lemma 5.1 (E), it corresponds to an edge $\ell$ of $\mathcal{V}$ which intersects $P$. Now, $\ell$ is a closed line segment, or a closed ray, or a line. By Lemma 6.1, $\ell$ is not tangent to $P$, and (by Lemma 6.1) $\ell$ has no endpoint on $\operatorname{bd} P$. Thus, $\ell$ intersects the interior of $P$. We claim that $\ell$ has at least one endpoint in $\operatorname{int} P$. Suppose, it does not. Then $\ell \cap \operatorname{bd} P$ is a pair of points and so, the faces of $P$ corresponding to indices in $I$ meet at more than one point. Since $|I| \geq 3$, it contradicts the assumption that $P$ is simple and standard. So, $\ell$ has either one or two endpoints in $\operatorname{int} P$. If it has two, then the two distinct 3-dimensional Delaunay cells corresponding to those endpoints (as in Lemma 5.1 (V)) are both members of $\mathcal{D}^t$ and contain the planar convex polygon $\operatorname{conv} c_I : i \in I$. If $\ell$ has one endpoint in $\operatorname{int} P$, then there is a unique 3-dimensional polyhedron in $\mathcal{D}^t$ (the one corresponding to that endpoint of $\ell$) that contains the planar convex polygon $\operatorname{conv} c_i : i \in I$. Moreover, in this case $\ell$ intersects $\operatorname{bd} P$ at a vertex of $P$. Since $P$ is simple, that vertex is contained in exactly three faces of $P$, and hence, $\operatorname{conv} c_i : i \in I$ is a triangle.

Next, working in the reverse direction, assume that $F_1, F_2, F_3 \in \mathcal{F}$ are faces of $P$ that meet at a vertex $v$ of $P$. Then $v$ is in exactly three Voronoi cells, $V_1, V_2$ and $V_3$. Thus, $\operatorname{conv} c_1, c_2, c_3 \in \mathcal{D}$, and $\ell := V_1 \cap V_2 \cap V_3$ is an edge of $\mathcal{V}$. By the above argument, $\ell$ has one endpoint in $P$ and so, $\operatorname{conv} c_1, c_2, c_3$ is a member of $\mathcal{D}^t$, and has the property that exactly one 3-dimensional member of $\mathcal{D}^t$ contains it.

Clearly, $P$ has at least one vertex and so, (by the previous paragraph) $\mathcal{D}^t$ contains at least one 3-dimensional polyhedron.

We recall that the *nerve* of a set family $\mathcal{G}$ is the abstract simplicial complex

$$\mathcal{N}(\mathcal{G}) := \{\{G_i : i \in I\} :$$
$$G_i \in \mathcal{G} \text{ for all } i \in I \text{ and } \bigcap_{i \in I} G_i \neq \emptyset\}.$$

Now, let $P = \mathbf{B}(C)$ be a simple and standard ball-polyhedron in $\mathbb{E}^3$ and let $\mathcal{F}$ denote the set of its faces. We define the following abstract 2-dimensional simplicial complex $\mathcal{S}$ on the vertex set $C$: Let $\mathcal{S}$ be the abstract simplicial complex generated by those triples of $C$ which are vertices of a triangle on $\operatorname{bd} Q$. Both $\mathcal{S}$ and the nerve $\mathcal{N}(\mathcal{F})$ of $\mathcal{F}$ are 2-dimensional abstract simplicial complexes with the property that any edge is contained in a 2-dimensional simplex. Indeed, $\mathcal{S}$ has this property by definition, while $\mathcal{N}(\mathcal{F})$ has it because $P$ is simple and standard. It follows by Lemma 6.3 that $\mathcal{S}$ is isomorphic to $\mathcal{N}(\mathcal{F})$. By Theorem 4.1, $\mathcal{N}(\mathcal{F})$ is isomorphic to the face-lattice of another standard ball-polyhedron: $P^*$. Since $P^*$ is a convex body in $\mathbb{E}^3$ (i.e., a compact convex set with non-empty interior in $\mathbb{E}^3$), the union of its faces is homeomorphic to the 2-sphere. Thus, $\mathcal{S}$ as an abstract simplicial complex is homeomorphic to the 2-sphere. On the other hand, $\operatorname{bd} Q$ is a geometric realization of $\mathcal{S}$. Thus, we have obtained that $\operatorname{bd} Q$ is a geometric simplicial complex which is homeomorphic to the 2-sphere. It follows that $Q$ is homeomorphic to the 3-ball.

Clearly, $Q$ is a weakly convex polyhedron as $C$ is in convex position. Also, $Q$ is the union of convex polyhedra and so, it is decomposable. On the other hand, $Q$ is also co-decomposable, as $\mathcal{D}^t$ is a sub-polyhedral complex of $\mathcal{D}$

(by Lemma 6.2), which is a family of convex polyhedra the union of which is $\mathrm{conv}\ Q = \mathrm{conv}\ C$.

So far, we proved that $Q$ is a weakly convex, decomposable, and co-decomposable polyhedron with triangular faces in $\mathbb{E}^3$. By Theorem 3.1, $Q$ is infinitesimally rigid. Since $\mathrm{bd}\, Q$ itself is a geometric simplicial complex therefore its edge graph is rigid (since infinitesimal rigidity implies rigidity). Finally, we recall that the edges of the polyhedron $Q$ correspond to the edges of the ball-polyhedron $P$, and the lengths of the edges of $Q$ determine (via a one-to-one mapping) the corresponding inner dihedral angles of $P$. It follows that $P$ is rigid with respect to its inner dihedral angles.

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## REFERENCES

[1] M. Aigner and G. M. Ziegler, *Proofs from The Book*, Fourth edition, Springer, Berlin, 2010.

[2] F. Aurenhammer and R. Klein, *Voronoi diagrams*, Handbook of computational geometry, North-Holland, Amsterdam, 2000, 201–290.

[3] K. Bezdek, *Classical Topics in Discrete Geometry*, CMS Books in Mathematics, Springer, New York, 2010.

[4] K. Bezdek and M. Naszódi, *Rigidity of ball-polyhedra in Euclidean 3-space*, European J. Combin. 27/2 (2005), 255–268.

[5] K. Bezdek, Zs. Lángi, M. Naszódi and P. Papez, *Ball-polyhedra*, Discrete Comput. Geom. 38/2 (2007), 201–230.

[6] I. Izmestiev and J.-M. Schlenker, *Infinitesimal rigidity of polyhedra with vertices in convex position*, Pacific J. Math. 248/1 (2010), 171-190.

[7] R. Seidel, *Exact upper bounds for the number of faces in d-dimensional Voronoi diagrams*, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., Amer. Math. Soc., Applied geometry and discrete mathematics, 4 (1991), 517–529.

[8] J. J. Stoker, *Geometric problems concerning polyhedra in the large*, Com. Pure and Applied Math. 21 (1968), 119–168.